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Tiling Representations of Zeckendorf Decompositions

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Abstract

Zeckendorf's theorem states that every positive integer can be decomposed uniquely into a sum of non-consecutive Fibonacci numbers (where $f_1 = 1$ and $f_2 = 2$). Previous work by Grabner and Tichy (1990) and Miller and Wang (2012) has found a generalization of Zeckendorf's theorem to a larger class of recurrent sequences, called *Positive Linear Recurrence Sequences* (PLRS's). We apply well-known tiling interpretations of recurrence sequences from Benjamin and Quinn (2003) to PLRS's. We exploit that tiling interpretation to create a new tiling interpretation specific to PLRS's that captures the behavior of the generalized Zeckendorf's theorem.

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Chapter 1

Introduction

The motivation for this thesis comes from research done during the summer of 2020 at Williams College, where I was able to explore generalizations of the Fibonacci numbers. The combinatorial Fibonacci numbers are defined by the recurrence relation $f_{n+1} = f_n + f_{n-1}$, with initial conditions $f_1 = 1$ and $f_2 = 2$. This generates the Fibonacci sequence $1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Zeckendorf's theorem states that every positive integer can be decomposed uniquely into a sum of non-consecutive Fibonacci numbers.

There are many generalizations of the Fibonacci numbers, which involve changing three parameters: the number of terms in the recurrence relation, the coefficients in the recurrence relation, and the initial conditions. We used a certain generalization, that is called *positive linear recurrence sequences*, or PLRS's, which have restrictions on what those three parameters we can modify can be. The purpose of the specific definition of a PLRS is to permit a generalized version of Zeckendorf's theorem to hold. We will cover all of these terms and results in detail in Chapter 2. However, the technical definition of a PLRS and the choices regarding what we can modify can seem arbitrary, abstract, or difficult to understand at first. That is why we use tilings. Tiling interpretations are tools that help you visualize number sequences; they show combinatorial results in a tangible way. Thus the main goal of this thesis is to develop a tiling interpretation of PLRS's that captures the behavior of the generalized Zeckendorf's theorem.

Work on the main goal, including many examples, is discussed in Chapter 3. After easing into decompositions and the tiling interpretation with first order recurrences, the Fibonacci numbers are generalized to L-bonacci numbers. The main result on L-bonacci numbers, which connects the generalized Zeckendorf's theorem to a tiling interpretation, is Proposition 3.7.

Then, second and third order recurrence relations with positive coefficients are explored. These are generalized to develop the main result for all PLRS's with positive coefficients, which also connects the generalized Zeckendorf's theorem to a tiling interpretation, in Proposition 3.9. The chapter concludes with two ways to extend the main result to allow coefficients of 0 in Section 3.7.

Later, Chapter 4 explores tiling results on the number of summands used to decompose a number. Chapter 5 notes some potential routes for future exploration. Finally, Appendix A contains Python functions to generate PLRS's, calculate decompositions, and interpret them as tilings.

Chapter 2

Background

The Fibonacci numbers are one of the most ubiquitous sequences in mathematics. They were famously used in 1202 by Leonardo of Pisa, commonly known as Fibonacci, to answer a puzzle he poses about the idealized population growth of pairs of rabbits in his *Liber Abaci* (see Fibonacci and Sigler (2002)). While this puzzle helped popularize this sequence, using the Fibonacci numbers as a solution requires resorting to some unrealistic conditions, such as that the rabbits will never die. The Fibonacci numbers are defined by the recurrence relation $F_{n+1} = F_n + F_{n-1}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. This gives the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \ldots.$$

Interestingly, it took hundreds of years until the Fibonacci sequence was given its name by Édouard Lucas in a number theory text (see Lucas (1891)). We will use the combinatorial Fibonacci numbers, defined by $f_n = F_{n+1}$; their indices are just shifted by one, which makes their combinatorial interpretation more straightforward. That is to say, the combinatorial Fibonacci numbers are defined by the recurrence relation $f_{n+1} = f_n + f_{n-1}$, with initial conditions $f_1 = 1$ and $f_2 = 2$. This generates the sequence

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \ldots.$$

Even before Fibonacci's time, as chronicled by Singh (1985), the Fibonacci numbers were used in ancient and medieval India. In Sanskrit prosody, certain kinds of poetry are based on syllables composed of 1 or 2 morae;[1]

[1]A *mora* (pl. *morae*) is a component of a syllable; typically there are one or two morae that correspond to one syllable. In some languages, there can be three morae in one syllable; in this example, only the cases where there are 1 or 2 morae are considered.

syllables of 1 mora are called "light" and syllables of 2 morae are called "heavy". The Fibonacci numbers' first recorded use in any form was by Ācārya Piṅgala (c. 480 - 410 B.C.) and the first person to record the recurrence relation was Ācārya Virahāṅka (c. 600 - 800 A.D.), according to Singh (1985). In Table 2.1, observe the relationship between the number of morae and the ways to combine light and heavy syllables. The connection between these morae and the Fibonacci numbers is that when writing prose with n morae, we have f_n ways to combine light and heavy syllables.

1 syllable	2 syllables	3 syllables	4 syllables	5 syllables
1	1 + 1	1 + 1 + 1	1 + 1 + 1 + 1	1 + 1 + 1 + 1 + 1
	2	1 + 2	1 + 1 + 2	1 + 1 + 1 + 2
		2 + 1	1 + 2 + 1	1 + 1 + 2 + 1
			2 + 1 + 1	1 + 2 + 1 + 1
			2 + 2	2 + 1 + 1 + 1
				1 + 2 + 2
				2 + 1 + 2
				2 + 2 + 1

Table 2.1 Table of combinations of light and heavy syllables to form polysyllabic groups with a given number of morae. Light syllables are represented by 1 and heavy syllables are represented by 2. Adapted from a table in Singh (1985).

This connection serves as a good introduction to tiling representations of Fibonacci numbers; in the next section, we will identify light syllables with squares, and heavy syllables with dominoes.

2.1 Standard Tiling Representations

2.1.1 Fibonacci Numbers

We can show that f_n counts the number of ways to tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes , as in Benjamin and Quinn (2003). To show that this tiling interpretation really counts the same thing as f_n, we need to check that it satisfies both the recurrence relation $f_{n+1} = f_n + f_{n-1}$ and has the tiling initial conditions $f_1 = 1$ and $f_2 = 2$.[2]

[2]Comparing the combinatorial Fibonacci numbers with the standard Fibonacci numbers, the recurrence relations $f_{n+1} = f_n + f_{n-1}$ and $F_{n+1} = F_n + F_{n-1}$ are equivalent; the initial conditions are $f_1 = 1 = F_2$ and $f_2 = 2 = F_3$.

Recurrence Relation. Using our tiling interpretation, we have that f_{n+1} counts the number of ways to tile a $1 \times (n+1)$ board with squares and dominoes. A note on notation: the # symbol indicates the number of ways to tile the following board.

$$f_{n+1} = \#\underbrace{\boxed{\qquad\quad\cdots\qquad\quad}}_{n+1}$$

We have that f_n counts the number of ways to tile a $1 \times n$ board with squares and dominoes. This is the same as the number of ways to tile a $1 \times (n+1)$ board with squares and dominoes, where the final tile is required to be a square (highlighted in red).

$$f_n = \#\underbrace{\boxed{\qquad\quad\cdots\qquad}\boxed{\;\;}}_{n+1}$$

We have that f_{n-1} counts the number of ways to tile a $1 \times (n-1)$ board with squares and dominoes. This is the same as the number of ways to tile a $1 \times (n+1)$ board with squares and dominoes, where the final tile is required to be a domino (highlighted in red).

$$f_{n-1} = \#\underbrace{\boxed{\qquad\cdots\qquad}\boxed{\qquad}}_{n+1}$$

Now, consider that for a $1 \times (n+1)$ board, its tiling must have either a square or a domino as the final tile. As these are the only two possibilities, they partition all possible tilings. As a result, we conclude that $f_{n+1} = f_n + f_{n-1}$, and this tiling interpretation satisfies the recurrence relation.

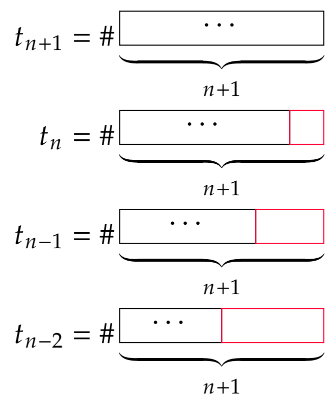
Initial Conditions. We verify that the initial conditions are correct. For a 1×1 board, there is one way to tile it, with a square $\boxed{\;\;}$, which gives $f_1 = 1$. For a 1×2 board, there are two ways to tile it, with two squares $\boxed{\;\;}\boxed{\;\;}$ or with one domino $\boxed{\qquad}$, which gives $f_2 = 2$.

2.1.2 Tribonacci Numbers

Next, we consider the Tribonacci numbers, which are an analogue of the Fibonacci numbers, but each term is a sum of the previous three instead of two terms. We can show that t_n counts the number of ways to tile a $1 \times n$ board with 1×1 squares $\boxed{\;\;}$, 1×2 dominoes $\boxed{\qquad}$, 1×3 trominoes $\boxed{\qquad\quad}$.

We also might not know what initial conditions we want, so let's take the initial conditions that arise naturally from this tiling interpretation.

Recurrence Relation. Using our tiling interpretation, we have that t_{n+1} counts the number of ways to tile a $1 \times (n+1)$ board with squares, dominoes, and trominoes. Similar to before, we have that a tiling must end in either a square, a domino, or a tromino. Counting the number of those possibilities respectively gives t_n, t_{n-1}, and t_{n-2}.



Indeed, they do satisfy $t_{n+1} = t_n + t_{n-1} + t_{n-2}$.

Initial Conditions. As we now have a third-order recurrence relation, we need three initial conditions. For a 1×1 board, there is one way to tile it, with a square, which gives $t_1 = 1$. For a 1×2 board, there are two ways to tile it, with two squares or with one domino, which gives $t_2 = 2$. For a 1×3 board, there are four ways to tiles it, with three squares, with a domino followed by a square, with a square followed by a domino, or with a tromino, which gives $t_3 = 4$. All together, they satisfy $t_1 = 1$, $t_2 = 2$, and $t_3 = 4$.

2.1.3 L-bonacci Numbers

Let's consider a more general recurrence relation, where each term in the sequence is the sum of an arbitrary number L of previous terms

$$h_{n+1} = h_n + h_{n-1} + \cdots + h_{n+1-L}.$$

We call these L-bonacci numbers. There is a tiling representation when $L \geq 1$, from Benjamin and Quinn (2003). If $L \geq 3$, to account for the h_{n-2} terms, we would use 1×3 trominoes. If $L \geq 4$, to account for the h_{n-3}

terms, we would use 1×4 tetrominoes . Extending this, if $L \geq i$, to account for the h_{n+1-i} terms, we would use $1 \times i$ tiles . So to represent all needed terms, we need tiles of dimensions 1×1 up through $1 \times L$.

2.1.4 Non-negative Coefficients

Let's consider a further generalization, where we introduce positive coefficients $c_i \geq 1$

$$h_{n+1} = c_1 h_n + c_2 h_{n-1} + \cdots + c_L h_{n+1-L}.$$

To represent a coefficient of $c_1 = 2$, we would use 2 colors for the 1×1 squares , . To represent a coefficient of $c_1 = 3$, we would use 3 colors for the 1×1 squares , , . In general, to represent a coefficient of $c_1 = i$, we would use i colors for the 1×1 squares , , , . . . , .

Moving on to the second coefficient, To represent a coefficient of $c_2 = 2$, we would use 2 colors for the 1×2 dominoes , . To represent a coefficient of $c_2 = 3$, we would use 3 colors for the 1×2 dominoes , , . In general, to represent a coefficient of $c_2 = i$, we would use i colors for the 1×2 dominoes , , , . . . , .

Generalizing this to all coefficients, to represent c_i in $\mathbb{Z}^+$, we can use c_i colors for the $1 \times i$ tiles. We also can have coefficients of 0, but we need to be careful. While possible to do in more generality,[3] we introduce the restriction that the first and last coefficients, i.e., c_1, c_L must be positive, and the intermediate coefficients may be 0. The tiling interpretation of having a coefficient of $c_i = 0$ is that there are zero tiles of size $1 \times i$ to use. We avoid allowing the first coefficient to be 0 to avoid sequences that are really comprised of different sequences that are interspersed.[4] We avoid allowing the last coefficient to be 0, so that L accurately reflects the depth of previous terms that actually affect subsequent terms in the sequence.

Lastly, we can use a variety of initial conditions by implementing special rules for the first tile, called *phases*, as in Benjamin and Quinn (2003). We will discuss phases in more detail as needed.

[3] For the curious reader, to represent coefficients in any commutative ring R, we can use "weights" for the $1 \times i$ tiles instead of colors; see Benjamin and Quinn (2003) for more details.

[4] For example, if we had the recurrence relation $h_{n+1} = 2h_{n-1}$, only every other term is related. We could have initial conditions $h_1 = 1$ and $h_2 = 0$, which would generate the sequence $\{1, 0, 2, 0, 4, 0, 8, 0 \ldots\}$, which we can see is really just two independent sequences, $\{1, 2, 4, 8, \ldots\}$ and $\{0, 0, 0, 0, \ldots\}$ that are interspersed.

2.2 Zeckendorf's Theorem

When we think about natural numbers, we typically think of them in terms of their unique decimal, or base-10, representation. Natural numbers also have a unique base-b representation, for any integer $b > 1$. To denote that a number is in a base other than base-10, we can use a subscript b following the number. For example, for the number 11, we can write it in base-2 as 1011_2, in base-3 as 102_3, in base-4 as 23_4, etc.

This raises the following question: can we represent numbers uniquely in other ways? In particular, can we use the Fibonacci numbers? As the number 1 is a Fibonacci number, it is clear that it is possible to write all natural numbers as some sum of Fibonacci numbers. However, we would want a representation to be unique. Let's consider some candidate decompositions of 11:

1. $11 = 8 + 3 = f_5 + f_3$
2. $11 = 5 + 3 + 2 + 1 = f_4 + f_3 + f_2 + f_1$
3. $11 = 5 + 5 + 1 = 2f_4 + f_1$

Note that the first representation is the most simple. Regarding the second representation, we could simplify with the recurrence relation by replacing $f_4 + f_3$ with f_5 and by replacing $f_2 + f_1$ with f_3. This suggests the introduction of some restriction that will prevent us from representations that could be simplified using the recurrence relation. As the recurrence relation applies to any consecutive Fibonacci numbers, we want to avoid any use of consecutive Fibonacci numbers. Lastly, the third representation uses a number twice. While this might not be immediately clear, we could split up one copy of f_4 using the recurrence relation into $f_3 + f_2$, which brings us to the second representation, where we could then use the recurrence relation twice to simplify to the first. These intuitions we built up about wanting to use representations that cannot be simplified using the recurrence relation are captured in the following theorem, from Zeckendorf (1972). [5]

[5] It is interesting to note that this result is so recent considering that the Fibonacci numbers for centuries prior, and the simplicity of the proof. Edouard Zeckendorf writes that he originally conceived of his eponymous theorem in 1939. It then appeared in publications by Lekkerkerker (1951) and Daykin (1960), before finally being published by Zeckendorf in Zeckendorf (1972). See Kimberling (1998) for more discussion of the history of Zeckendorf's theorem, and a biography of Zeckendorf.

Theorem 2.1 (Zeckendorf's Theorem)**.** *Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers, when indexing from* $\{1, 2, 3, 5, \dots\}$*. We call this unique decomposition the* Zeckendorf decomposition*.*

Proof. First, we show existence, by strong induction. Suppose that $m \in \mathbb{N}^+$. If m is a Fibonacci number, we're done; else, there exists $j \in \mathbb{N}^+$ such that $f_j < m < f_{j+1}$. For our base cases, we use $m = 1, 2, 3$, since they are all Fibonacci numbers, and hence have trivial decompositions. So suppose that each $n < m$ has a decomposition into Fibonacci numbers. Then, consider $\tilde{n} = m - f_j$. As $\tilde{n} < m$, then by the induction hypothesis, $\tilde{n}$ has a decomposition into Fibonacci numbers. Observe that $\tilde{n} < f_{j+1} - f_j = f_{j-1}$, so the decomposition doesn't use f_{j-1}. Thus, we may combine the decomposition for $\tilde{n}$ with f_j to get a valid decomposition with only nonadjacent Fibonacci numbers for $\tilde{n} + f_j = m$.

Second, we show uniqueness. Suppose for the sake of contradiction that two different representations of a natural number m into nonconsecutive Fibonacci numbers exist,

$$\begin{aligned} m &= f_{i_1} + f_{i_2} + \cdots + f_{i_k} \\ m &= f_{j_1} + f_{j_2} + \cdots + f_{j_l}. \end{aligned}$$

We can write the sets of indices as $I = \{i_1, i_2, \dots, i_k\}$ and $J = \{j_1, j_2, \dots, j_l\}$. Let $I' = I/J$ and $J' = J/I$, the set of indices that are unique to each decomposition. As the decompositions are not the same, $I', J' \neq \varnothing$. We can consider that

$$\sum_{i \in I} f_i - \sum_{i \in I \cap J} f_i = \sum_{j \in J} f_j - \sum_{j \in I \cap J} f_j, \text{ i.e., } \sum_{i \in I'} f_i = \sum_{j \in J'} f_j.$$

Next, we pick the largest Fibonacci numbers that have indices in I', J': let $f_{i_{\max}} = \max\{f_i : i \in I\}$ and $f_{j_{\max}} = \max\{f_j : j \in J\}$. As $I' \cap J' = \varnothing$, then either $f_{i_{\max}} > f_{j_{\max}}$ or $f_{i_{\max}} < f_{j_{\max}}$. Without loss of generality, let $f_{i_{\max}} < f_{j_{\max}}$. It can be shown by induction that the sum of every other Fibonacci number up to some f_k, starting from $f_1 = 1$ or $f_2 = 2$ depending on parity, is one less than f_{k+1}. As this is the closest that the Fibonacci numbers could be to each other in a Zeckendorf decomposition, we have that

$$\sum_{i \in I'} f_i < f_{i_{\max}+1} \leq f_{j_{\max}}.$$

However, this contradicts that

$$\sum_{i \in I'} f_i = \sum_{j \in J'} f_j,$$

since $f_{j_{\max}} \in \{f_j : j \in J'\}$. Hence, it is not possible for there to be two different representations of a natural number with nonconsecutive Fibonacci numbers. □

Remark 2.2. Zeckendorf's theorem results in us using $f_5 + f_3 = 8 + 3$ as the unique decomposition of 11.

2.3 Generalized Zeckendorf's Theorem

In the literature, there exist many generalizations of Zeckendorf's theorem to more sequences than just the Fibonacci numbers. When generalizing this theorem, there are three main types of changes that can be made to the underlying recurrence sequence. The first is that the depth of the recurrence can be increased, the second is the the coefficients can be changed, and the third is that the initial conditions can be changed. However, when making a generalization of Zeckendorf's theorem, we must be careful to consider what properties of the original theorem we wish to preserve. For example, Zeckendorf also wrote about a Zeckendorf's theorem analogue for Lucas numbers in Zeckendorf (1972), however, his formulation lost uniqueness– some numbers have two representations. For additional generalizations of Zeckendorf's theorem see Hoggatt (1972); Keller (1972); Lengyel (2006).

We choose to have the preservation of unique decompositions as our top priority. In particular, this property is what allows these sequences to be used as bases of enumeration; see Fraenkel (1985) and Gewurz and Merola (2012). It will also be essential for our development of tiling representations. Ultimately, the generalized Zeckendorf's theorem that we use here is from Miller and Wang (2012). This allows us great flexibility with respect to the depth of the recurrence and the coefficients, while the trade off is that we lose control over the initial conditions, which are forced to be specific values depending on the coefficients.

We begin with an important definition that establishes exactly what kinds of sequences we have a generalized Zeckendorf's theorem for.

Definition 2.3. We say a sequence $\{h_n\}_{n=1}^{\infty}$ of positive integers is a **Positive Linear Recurrence Sequence (PLRS)** if the following properties hold:

1. *Recurrence relation*: There are non-negative integers $L, c_1, \ldots, c_L$ such that
$$h_{n+1} \;=\; c_1 h_n + \cdots + c_L h_{n+1-L},$$
with L, c_1 and c_L positive.

2. *Initial conditions*: $h_1 = 1$, and for $1 \leq n < L$ we have
$$h_{n+1} \;=\; c_1 h_n + c_2 h_{n-1} + \cdots + c_n h_1 + 1.$$

The coefficient c_1 must be positive to prevent the situation where there are multiple independent subsequences that don't interact with each other, such as what would happen if $h_{n+1} = 2h_{n-1}$, where there could be two interspersed subsequences. The coefficient c_L must be positive to prevent the inclusion of an arbitrary number of coefficients of zero at the end, which would permit multiple values of L for one sequence. The "+1" is included in the definition of the initial conditions to force the PLRS to grow quickly enough so that there are no repeated terms, and more generally, because it permits a relatively simple generalized Zeckendorf's theorem. Now let's see an example of a PLRS.

Example 2.4. We define a PLRS by its coefficients. Let there be $L = 3$ coefficients, $c_1 = 1$, $c_2 = 4$, and $c_3 = 9$. Next, we need to determine the initial conditions. We are given that $h_1 = 1$. Then, by the definition, $h_2 = c_1h_1+1 = 2$, and $h_3 = c_1h_2 + c_2h_1 + 1 = 7$. Now, we have three initial conditions, which are sufficient to use the recurrence relation $h_{n+1} = c_1h_n + c_2h_{n-1} + c_3h_{n-2}$. Thus, $h_4 = c_1h_3 + c_2h_2 + c_3h_1 = 24$. Repeatedly applying the recurrence relation allows us to generate the PLRS $\{1, 2, 7, 24, 70, 229, \ldots\}$.

A decomposition of a positive integer N is a sum of positive integers that sum to N. The decomposition is a formal object in the sense that which numbers are summed together, and how many times, is essential information (the order does not matter). (We also write each unique summand only once, multiplied by the appropriate coefficient.) In Section 2.2, we saw that there may be multiple possible decompositions for N into Fibonacci numbers. In order to get unique decompositions, Zeckendorf's theorem gives a decomposition rule, which is that only nonconsecutive Fibonacci numbers can be used. Shortly, we will define a *legal decomposition*, which is a decomposition that obeys certain rules designed to create unique decompositions. The definition generalizes the Fibonacci numbers' rule, by making use of *decomposition blocks*. A decomposition block is an ordered sequence of i coefficients a_i (for $1 \leq i \leq L$), where the coefficients are multipliers for a subsection of the PLRS. There are c_i distinct decomposition blocks of length i. (Thus the total number of distinct decomposition blocks is $c_1 + c_2 + \cdots + c_L$.)

Definition 2.5 (Legal decompositions)**.** We call a decomposition $\sum_{i=1}^{m} a_i h_{m+1-i}$ of a positive integer N *legal* if $a_1 > 0$, the other $a_i \geq 0$, and one of the following two conditions holds:

1. We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
2. There exists $s \in \{1, \ldots, L\}$ such that

$$a_1 \ = \ c_1,\ a_2 \ = \ c_2,\ \cdots,\ a_{s-1} \ = \ c_{s-1} \text{ and } a_s < c_s, \tag{2.1}$$

[6]

$a_{s+1}, \ldots, a_{s+\ell} \ = \ 0$ for some $\ell \geq 0$, and the remaining decomposition $\sum_{i=s+\ell+1}^{m} a_i h_{m+1-i}$ is legal or empty[7].

To understand the main idea of this definition, focus on just the second condition. The second condition states that if there are $s \in \{1, \ldots, L\}$ coefficients used in a decomposition block (the a_i's), then the first $s-1$ of those must match the first $s-1$ coefficients used to generate the PLRS (the c_i's), and for the sth coefficient, $a_s < c_s$. Also, the remaining amount left to decompose must decompose legally as well, after a gap of $\ell \geq 0$ terms in the sequence. The second condition is by far the more important one, because only the second condition can be applied repeatedly within the decomposition of one number, (since it allows for the remainder of the decomposition to be calculated recursively). When the first condition is used, the decomposition must end. The first condition states that all the m coefficients used in the decomposition block (the a_i's) must match the first m coefficients used to generate the PLRS (the c_i's).[8]

[6]Clarifying this in the case of small s, if $s = 1$, then the condition is $a_1 < c_1$. If $s = 2$, then the condition is $a_1 = c_1$ and $a_2 < c_2$. If $s \geq 3$, then the condition is $a_1 = c_1, a_2 = c_2, \ldots, a_{s-1} = c_{s-1}$ and $a_s < c_s$.

[7]That the remaining decomposition is legal or empty can be formalized as $\{b_i\}_{i=1}^{m-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is legal or empty.

[8]At first glance, it may seem that the both conditions could be combined into one like the second condition, but with $a_s \leq c_s$ instead of $a_s < c_s$. However, the conditions are written the way they are to prohibit matching all L coefficients, as then the recurrence relation should be used to create a larger term. Writing the conditions separately clarifies that only all but one of the coefficients can be "maxed-out". So our decomposition blocks are ordered sequences of coefficients (the a_i's) that can be used to satisfy the second condition. We will eventually see that the decomposition blocks also work for the first condition, but with some additional modifications or rules, which depend on the initial conditions.

One way to think about a decomposition block is like a mold that we can fill up, but not exceed, and where each position has to be filled in order to begin filling the next position. Then, once one decomposition block, or mold, is ended, a new one can begin. Let's explore some examples to understand how it works. Later on, we will combine the ideas of these molds with the tiling interpretation presented in Chapter 3 to create a new tiling interpretation of the decompositions.

Notation 2.6. We will use the notation $[c_1, \ldots, c_L]$, which is the collection of all L coefficients, to represent the PLRS $h_{n+1} = c_1 h_n + \cdots + c_L h_{n+1-L}$.

Example 2.7. As in the previous example, define a PLRS by the coefficients $[c_1, c_2, c_3] = [1, 4, 9]$. The PLRS is $\{1, 2, 7, 24, 70, 229, \ldots\}$. Say we want to create a decomposition for 21. We start a decomposition block with the largest number from the sequence that is less than or equal to the number we wish to decompose. In the case of 21, the largest number in the sequence less than or equal to it is 7. The first coefficient is 1, so we can only use up to one 7 within a decomposition block. So we move to the previous term in the sequence, 2. We can use up to four 2's, since the second coefficient is 4. As $21 - 7 - 4 \cdot 2 = 6 \geq 0$, we choose to use all four 2's in the decomposition. Then, we go to the third position in the decomposition block, where we can use up to nine 1's. To complete the decomposition, we only need to use six of the 1's. So we get that the decomposition of $21 = 7 + 4 \cdot 2 + 6 \cdot 1$. See Figure 2.1 for a visual interpretation of how the coefficients fit into the mold.

Multiple decomposition blocks may be necessary. For 134, it has decomposition $[70 + 2 \cdot 24] + [7 + 4 \cdot 2 + 1]$; the first decomposition block has coefficients $1, 2$ which are dominated by $1, 4$ and the second decomposition block has coefficients $1, 4, 1$ which are dominated by $1, 4, 9$. This shows why decomposition blocks need to be able to terminate early, because if we forced the first decomposition block to be $1, 2, 0$ (which is dominated by $1, 4, 9$) then we wouldn't have been able to use any 7's in our decomposition–which we need.

Here is another example, where $c_1 > 1$. This example shows two consecutive decomposition blocks.

Example 2.8. Define a PLRS by the coefficients $[2, 1, 1]$. Then the PLRS is $\{1, 3, 8, 20, 51, \ldots\}$. Say we want to decompose 4. The largest sequence term (or multiple of sequence term, by a factor from $1, \ldots, c_1$) that is less than or equal to 4 in this case is 3. It's clear that we will want the decomposition $4 = 3 + 1$, but it may not be immediately obvious that this actually uses two

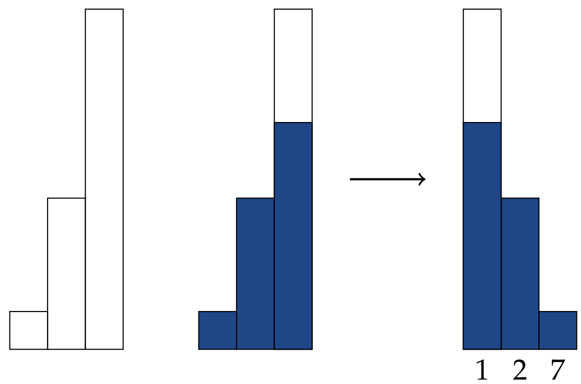


Figure 2.1 The figure on the left is an empty "mold" that can be used to visualize a decomposition block for a PLRS generated by the coefficients $[1, 4, 9]$. The figure in the middle is representing the particular coefficients $1, 4, 6$ that are used in the decomposition of the number 21. Each coefficient used in a decomposition must be maximized in order for the next coefficient to be used. The figure on the right shows the decomposition blocked flipped, so that the decomposition coefficients (represented by the height of the blue region) are in the correct order for the sequence when written from left to right. Lastly, when this decomposition block's coefficients are applied to the first three sequence terms $1, 2, 7$, we see that the number $6 \cdot 1 + 4 \cdot 2 + 1 \cdot 7 = 21$ is represented.

decomposition blocks. For a decomposition block to contain more than one unique sequence term, all sequence terms, except for the last one, must be maximized to the coefficients $[2, 1, 1]$. So since we aren't using 3 twice, we can't then put a different sequence term in the same decomposition block. However, we can stop a decomposition block at any position, and start another one immediately. As a result, the decomposition is $4 = [3] + [1]$, with two distinct decomposition blocks.

Now, we state the main theorem of this section. It is originally due to Grabner and Tichy (1990) and this formulation is from Miller and Wang (2012).

Theorem 2.9 (Generalized Zeckendorf's Theorem for PLRS's)**.** *Let $\{h_n\}_{n=1}^{\infty}$ be a* Positive Linear Recurrence Sequence*. Then there is a unique legal decomposition for each positive integer $N \geq 0$.*

Chapter 3

Tilings of Generalized Zeckendorf Decompositions

In this chapter, we introduce *decomposition tilings*, which are a modification of the standard tiling representations we saw in Chapter 2. By designing decomposition tilings so that they correspond to how decomposition blocks work, they capture the behavior of the generalized Zeckendorf's theorem. We start with first order recurrences in Section 3.1 to introduce decompositions and our decomposition tiling interpretation. We show how to extend this tiling interpretation to the Fibonacci numbers in Section 3.2. We then generalize our tilings to a simple generation of the Fibonacci numbers, which we call L-bonacci numbers, in Section 3.3. There, we check that our work is correct, in the sense that the tiling interpretation corresponds to the generalized Zeckendorf's theorem, with Proposition 3.7. L-bonacci numbers are only generated by coefficients of 1. So we extend our tiling interpretation to second order and third order recurrences with arbitrary positive coefficients in Sections 3.4 and 3.5. Then, we generalize our results to all PLRS's with positive coefficients in Section 3.6. We also check that our work is correct again with Proposition 3.9. Finally, we discuss two ways to modify the tiling interpretation to permit coefficients of 0 in Section 3.7.

3.1 First Order Recurrences

Let's start with first order recurrences. All first order PLRS's can be written as $h_{n+1} = c_1 h_n$, and have the initial condition of $h_1 = 1$. Since there's only one coefficient, we can forgo the subscript and write $c = c_1$. Ignoring the case of the trivial sequence generated by $c = 1$, we can note that $c = 2$ generates the sequence

$$\{1, 2, 4, 8, 16, 32, \ldots\},$$

that $c = 3$ generates the sequence

$$\{1, 3, 9, 27, 81, 243, \ldots\},$$

that $c = 4$ generates the sequence

$$\{1, 4, 16, 64, 256, 1024, \ldots\},$$

and in general, c generates the sequence

$$\{1, c, c^2, c^3, c^4, c^5 \ldots\}.$$

Notice that this kind of sequence is related to the base-c representation of a number. For example, if $c = 10$, we have the sequence

$$\{1, 10, 10^2, 10^3, 10^4, 10^5, \ldots\}.$$

Then, we can write the decimal representation of any number. For example, we can write 246 as $2 \cdot 10^2 + 4 \cdot 10 + 6 \cdot 1$. We can also express 246 in other bases, for example, in base-2 (binary), it is 11110110_2, which is $1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$. In base-3 (ternary), it is 100010_3, which is $1 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0$. In base-4 (quaternary), it is 3312_4, which is $3 \cdot 4^3 + 3 \cdot 4^2 + 1 \cdot 4^1 + 2 \cdot 4^0$.

The basic properties of a base-c representation of a number are that for each power of c, it can be multiplied by a coefficient in $\{0, 1, \ldots, c - 1\}$. A coefficient can't be c or higher, because then you should use a larger power of c instead. The most straightforward way to represent numbers in base-c using tilings would be to use c squares, each with a *filter*, which acts as a multiplier of $\{\times 0, \times 1, \ldots, \times c - 1\}$.[1] For the purpose of enumerating the total number of possible tilings on a board of a given length, these filters can correspond to a color, and are counted as described in Chapter 2. Let's now see an example.

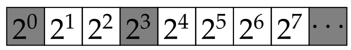
Example 3.1 (Binary Tilings)**.** To represent the binary decomposition of a number with tilings, we will be tiling a board that is labelled with the sequence of powers of 2.

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	$\cdots$

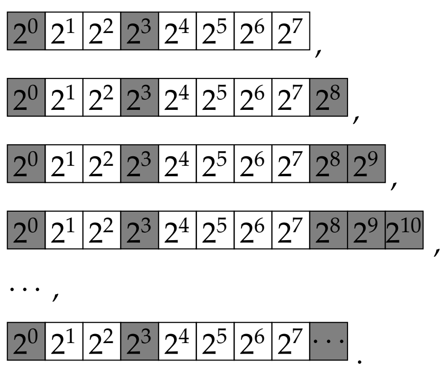
[1]These filters are inspired by the double letter and triple letter score spaces on a Scrabble game board (which are blue and green respectively).

One natural question is how long does the board need to be? Well, the board needs to be at least long enough so that all nonzero coefficients are able to be represented on the board. Let's look at the tiling representation for 246 and then return to this question.

We previously saw that $246_{10} = 11110110_2 = 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$. To capture this on our board, we use transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.



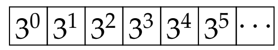
Then the way to "read" this tiling is to ignore anything on the board that is covered by an opaque tile (since the opaque tile is blocking it from being read), and then add all of the uncovered entries. As we know that binary representations of numbers are unique, we just need to make sure that there is a unique tiling representation of any number. Now with this tiling interpretation, any positions that are labelled with 2^n with $n > 7$ must be covered by an opaque tile. As a result, all of the following tilings can be considered equivalent:



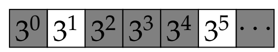
As we want a unique tiling representation, we will use the last board, which is semi-infinite (infinite in one direction), as this will be long enough for all cases. However, in practice, we can omit discussion of the length of the board, as long as all non-opaque tiles are shown, with the understanding that any drawings are equivalent to a semi-infinite board.

Now, let's look at the sightly more complicated example of a ternary tiling representation.

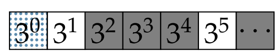
Example 3.2 (Ternary Tilings)**.** To represent the ternary decomposition of a number with tilings, we will be tiling a board that is labelled with the sequence of powers of 3.



We previously saw that $246_{10} = 100010_3 = 1 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0$. To capture this on our board, we use transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.

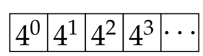


What about if there is a coefficient of 2? Observe that $248_{10} = 100012_3 = 1 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0$. To capture this on our board, we use transparent tiles with a light blue $\times 2$ filter to represent a coefficient of 2, transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.

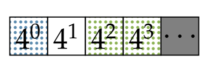


We can continue to add more filters, as in the example of a quaternary tiling representation.

Example 3.3 (Quaternary Tilings)**.** To represent the ternary decomposition of a number with tilings, we will be tiling a board that is labelled with the sequence of powers of 4.



We previously saw that $246_{10} = 3312_4 = 3 \cdot 4^3 + 3 \cdot 4^2 + 1 \cdot 4^1 + 2 \cdot 4^0$. To capture this on our board, we use transparent tiles with a light green $\times 3$ filter to represent a coefficient of 3, we use transparent tiles with a light blue $\times 2$ filter to represent a coefficient of 2, transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.



We can generalize the results from this section in the following result. While very simple, it lays an important foundation for generalizing to more complicated and interesting sequences.

Proposition 3.4. *For any first order PLRS $\{h_n\}_n$, defined by $h_{n+1} = ch_n$, with $c > 1$, the unique decompositions of every positive integer guaranteed by the generalized Zeckendorf's theorem have a one-to-one correspondence with a decomposition tiling representation.*

Proof. A first order PLRS is $h_{n+1} = ch_n$. Let x be the number we wish to represent. Begin with a semi-infinite board and label each position as follows[2]

c^0	c^1	c^2	c^3	c^4	c^5	$\cdots$

Recall that every positive integer x has a unique base-c representation for all natural numbers $c \geq 2$. So we can write $x = \alpha_0 + \alpha_1 c + \alpha_2 c^2 + \alpha_3 c^3 + \alpha_4 c^4 + \alpha_5 c^5 + \cdots$ with a unique sequence of $\{\alpha_i\}_i$. Then, when $\alpha_i = 0$, use an opaque square to cover that position. When $\alpha_i = 1$, use a transparent square to cover that position. When $\alpha_i = 2, 3, 4, \ldots, c-1$, use a transparent square with a colored filter , , , . . . , to cover that position, where each colored filter represents a multiplier of $\times 2, \times 3, \times 4, \ldots, \times c - 1$. As this tiling is in one-to-one correspondence with the unique base-c representation, it is unique. □

3.2 Fibonacci Numbers

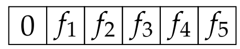
In Chapter 2, we saw the standard tiling representations of the Fibonacci numbers, using squares and dominoes. We also know that Zeckendorf's theorem says that every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers. How can we combine these two concepts?

We apply the concepts of transparency and filters that were just introduced to our square and domino tilings from Chapter 2. Squares are opaque , and dominoes are opaque for their first half and transparent for their second half . Note that the squares have a purely cosmetic addition of diagonal black lines; this is to help distinguish between spaces on the board that are covered by opaque squares and by opaque parts of dominoes. The purpose of the introduction of this type of tiling that is different from the

[2]We we later use boards where there is a zeroth position on the board labeled with 0 before the positions $i > 0$ are labelled with h_i. The board used here without a 0 is equivalent to a board with a 0, when we require that an opaque square is used to cover to 0, since that restriction means that no additional choices are introduced for the ways to tile the board.

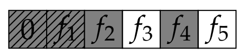
standard tilings of Chapter 2 is to capture the behavior of the decomposition rule. We begin with the Fibonacci numbers, because their decomposition rule is easy to understand–it says that no consecutive Fibonacci numbers may be used. So using these dominoes that are half-opaque and half-transparent prevents any consecutive Fibonacci numbers from ever being used.

We label the board that we are tiling as follows.



Note that we are labeling the first square with a 0, which we had not done with first-order recurrences. This is necessary now to allow a domino's transparent half to be placed over the first sequence term f_1, so that it can be used in a decomposition.

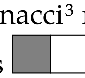
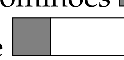
So to represent $11 = 8 + 3 = f_5 + f_3$, we create the following tiling which is two squares followed by two dominoes.



Now, let's look at more examples in Figure 3.1. Recall that we are labelling the underlying strip with the sequence $\{f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, \ldots\}$. Also, $\sum_{\varnothing}$ denotes the empty sum, which is equal to the additive identity, 0.

3.3 *L*-bonacci Numbers

One straightforward generalization of the Fibonacci numbers that we were introduced to in Chapter 2 is the L-bonacci numbers, which are defined by $h_{n+1} = h_n + h_{n-1} + \cdots + h_{n-L+1}$. Note that these are a proper subset of the set of all PLRS's, with the restriction that we are using only coefficients of 1. We also use the initial conditions given by the definition of a PLRS.

First, we consider the Tribonacci[3] numbers $t_{n+1} = t_n + t_{n-1} + t_{n-2}$. Again we use squares , dominoes , and now we add in trominoes with the first third opaque . The reason why we need the tromino is because the decomposition rule for Tribonacci numbers is that no three consecutive Tribonacci numbers can be used. The rule is a consequence of the fact that if there were three consecutive Tribonacci numbers used, they

[3]Tribonacci is a commonly used name for the 3-bonacci numbers.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= f_1 \\
2 &= f_2 \\
3 &= f_3 \\
4 &= f_3 + f_1 \\
5 &= f_4 \\
6 &= f_4 + f_1 \\
7 &= f_4 + f_2 \\
8 &= f_5 \\
9 &= f_5 + f_1 \\
10 &= f_5 + f_2 \\
11 &= f_5 + f_3 \\
12 &= f_5 + f_3 + f_1
\end{aligned}
$$

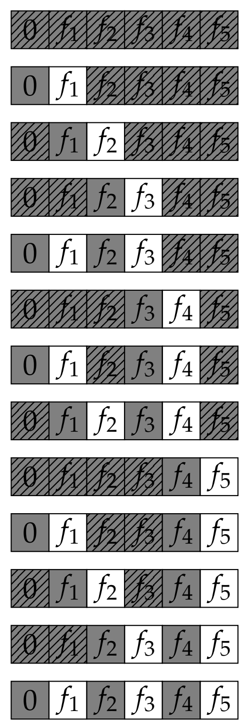


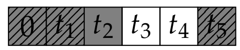
Figure 3.1 Examples of decomposition tilings for the Fibonacci numbers, i.e., the PLRS generated by $[1, 1]$.

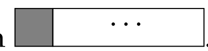
could be replaced with the next term in the sequence by using the recurrence relation–so uniqueness would be lost.

Again we use labels on the board that we are tiling:

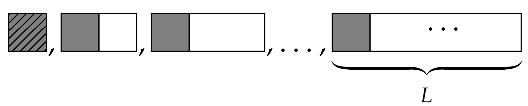
0	t_1	t_2	t_3	t_4	t_5

So to represent $11 = 7 + 4 = t_4 + t_3$, we draw:



We can extended this method to all L-bonacci numbers defined by $h_{n+1} = h_n + \cdots + h_{n+1-L}$, using tiles of the form . We state this precisely in the following result, which we also prove aligns with the existing definition of decomposition blocks.

Tiling Interpretation 3.5 (Decomposition Tilings for L-bonacci Sequences)**.** Consider any PLRS generated by $[\underbrace{1, \ldots, 1}_{L}]$. Construct a semi-infinite strip, and label the initial 1×1 space with a 0. Then label each subsequent space by the terms of the PLRS, beginning with $h_1 = 1$. Tile the strip with $1 \times i$ tiles where the leftmost 1×1 part of each tile is opaque, and the remaining $1 \times (i - 1)$ part of each tile is transparent, for all $i \in \{1, \ldots, L\}$, i.e.,



The only restriction on tile placement is that a tile cannot be placed such that a transparent portion covers the initial 0.[4]

Next, we describe the greedy algorithm for determining a tiling. An implementation of this algorithm (fully generalized to all PLRS's) appears in Appendix A.

Algorithm 3.6. The PLRS we are working with is $h_{n+1} = h_n + \cdots + h_{n-L+1}$. By the generalized Zeckendorf's theorem, any positive integer x has a unique decomposition into a sum of L-bonacci numbers such that there are never L consecutive L-bonacci numbers used. We will be tiling the following board

[4] In this tiling interpretation, there are no tiles that begin with a non-opaque portion, so we do not need to worry about this restriction. However, it is stated here, because it will be relevant when generalizations allow for tiles that begin with non-opaque portions.

0	h_1	h_2	h_3	h_4	h_5	$\cdots$

We proceed by decomposition blocks using the greedy algorithm. First, find the maximal h_N such that $h_N \leq x$. Then, we know that h_N will be the label on the rightmost space that will not be covered by an opaque tile. [We cover all positions on the board with labels h_m for $m > N$ with opaque squares.] Next, we determine what size tile will be the tile that has its rightmost (and transparent) part covering h_N, by looking at $h_{N-1}, h_{N-2}, \ldots$ until we find the last consecutive L-bonacci number that is used in the decomposition. The last possible consecutive L-bonacci number that is used is not before h_{N-L+2}, because the inclusion of h_{N-L+1} would allow the recurrence relation to be used, implying that $x \geq h_{N+1}$. Say that the number of consecutive L-bonacci numbers used was j. Then, we place a $1 \times (j+1)$ tile on our board so that those j consecutive L-bonacci numbers are visible under the transparent portion, and the opaque portion covers h_{N-j}. Now, we have finished the first decomposition block, and our board looks like[5]

?	h_{N-j}	h_{N-j+1}	$\cdots$	h_{N-1}	h_N	h_{N+1}	$\cdots$

.

All that is left is to tile the region labelled with a question mark. To do so, we calculate

$$y = x - (h_N + \cdots + h_{N-j+1}),$$

which is the part of x that hasn't been accounted for yet. We then proceed with the next decomposition block as if y were the number that we were trying to decompose, first by finding the largest L-bonacci number that is less than or equal to y, which becomes the rightmost part of the next non-square tile used. If that L-bonacci number happens to be h_{N-j-1}, then the next non-square tile is adjacent to the last; if not, then there can be any number of squares indicating positions skipped before the next non-square tile (and corresponding decomposition block) begins. We keep applying this method until the part of x that hasn't been accounted for yet is 0, and add additional opaque squares to cover all remaining uncovered positions on the board.

Finally, we show that our tiling interpretation corresponds to the unique decompositions given by the generalized Zeckendorf's theorem, which we

[5]The board is larger only to allow the subscripts to be more readable; it is equivalent to the small sized boards.

prove in two ways. Both proofs, however, rely on the fact that tiles correspond to the decomposition blocks used in legal decompositions.

Proposition 3.7. *The unique decomposition of any positive integer guaranteed by the generalized Zeckendorf's theorem* (*Theorem* 2.9) *into a sum of terms of a PLRS, generated by L coefficients of* 1*, has a one-to-one correspondence with a decomposition tiling representation given by Tiling interpretation* 3.5*.*

Proof. This tiling is unique since we show there exists a bijection between the tilings and the numbers that they decomposing. Define T_∞ as the collection of all L-bonnaci decomposition tilings (as previously defined) of a semi-infinite board. Let $\tau_n \in T_\infty$ be the tiling representation of n. Let $\rho : \mathbb{N} \to T_\infty$ be defined by $\rho(n) = \tau_n$. Let T_m be the collection of all tilings of a board of infinite length, where all tiles beginning at position m and afterwards are opaque squares (where 0 labels the zeroth position, and each h_m labels the mth position). We claim that $\rho([0, h_m - 1]) = T_m$. Since no L-bonacci number h_m or greater is needed in a decomposition of a number in $[0, h_m - 1]$, $\rho([0, h_m - 1]) \subseteq T_m$. Now for any $\tau_i \in T_m$, it represents the value $i \in \mathbb{N}$. Note that $i \in [0, h_m - 1]$ since if not, $i \geq h_m$, which implies that h_m (or a greater term) appears in a decomposition block (by the generalized Zeckendorf's theorem), and thus a non-opaque tile occurs at position m (or later), which contradicts that $\tau_i \in T_m$. Thus $\rho([0, h_m - 1]) \supseteq T_m$, and we conclude that $\rho([0, h_m - 1]) = T_m$.

Consider ρ with the domain restricted to $[0, h_m - 1]$, and call it ρ_m. By the argument we just made, ρ_m is surjective by construction, because for any $\tau_i \in T_m$, we can determine the value it represents, i, such that $\rho(i) = \tau_i$. Note that $|[0, h_m - 1]| = h_m$. Also, $|T_m| = h_m$, as h_m exactly counts the number of tilings in T_m. Hence, $|[0, h_m - 1]| = |T_m|$. As we have a surjection ρ_m from two finite sets of the same size, we have a bijection.[6] As ρ_m is a bijection for all finite $m \in \mathbb{N}$, we can find an m such that $[0, h_m - 1]$ includes n, for any natural number n. □

Proof. This second proof draws directly on the definition of a decomposition block, to show a correspondence between decomposition blocks and the tiles that can be used. Decomposition blocks were introduced in the context

[6]To help visualize the bijection between $[0, h_m - 1]$ and T_m, see Figure 3.2, which expands on the familiar Fibonacci example. Use the figure to see how the tilings that are (at least partly) within a colored boundary can be used to count both $|[0, h_m - 1]|$ (via the numbers to the left of the figure) and $|T_m|$ (via the tilings).

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing}\\
1 &= f_1\\
2 &= f_2\\
3 &= f_3\\
4 &= f_3 + f_1\\
5 &= f_4\\
6 &= f_4 + f_1\\
7 &= f_4 + f_2\\
8 &= f_5\\
9 &= f_5 + f_1\\
10 &= f_5 + f_2\\
11 &= f_5 + f_3\\
12 &= f_5 + f_3 + f_1
\end{aligned}
$$

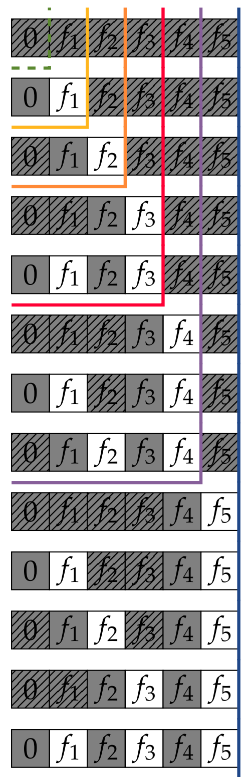
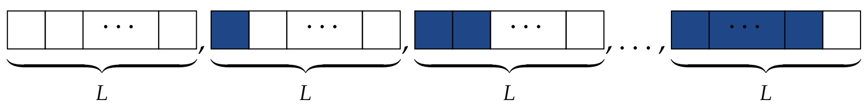
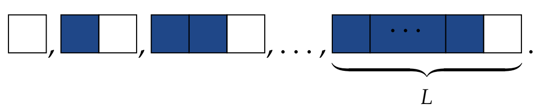


Figure 3.2 Examples of decomposition tilings for the Fibonacci numbers $\{f_n\}$, i.e., the PLRS generated by $[1, 1]$. The colored lines mark the T_m as follows. The vertical portion of a colored line to the immediate left of the spaces labelled f_m shows the boundary between tiles that are not necessarily opaque (to the left of the line) and those tiles that must be opaque (to the right of the line). Then the horizontal portion of each line is the lower boundary for which tilings are included in T_m. Specifically, the green lines show T_1 (the green line is dashed, since it is unique in that it marks no tilings that contain a non-opaque tile), the yellow lines show T_2, the orange lines show T_3, the red lines show T_4, the purple lines show T_5, and the blue lines show T_6. Also, observe that the height (the number of tilings the line passes through) of the vertical line for T_m, i.e., $|T_m|$ is f_m. Finally, the recursive structure of the tilings can be seen as follows. Consider the tiles that have a non-opaque tile in the position immediately to the left of the vertical line. For example, if we take the blue line, that is the tiles $\tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}$. Consider how if the final domino is removed from each of those and replaced with two opaque squares, they become exactly the tilings contained within the line that is moved left twice, in this case, the red line.

of legal decompositions in Definition 2.5. As a decomposition block's coefficients act as multipliers for subsections of the PLRS, they are equivalent to using the tilings that we have defined in Tiling Interpretation 3.5 as we show here. Note that in decompositions blocks, a blue color means that it being used, while white is not. The first step is to consider all possible decomposition blocks. We create all possible decomposition blocks by taking an empty mold, where the height of each position corresponds to a coefficient used to generate the PLRS. Since all coefficients are 1, since we are considering only L-bonacci numbers, the decomposition blocks (before any modifications) are

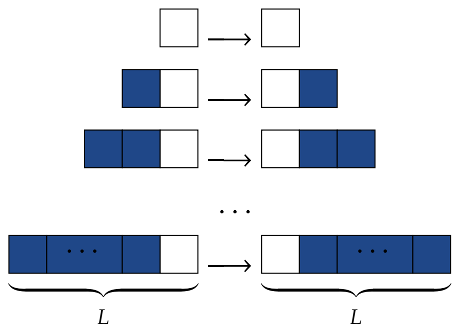


However, we can end a decomposition block early. Let a_i be a decomposition block coefficient and c_i be a PLRS coefficient. Since the second condition of Definition 2.5 is what is able to be used repeatedly, we model the decomposition blocks after that. That condition is that there exists $s \in \{1, \ldots, L\}$ such that $a_1 = c_1,\ a_2 = c_2,\ \ldots,\ a_{s-1} = c_{s-1},\ a_s < c_s$, and $a_{s+1}, \ldots, a_{s+\ell} = 0$ for some $\ell \geq 0$. Thus, if the first $s - 1$ coefficients are matched, the decomposition block cannot have width of just $s - 1$, it then actually also must have a coefficient $a_s = 0$, which satisfies $a_s < c_s$.[7] In the case of the L-bonacci numbers, there is no possibility of coefficients being greater than 1, so there will never be a partially full location. Thus all decomposition blocks will end in an empty space. The minimized decompositions blocks then become

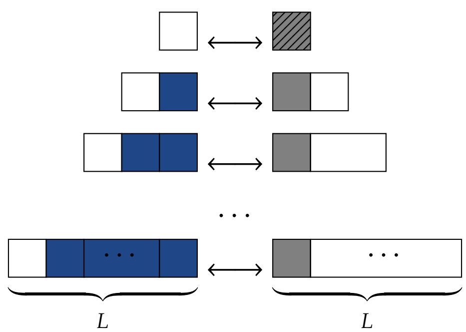


Finally, we reverse all of the decomposition blocks, because according to the definition, the PLRS $\{h_n\}_{n=1}^{\infty}$ and the decomposition blocks (with coefficients a_i) increment in opposite directions (since a decomposition is written $N = \sum_{i=1}^{m} a_i h_{m+1-i}$). By reversing the decomposition blocks, we can orient them correctly on the PLRS going from left to right. The reversal is as follows.

[7] Informally, when a portion of a decomposition mold is completely maxed out, it can't terminate immediately; it needs one empty position afterwards. However, if the final portion of a decomposition mold is partially full, there is no need to add additional empty positions following it.



Now that we have the decomposition blocks, we can see that they correspond to the tilings for L-bonacci numbers, by mapping white squares to opaque (gray) squares, and mapping blue squares to transparent squares.



Since we are using tilings to represent decomposition blocks over the same PLRS (with the addition of an initial zero), they are interchangeable representations. As the decomposition of a positive integer into decomposition blocks is given uniquely by the generalized Zeckendorf's theorem, the tiling interpretation also faithfully represents the decomposition tiling representation. □

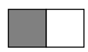
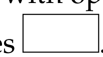
In this generalization to L-bonacci numbers, we took advantage of the simple decomposition rule. A natural question to ask is how can we extend decomposition tilings to include coefficients $c_i > 1$? We begin to address this question in the next section, where we answer it for second order recurrences.

3.4 Second Order Recurrences

In Section 3.2, we saw the tiling interpretation for coefficients $[1, 1]$. We wish to generalize this to all second order recurrences, i.e., those with coefficients

$[a, b]$ for all positive integers a, b. First, we show how to develop $[1, b]$.

3.4.1 Sequences Generated by $[1, b]$

For $b = 2$, we get the sequence $\{1, 2, 4, 8, 16, 32, \ldots\}$.[8] We cannot use exactly the same types of tiles as for the Fibonacci numbers, because legal decompositions for this sequence permit an arbitrary number of terms to be used in a row. It makes since, however, to start with the squares  and dominoes  that we used for the Fibonacci numbers. Now, we need to add an additional tile to allow for multiple consecutive sequence terms in a decomposition. Some tiles to consider are transparent squares, transparent dominoes, or transparent trominoes (or longer). If we were to include transparent squares, then we would be able to create every tiling with just the opaque squares and transparent squares, which leaves the half-opaque and half-transparent dominoes redundant (and thus, all possible tilings wouldn't count what we want it to count). On the other hand, if we were to use transparent trominoes (or longer), then there would be no way to have just two transparent spaces in a row, with opaque space on either side. That is why we add transparent dominoes . Using these tiles, examples of the Zeckendorf decomposition tilings for 0 to 16 are shown in Figure 3.3.

Observe that there are no transparent dominoes that cover the 0 on the board. Since placing a half opaque-half transparent domino or a transparent domino as the first tile on the board would both contribute a value of 1, we have the rule that any part of a tile that covers a 0 must be opaque. Since 0 only occurs as the first label on the board, this rule only restricts what the initial tile can be. (This is where we use phased tiles, where initial conditions are affected by the possibilities for the initial tile only. After the initial tile, the recurrence relation takes over.)

The boundaries between tiles may appear ambiguous at first. However, the greedy algorithm allows us to determine which tiles are used, by proceeding from the rightmost non-trivial tile, and identifying the tile that is the longest, and with the greatest multipliers that work, and proceeding recursively.

Next, consider $b = 3$, which generates the sequence $\{1, 2, 5, 11, 26, \ldots\}$.

[8]We saw this sequence before; it is also used in the binary decomposition of numbers in Section 3.1. Compare the Zeckendorf tilings of the two, and while they appear very similar at first, the sequence generated by [2] uses only opaque and transparent squares, while the sequence generated by [1, 2] uses opaque squares and two types of dominoes.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 2 \\
3 &= 2+1 \\
4 &= 4 \\
5 &= 4+1 \\
6 &= 4+2 \\
7 &= 4+2+1 \\
8 &= 8 \\
9 &= 8+1 \\
10 &= 8+2 \\
11 &= 8+2+1 \\
12 &= 8+4 \\
13 &= 8+4+1 \\
14 &= 8+4+2 \\
15 &= 8+4+2+1 \\
16 &= 16
\end{aligned}
$$

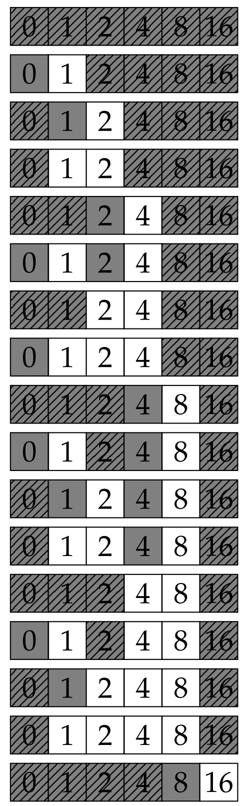
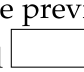
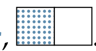
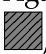
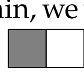
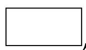
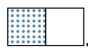
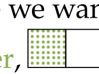
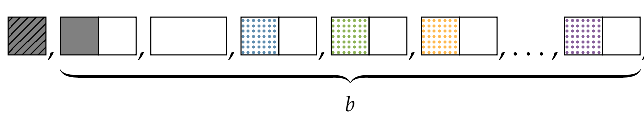


Figure 3.3 Examples of decomposition tilings for the PLRS generated by $[1, 2]$.

Again, we will use the previous three tiles from the previous case (when $b = 2$), , , and . However, the decomposition rule also allows for a number to be used twice sometimes, for example, the decomposition of 4 is $2 + 1 + 1$. Specifically, from the second condition of Definition 2.5, within a decomposition block, a coefficient in a subsequent position beyond the first can only be used if all previous positions' coefficients are maximal. So what that means here is within one decomposition block (analogous to one tile), we only want a light blue $\times 2$ filter to be used when a coefficient of one (the maximal first coefficient from $[1, 3]$) was already used on the first part of the tile. Since we approach tilings from the largest values first and work recursively, that means that the additional tile we want is a transparent domino, with the left half with a light blue $\times 2$ filter, . The Zeckendorf decomposition tilings for 0 to 16 are in Figure 3.4.

Next, consider $b = 4$, which generates the sequence $\{1, 2, 6, 14, 38, \ldots\}$. Again, we will use the previous four tiles from the previous case (when $b = 3$), , , , and . However, the decomposition rule also allows for a number to be used thrice sometimes, for example, the decomposition of 5 is $2 + 1 + 1 + 1$. Again, from the second condition of Definition 2.5, within a decomposition block, a coefficient in a subsequent position beyond the first can only be used if all previous positions' coefficients are maximal. So what that means here is within one decomposition block (analogous to one tile), we only want a light green $\times 3$ filter to be used when a coefficient of one (the maximal first coefficient from $[1, 4]$) was already used on the first part of the tile. That means that the additional tile we want is a transparent domino, with the left half with a light green $\times 3$ filter, . The Zeckendorf decomposition tilings for 0 to 16 are in Figure 3.5.

When moving from the case of $b = 3$ to $b = 4$, it was fairly straightforward once we knew to just add another domino with a new filter color (and with a multiplier of $b - 1$) on the left half and the right half transparent. Generalizing this, the way to create Zeckendorf decomposition tilings for any PLRS generated by coefficients $[1, b]$ is by using tiles



where there is one square, and there are b dominoes, which have filters (if $b > 2$), beginning at $\times 2$ up to $\times b - 1$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 2 \\
3 &= 2 + 1 \\
4 &= 2 + 1 + 1 \\
5 &= 5 \\
6 &= 5 + 1 \\
7 &= 5 + 2 \\
8 &= 5 + 2 + 1 \\
9 &= 5 + 2 + 2 \\
10 &= 5 + 2 + 2 + 1 \\
11 &= 11 \\
12 &= 11 + 1 \\
13 &= 11 + 2 \\
14 &= 11 + 2 + 1 \\
15 &= 11 + 2 + 1 + 1 \\
16 &= 11 + 5
\end{aligned}
$$

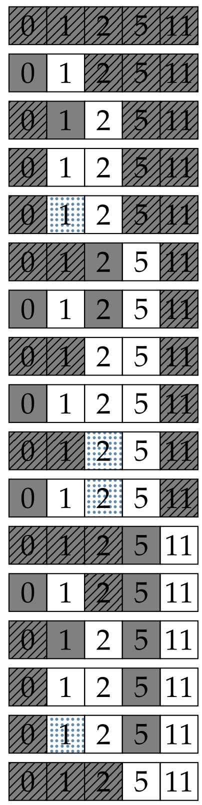


Figure 3.4 Examples of decomposition tilings for the PLRS generated by $[1, 3]$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 2 \\
3 &= 2 + 1 \\
4 &= 2 + 1 + 1 \\
5 &= 2 + 1 + 1 + 1 \\
6 &= 6 \\
7 &= 6 + 1 \\
8 &= 6 + 2 \\
9 &= 6 + 2 + 1 \\
10 &= 6 + 2 + 2 \\
11 &= 6 + 2 + 2 + 1 \\
12 &= 6 + 2 + 2 + 2 \\
13 &= 6 + 2 + 2 + 2 + 1 \\
14 &= 14 \\
15 &= 14 + 1 \\
16 &= 14 + 2
\end{aligned}
$$

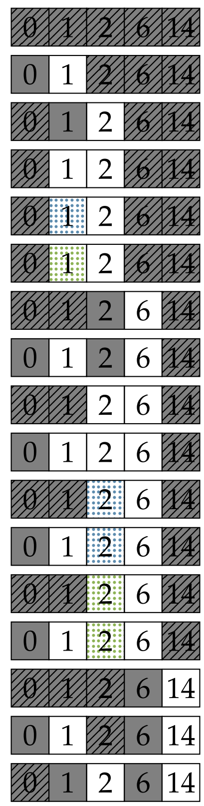


Figure 3.5 Examples of decomposition tilings for the PLRS generated by $[1, 4]$.

3.4.2 Sequences Generated by $[2, b]$

Next, we consider the case of PLRS's generated by $[2, b]$. We naturally wish to see if we can use the tiles that were used for $[1, b]$, with any necessary modifications. The key part of the definition of a legal decomposition that comes into play here is that in order to have multiple nonzero coefficients in a row that are part of the same decomposition block/tile, all coefficients except for the last must be maximized. Since we only are dealing with two coefficients here, then for any domino that has no opaque regions, the rightmost part must have a light blue $\times 2$ filter (from the 2 in $[2, b]$).

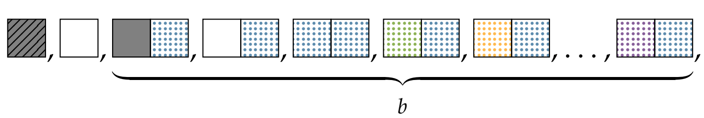
As our first example of $[2, b]$, let's consider the case when $b = 1$, which generates the sequence $\{1, 3, 7, 17, 41, \ldots\}$. We use the squares and dominoes, where we change the rightmost part to have a $\times 2$ filter . Now, we need to add a transparent tile with no filter to allow for decompositions that just use sequence terms once. We use the squares , since they can appear adjacent to each other any number of times, by repeatedly starting new decomposition blocks. The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by coefficients $[2, 1]$ are in Figure 3.6.

Next, let $b = 2$. We just need to add another domino that is transparent, with a light blue $\times 2$ filter on the right half. So, we use , , , . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by coefficients $[2, 2]$, are in Figure 3.7.

Next, let $b = 3$. We just need to add another domino that is transparent, with a light blue $\times 2$ filter on both the left and the right halves. So, we use , , , , . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by coefficients $[2, 3]$, are in Figure 3.8.

Next, let $b = 4$. We just need to add another domino that is transparent, with a light green $\times 3$ filter on the left half and a light blue $\times 2$ filter on the right half. So, the tiles are , , , , , . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by coefficients $[2, 4]$, are in Figure 3.9.

Generalizing this pattern, we get the following tiles for $[2, b]$. We use tiles



where there are two squares, and there are b dominoes, which have filters (if

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1 + 1 \\
3 &= 3 \\
4 &= 3 + 1 \\
5 &= 3 + 1 + 1 \\
6 &= 3 + 3 \\
7 &= 7 \\
8 &= 7 + 1 \\
9 &= 7 + 1 + 1 \\
10 &= 7 + 3 \\
11 &= 7 + 3 + 1 \\
12 &= 7 + 3 + 1 + 1 \\
13 &= 7 + 3 + 3 \\
14 &= 7 + 7 \\
15 &= 7 + 7 + 1 \\
16 &= 7 + 7 + 1 + 1 \\
17 &= 17 \\
18 &= 17 + 1 \\
19 &= 17 + 1 + 1 \\
20 &= 17 + 3
\end{aligned}
$$

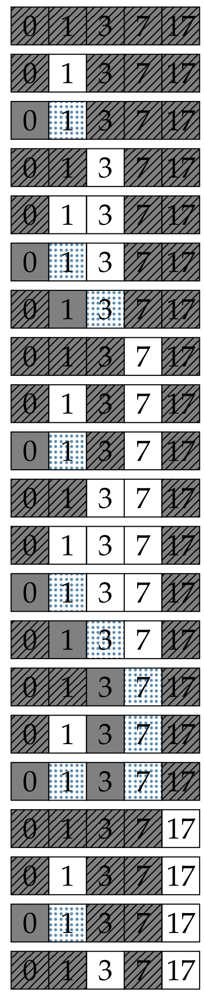


Figure 3.6 Examples of decomposition tilings for the PLRS generated by $[2, 1]$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1 + 1 \\
3 &= 3 \\
4 &= 3 + 1 \\
5 &= 3 + 1 + 1 \\
6 &= 3 + 3 \\
7 &= 3 + 3 + 1 \\
8 &= 8 \\
9 &= 8 + 1 \\
10 &= 8 + 1 + 1 \\
11 &= 8 + 3 \\
12 &= 8 + 3 + 1 \\
13 &= 8 + 3 + 1 + 1 \\
14 &= 8 + 3 + 3 \\
15 &= 8 + 3 + 3 + 1 \\
16 &= 8 + 8 \\
17 &= 8 + 8 + 1 \\
18 &= 8 + 8 + 1 + 1 \\
19 &= 8 + 8 + 3 \\
20 &= 8 + 8 + 3 + 1
\end{aligned}
$$

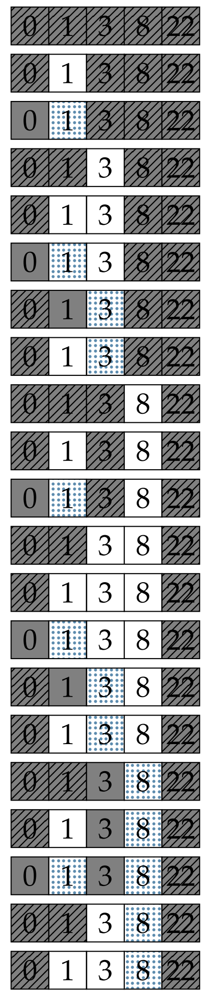


Figure 3.7 Examples of decomposition tilings for the PLRS generated by $[2, 2]$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1 + 1 \\
3 &= 3 \\
4 &= 3 + 1 \\
5 &= 3 + 1 + 1 \\
6 &= 3 + 3 \\
7 &= 3 + 3 + 1 \\
8 &= 3 + 3 + 1 + 1 \\
9 &= 9 \\
10 &= 9 + 1 \\
11 &= 9 + 1 + 1 \\
12 &= 9 + 3 \\
13 &= 9 + 3 + 1 \\
14 &= 9 + 3 + 1 + 1 \\
15 &= 9 + 3 + 3 \\
16 &= 9 + 3 + 3 + 1 \\
17 &= 9 + 3 + 3 + 1 + 1 \\
18 &= 9 + 9 \\
19 &= 9 + 9 + 1 \\
20 &= 9 + 9 + 1 + 1
\end{aligned}
$$

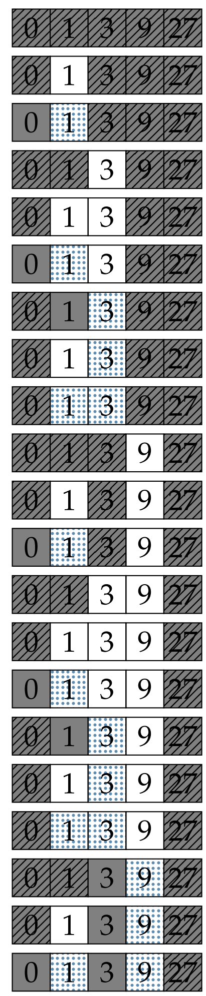


Figure 3.8 Examples of decomposition tilings for the PLRS generated by $[2, 3]$.

$0 = \sum_{\varnothing}$
$1 = 1$
$2 = 1 + 1$
$3 = 3$
$4 = 3 + 1$
$5 = 3 + 1 + 1$
$6 = 3 + 3$
$7 = 3 + 3 + 1$
$8 = 3 + 3 + 1 + 1$
$9 = 3 + 3 + 1 + 1 + 1$
$10 = 10$
$11 = 10 + 1$
$12 = 10 + 1 + 1$
$13 = 10 + 3$
$14 = 10 + 3 + 1$
$15 = 10 + 3 + 1 + 1$
$16 = 10 + 3 + 3$
$17 = 10 + 3 + 3 + 1$
$18 = 10 + 3 + 3 + 1 + 1$
$19 = 10 + 3 + 3 + 1 + 1 + 1$
$20 = 10 + 10$

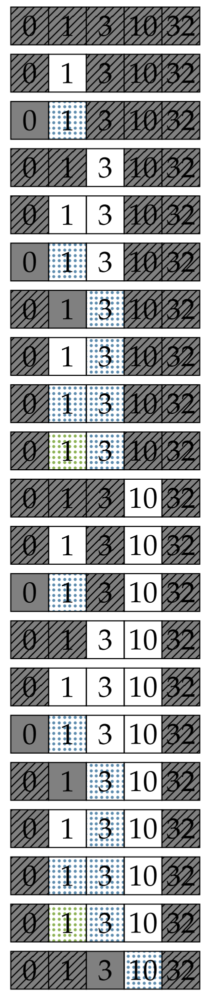
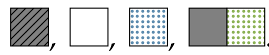


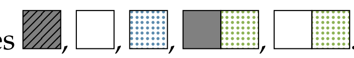
Figure 3.9 Examples of decomposition tilings for the PLRS generated by $[2, 4]$.

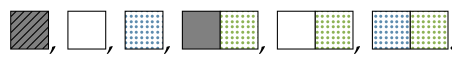
$b > 2$), beginning at $\times 2$ up to $\times b-1$.

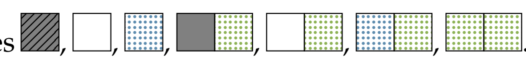
3.4.3 Sequences Generated by $[3, b]$

Next, we consider the case of PLRS's generated by $[3, b]$. By the decomposition rule, for a domino, the right half must have a light green $\times 3$ filter. So we can reuse the tiles from the $[2, b]$ case, where we modify the right half of all dominoes to have a light green $\times 3$ filter. We also need to add in a square with a light blue $\times 2$ filter, since we can use that in an unrestricted manner now.

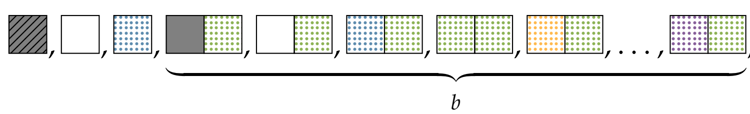
First, let $b = 1$. We use tiles . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by $[3, 1]$, are in Figure 3.10.

Next, let $b = 2$. We use tiles . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by $[3, 2]$, are in Figure 3.11.

Next, let $b = 3$. We use tiles . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by $[3, 3]$, are in Figure 3.12.

Next, let $b = 4$. We use tiles . The Zeckendorf decomposition tilings for 0 to 20, using the sequence generated by $[3, 4]$, are in Figure 3.13.

Generalizing this pattern, we get the following tiles for $[3, b]$. We use the $3 + b$ tiles



where there are an appropriate number of tiles with dominoes with filters on their left half, from 0 (opaque) up to $\times b-1$.

3.4.4 Sequences Generated by $[a, b]$

When comparing cases $[1, b]$, $[2, b]$, and $[3, b]$, we see that as the first coefficient increases by one, we make two changes to the set of tiles we use. First, we add an additional transparent square tile with a $\times a-1$ filter. Second, we increase the multiplier of the filter on the right half of all dominoes by 1.

$0 = \sum_{\varnothing}$
$1 = 1$
$2 = 1 + 1$
$3 = 1 + 1 + 1$
$4 = 4$
$5 = 4 + 1$
$6 = 4 + 1 + 1$
$7 = 4 + 1 + 1 + 1$
$8 = 4 + 4$
$9 = 4 + 4 + 1$
$10 = 4 + 4 + 1 + 1$
$11 = 4 + 4 + 1 + 1 + 1$
$12 = 4 + 4 + 4$
$13 = 13$
$14 = 13 + 1$
$15 = 13 + 1 + 1$
$16 = 13 + 1 + 1 + 1$
$17 = 13 + 4$
$18 = 13 + 4 + 1$
$19 = 13 + 4 + 1 + 1$
$20 = 13 + 4 + 1 + 1 + 1$

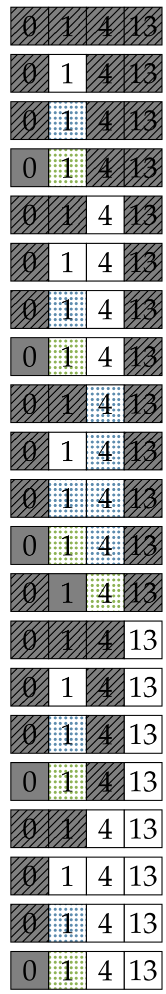


Figure 3.10 Examples of decomposition tilings for the PLRS generated by $[3, 1]$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1 + 1 \\
3 &= 1 + 1 + 1 \\
4 &= 4 \\
5 &= 4 + 1 \\
6 &= 4 + 1 + 1 \\
7 &= 4 + 1 + 1 + 1 \\
8 &= 4 + 4 \\
9 &= 4 + 4 + 1 \\
10 &= 4 + 4 + 1 + 1 \\
11 &= 4 + 4 + 1 + 1 + 1 \\
12 &= 4 + 4 + 4 \\
13 &= 4 + 4 + 4 + 1 \\
14 &= 14 \\
15 &= 14 + 1 \\
16 &= 14 + 1 + 1 \\
17 &= 14 + 1 + 1 + 1 \\
18 &= 14 + 4 \\
19 &= 14 + 4 + 1 \\
20 &= 14 + 4 + 1 + 1
\end{aligned}
$$

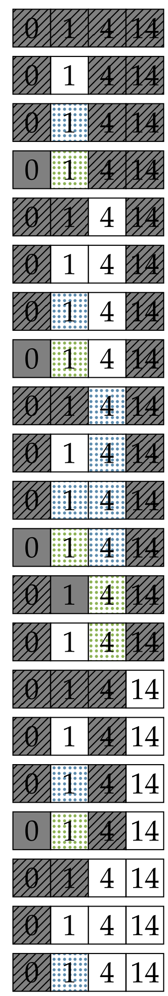


Figure 3.11 Examples of decomposition tilings for the PLRS generated by $[3, 2]$.

$$0 = \textstyle\sum_{\varnothing}$$
$$1 = 1$$
$$2 = 1 + 1$$
$$3 = 1 + 1 + 1$$
$$4 = 4$$
$$5 = 4 + 1$$
$$6 = 4 + 1 + 1$$
$$7 = 4 + 1 + 1 + 1$$
$$8 = 4 + 4$$
$$9 = 4 + 4 + 1$$
$$10 = 4 + 4 + 1 + 1$$
$$11 = 4 + 4 + 1 + 1 + 1$$
$$12 = 4 + 4 + 4$$
$$13 = 4 + 4 + 4 + 1$$
$$14 = 4 + 4 + 4 + 1 + 1$$
$$15 = 15$$
$$16 = 15 + 1$$
$$17 = 15 + 1 + 1$$
$$18 = 15 + 1 + 1 + 1$$
$$19 = 15 + 4$$
$$20 = 15 + 4 + 1$$

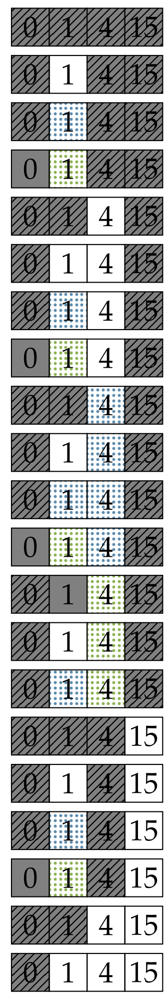


Figure 3.12 Examples of decomposition tilings for the PLRS generated by $[3, 3]$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing}\\
1 &= 1\\
2 &= 1+1\\
3 &= 1+1+1\\
4 &= 4\\
5 &= 4+1\\
6 &= 4+1+1\\
7 &= 4+1+1+1\\
8 &= 4+4\\
9 &= 4+4+1\\
10 &= 4+4+1+1\\
11 &= 4+4+1+1+1\\
12 &= 4+4+4\\
13 &= 4+4+4+1\\
14 &= 4+4+4+1+1\\
15 &= 4+4+4+1+1+1\\
16 &= 16\\
17 &= 16+1\\
18 &= 16+1+1\\
19 &= 16+1+1+1\\
20 &= 16+4
\end{aligned}
$$

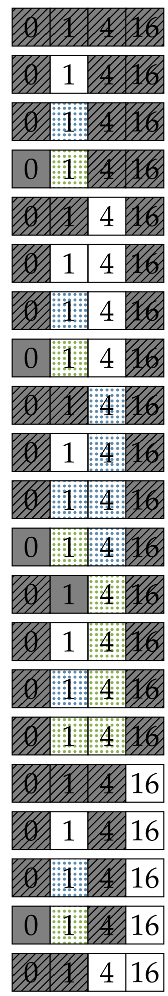
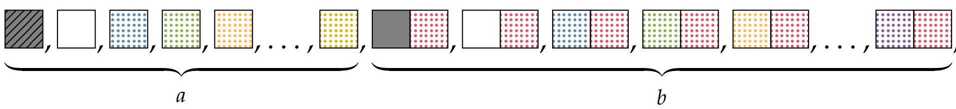


Figure 3.13 Examples of decomposition tilings for the PLRS generated by $[3,4]$.

So, in general, we use the $a + b$ types of tiles



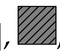
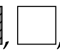
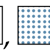
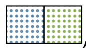
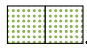
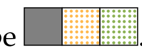
where there are a square tiles available, with multipliers from 0 (opaque) up to $\times a - 1$. Additionally, there are b domino tiles available, which all have a $\times a$ multiplier on the right half, and multipliers ranging from 0 (opaque) up to $\times b - 1$.

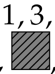
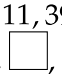
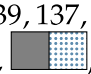
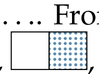
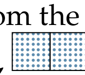
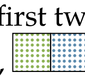
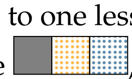
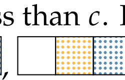
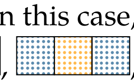
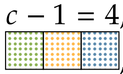
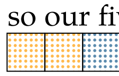
3.5 Third Order Recurrences

After understanding the decomposition tiling representation for all second order PLRS's, we can use the same methods to extend to higher order recurrences. Note that by the definition of a PLRS, the coefficients it is generated by cannot start nor end in a zero. So when dealing with first and second order recurrences, there never was a coefficient of zero. However, when the recurrence is of third order or higher, there can be zeros as coefficients for any of the middle coefficients. Such a zero complicates the situation slightly, so we first address recurrences with only positive coefficients, before also dealing with those with non-negative coefficients.

3.5.1 Positive Coefficients

Our first foray into third-order recurrences actually already happened when we considered L-bonacci numbers in Section 3.3. However, then we were just extending the Fibonacci numbers, now, we seek to extend from the general second-order recurrences. In the case of a second order recurrence $[a, b]$, we used a square tiles and b dominoes. So, we expect in the third order case of $[a, b, c]$ to use a square tiles, b dominoes, and c trominoes. Regarding what transparencies and filters to use, we want to reuse the same a squares and b dominoes as in the case of $[a, b]$. Recall that in the case of $[a, b]$, the squares have multipliers from 0 to $a - 1$, and the dominoes have a right-half multiplier of a and a left-half multiplier of 0 to $b - 1$. So for trominoes, we would like them to have a right-third multiplier of a, a middle-third multiplier of b, and a left-third multiplier of 0 to $c - 1$. This is because of how the definition of a legal decomposition block requires that for a subsequent coefficient to be used within a decomposition block, all previous coefficients have to be maximized. Now that we have some ideas for what third-order tilings should look like, let's see a couple examples that implement these ideas.

Consider the sequence generated by $[3, 4, 1]$, which is $1, 4, 17, 68, 276, \ldots$. From the first two coefficients, we use the tiles for $[3, 4]$, , , , , , , . Then, according to the idea we just proposed, we also add in one length 3 tile, specifically, the last two positions will be maximized with the first two coefficients, and the first position will range from a multiplier of 0 up to one less than c. In this case, $c - 1$ and 0 are the same, so our one additional tile will be . As before, we use opaque tiles to represent a coefficient of 0, transparent tiles to represent a coefficient of 1, transparent tiles with a light blue ×2 filter to represent a coefficient of 2, transparent tiles with a light green ×3 filter to represent a coefficient of 3, and transparent tiles with an orange ×4 filter to represent a coefficient of 4. See examples of these tilings in Figure 3.14.

Next, we will see an example with a greater value for c, so we can have multiple types of trominoes. Consider the sequence generated by $[2, 4, 5]$, which is $1, 3, 11, 39, 137, \ldots$. From the first two coefficients, we use the tiles for $[2, 4]$, , , , , , . Then, according to the idea we just proposed, we also add in 5 length 3 tiles, specifically, the last two positions will be maximized with the first two coefficients, and the first position will range from a multiplier of 0 up to one less than c. In this case, $c - 1 = 4$, so our five additional tiles will be , , , , . See examples of these tilings in Figures 3.15 and 3.16.

3.6 The General Decomposition Tiling Interpretation

We can generalize the decomposition tiling representation that we have explored so far to recurrences of any order.

Tiling Interpretation 3.8 (Decomposition Tilings for Positive Sequences)**.** Consider any PLRS generated by positive coefficients $[a, b, c, d, \ldots, y, z]$,[9]. Construct a semi-infinite strip, and label the first 1×1 space with a 0. Then label each subsequent space by the terms of the PLRS, beginning with 1. To tile the strip, we use the $a + b + c + d + \cdots + z$ tiles shown in Figure 3.17. For

[9]Note that this is just a relabelling of the coefficients $[c_1, \ldots, c_L]$. As a result, z and y just represent the last and second to last coefficients, but the number of coefficients can be any positive integer, i.e., it is not limited to the letters of the alphabet.

$1 = 1$
$2 = 1 + 1$
$3 = 1 + 1 + 1$
$4 = 4$
$5 = 4 + 1$
$6 = 4 + 1 + 1$
$7 = 4 + 1 + 1 + 1$
$8 = 4 + 4$
$9 = 4 + 4 + 1$
$10 = 4 + 4 + 1 + 1$
$11 = 4 + 4 + 1 + 1 + 1$
$12 = 4 + 4 + 4$
$13 = 4 + 4 + 4 + 1$
$14 = 4 + 4 + 4 + 1 + 1$
$15 = 4 + 4 + 4 + 1 + 1 + 1$
$16 = 4 + 4 + 4 + 1 + 1 + 1 + 1$
$17 = 17$
$18 = 17 + 1$
$19 = 17 + 1 + 1$
$20 = 17 + 1 + 1 + 1$

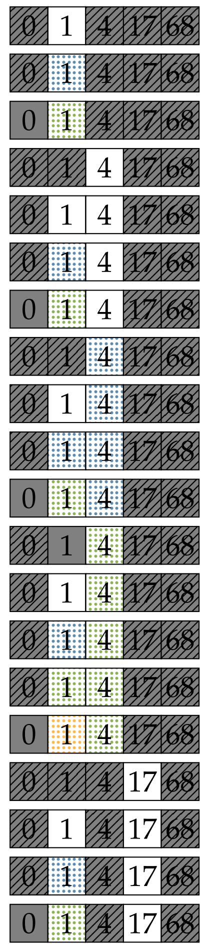


Figure 3.14 Examples of decomposition tilings for the PLRS generated by $[3, 4, 1]$.

$$
\begin{aligned}
1 &= 1\\
2 &= 1+1\\
3 &= 3\\
4 &= 3+1\\
5 &= 3+1+1\\
6 &= 3+3\\
7 &= 3+3+1\\
8 &= 3+3+1+1\\
9 &= 3+3+1+1+1\\
10 &= 3+3+1+1+1+1\\
11 &= 11\\
12 &= 11+1\\
13 &= 11+1+1\\
14 &= 11+3\\
15 &= 11+3+1\\
16 &= 11+3+1+1\\
17 &= 11+3+3\\
18 &= 11+3+3+1\\
19 &= 11+3+3+1+1\\
20 &= 11+3+3+1+1+1
\end{aligned}
$$

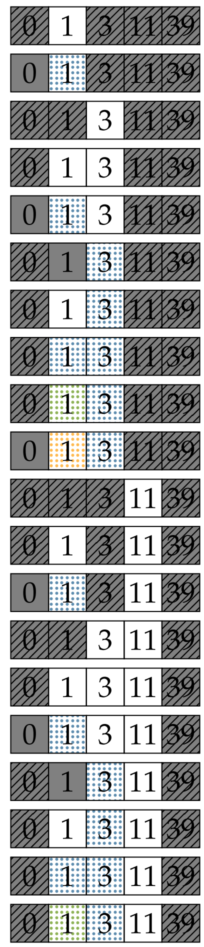


Figure 3.15 Examples of decomposition tilings for the PLRS generated by $[2, 4, 5]$ (from 1 to 20).

$21 = 11 + 3 + 3 + 1 + 1 + 1 + 1$
$22 = 11 + 11$
$23 = 11 + 11 + 1$
$24 = 11 + 11 + 1 + 1$
$25 = 11 + 11 + 3$
$26 = 11 + 11 + 3 + 1$
$27 = 11 + 11 + 3 + 1 + 1$
$28 = 11 + 11 + 3 + 3$
$29 = 11 + 11 + 3 + 3 + 1$
$30 = 11 + 11 + 3 + 3 + 1 + 1$
$31 = 11 + 11 + 3 + 3 + 3$
$32 = 11 + 11 + 3 + 3 + 3 + 1$
$33 = 11 + 11 + 3 + 3 + 3 + 1 + 1$
$34 = 11 + 11 + 3 + 3 + 3 + 3$
$35 = 11 + 11 + 3 + 3 + 3 + 3 + 1$
$36 = 11 + 11 + 3 + 3 + 3 + 3 + 1 + 1$
$37 = 11 + 11 + 3 + 3 + 3 + 3 + 1 + 1 + 1$
$38 = 11 + 11 + 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1$
$39 = 39$
$40 = 39 + 1$

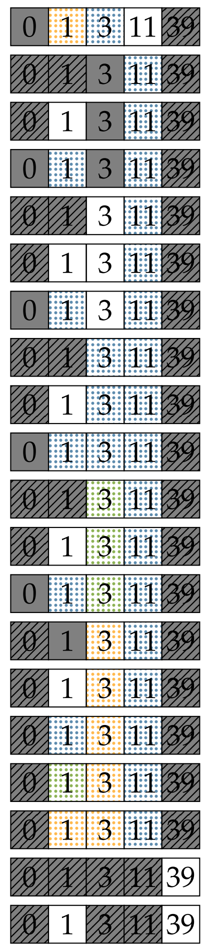


Figure 3.16 Examples of decomposition tilings for the PLRS generated by $[2, 4, 5]$ (from 21 to 40).

readability, the tiles with filters are labeled above by their multipliers. The only restriction on tile placement is that a tile cannot be placed such that a transparent portion (including transparent with a filter) covers the initial 0.

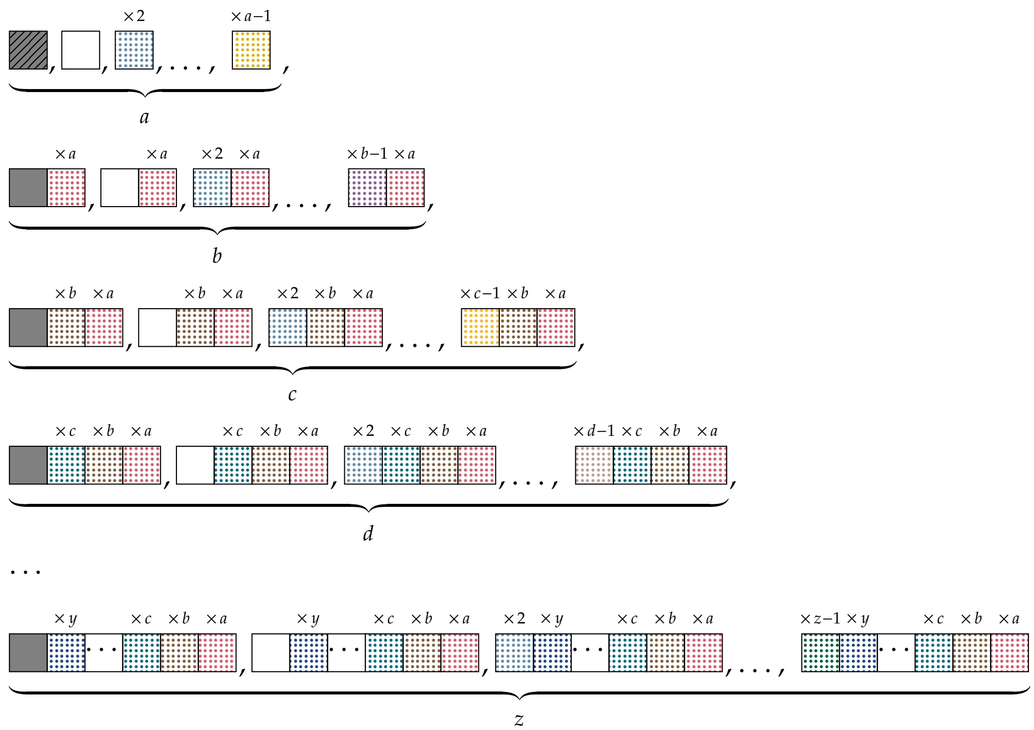


Figure 3.17 The tiles needed for creating decomposition tilings, for a general PLRS $[a, b, c, d, \ldots, y, z]$.

Using the labelling $[c_1, \ldots, c_L] = [a, b, c, d, \ldots, y, z]$, notice that there are the same number of tiles of a particular length c_i, as the ith coefficient used to generate the PLRS. Specifically, those c_i tiles will all have the same rightmost $c_i - 1$ components, which are, from the right, $c_1, c_2, \ldots, c_{i-1}$, and then the leftmost position ranges from a multiplier of 0 up to $c_i - 1$.

Now we state the following proposition, which establishes the correctness of the connection between this tiling interpretation and the generalized Zeckendorf's theorem.

Proposition 3.9. *The unique decomposition of any positive integer guaranteed by the generalized Zeckendorf's theorem* (*Theorem* 2.9) *into a sum of terms of a*

PLRS, generated by positive coefficients, has a one-to-one correspondence with a decomposition tiling representation given by Tiling interpretation 3.8.

Proof. This proof is a generalized version of the second proof of Proposition 3.7. This proof draws directly on the definition of a decomposition block, to show a correspondence between decomposition blocks and the tiles that can be used. Decomposition blocks were introduced in the context of legal decompositions in Definition 2.5. As a decomposition block's coefficients act as multipliers for subsections of the PLRS, they are equivalent to using the tilings that we have defined in Tiling Interpretation 3.8 as we show here. Note that in decompositions blocks, a blue color means that it being used, while white is not. The first step is to consider all possible decomposition blocks. We create all possible decomposition blocks by taking an empty mold, where the height of each position corresponds to a coefficient used to generate the PLRS. The height of the ith column is c_i. The decomposition blocks (before any modifications) are in Figure 3.18.

However, we can end a decomposition block early. Let a_i be a decomposition block coefficient (that is how high the blue is in a column) and c_i be a PLRS coefficient. Since the second condition of Definition 2.5 is what is able to be used repeatedly, we model the decomposition blocks after that. That condition is that there exists $s \in \{1, \dots, L\}$ such that $a_1 = c_1,\ a_2 = c_2,\ \dots,\ a_{s-1} = c_{s-1},\ a_s < c_s$, and $a_{s+1}, \dots, a_{s+\ell} = 0$ for some $\ell \geq 0$. Thus, if the first $s - 1$ coefficients are matched, the decomposition block cannot have width of just $s - 1$, it then actually also must have a coefficient $a_s = 0$, which satisfies $a_s < c_s$. In this general case, whenever there is a coefficient greater than 1, there is the possibility of a partially full location. Thus all decomposition blocks will end in partially full column, or in an empty space. The minimized decompositions blocks are shown in Figure 3.19.

Finally, we reverse all of the decomposition blocks, because according to the definition, the PLRS $\{h_n\}_{n=1}^{\infty}$ and the decomposition blocks (with coefficients a_i) increment in opposite directions (since a decomposition is written $N = \sum_{i=1}^{m} a_i h_{m+1-i}$). By reversing the decomposition blocks, we can orient them correctly on the PLRS going from left to right. The reversed decomposition blocks are shown in Figure 3.20.

Now that we have the decomposition blocks, we can see that they correspond to the tilings for the PLRS generated by $[c_1, \dots, c_L]$, by mapping white columns to opaque (gray) squares, and mapping blue columns (including partially blue columns) to transparent squares, where the height of each

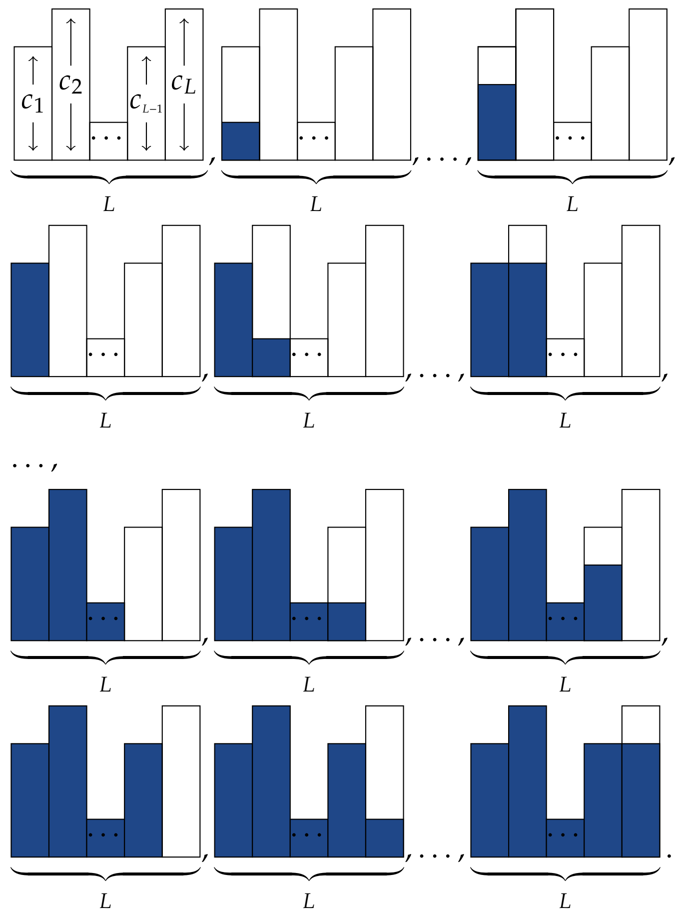


Figure 3.18 Arbitrary decomposition blocks before any modifications. Note that the height of each column i is c_i. The width of each block is L.

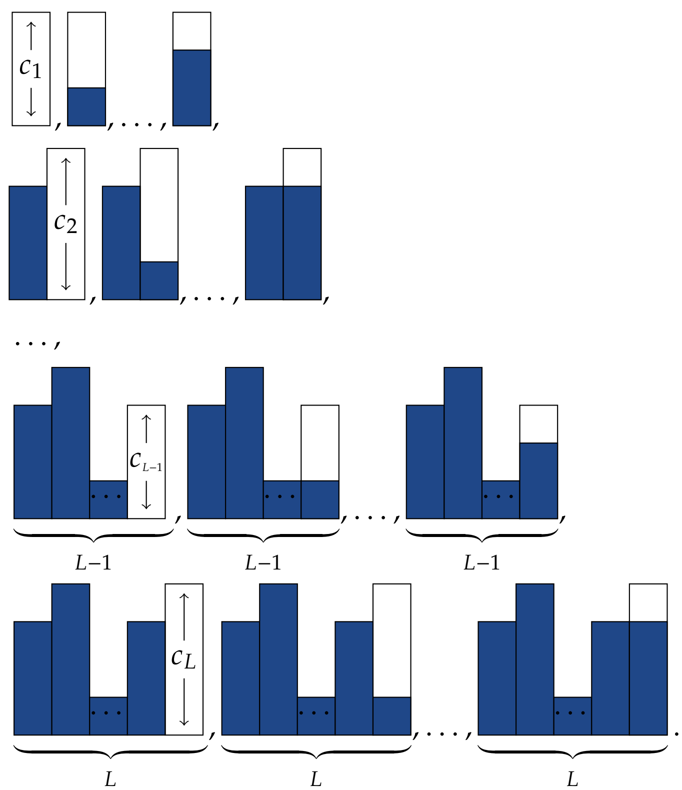


Figure 3.19 Arbitrary decomposition blocks that have had unnecessary entirely white columns removed. An entirely white column only remains if the column preceding it is completely full (blue).

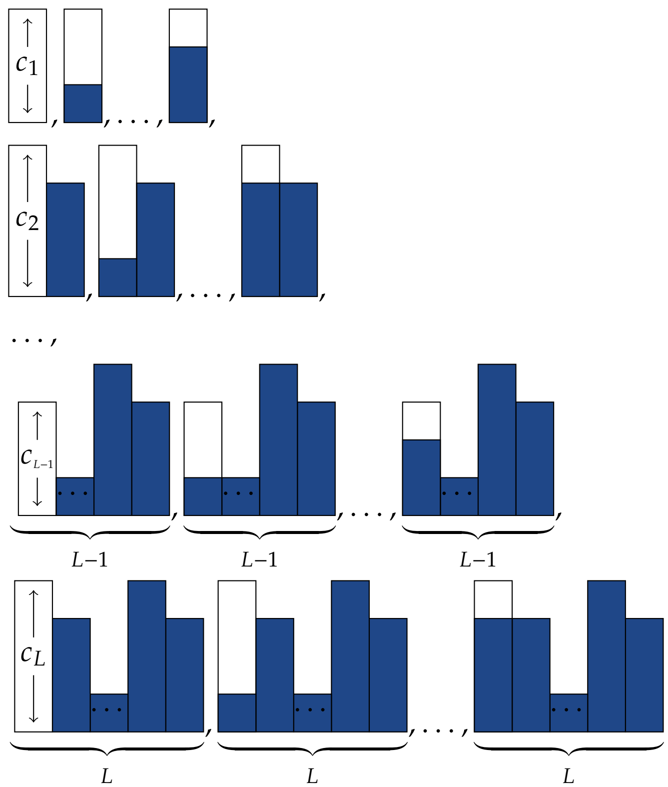


Figure 3.20 Arbitrary decomposition blocks that have had unnecessary entirely white columns removed, and have been reversed.

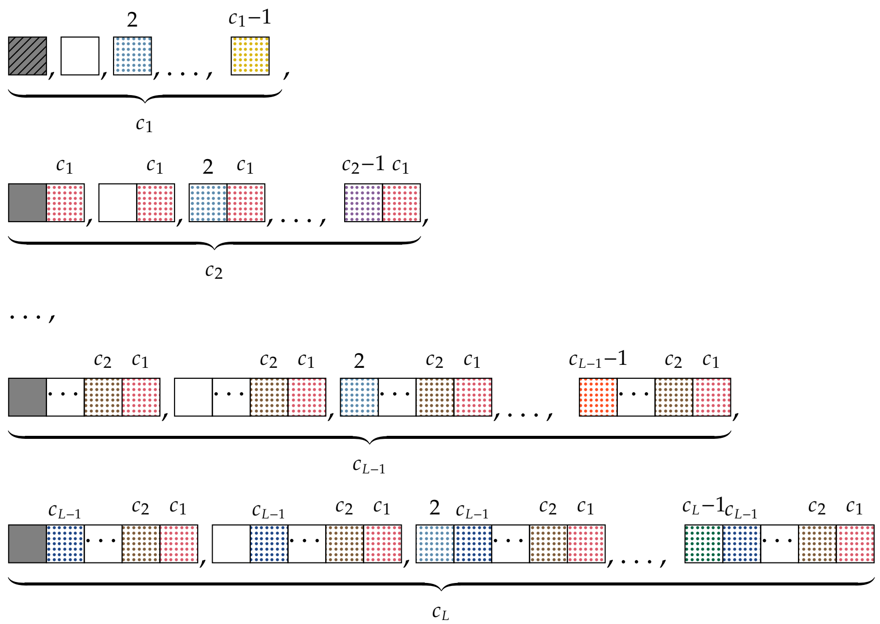


Figure 3.21 Tiles that correspond to arbitrary decomposition blocks. The numbers above the tiles are the filter multipliers.

blue column corresponds to the multiplier of the filter. That is, if the blue column is 1 unit high, it becomes a transparent square with no filter, if it is 2 units high, it becomes a transparent square with a light blue ×2 filter, if it is 3 units high, it becomes a transparent square with a light green ×3 filter, etc. The aforementioned decomposition blocks then correspond to the tiles in Figure 3.21. These tiles are the same as the tiles in Figure 3.17, which are used in Tiling Interpretation 3.8, up to a relabelling via $[a, b, \ldots, y, z] = [c_1, c_2, \ldots, c_{L-1}, c_L]$.[10]

Since we are using tilings to represent decomposition blocks over the same PLRS (with the addition of an initial zero), they are interchangeable representations. As the decomposition of a positive integer into decomposition blocks is given uniquely by the generalized Zeckendorf's theorem, the tiling interpretation also faithfully represents the decomposition tiling representation. □

[10] Additionally, the tiles of arbitrary length can have a different width shown explicitly. Also, here the × symbol is not used on the labels of the multipliers to increase readability.

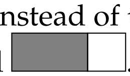
Up through now, we have restricted ourselves to only positive coefficients. In the next section, we will see how this tiling interpretation can be easily extended to allow coefficients of 0 as well.

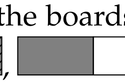
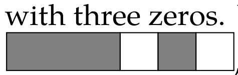
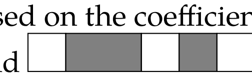
3.7 Zero as a Coefficient

Now, let's consider a simple example with a zero, the PLRS generated by $[1, 0, 1]$, which has terms $\{1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots\}$. We want to keep our existing method of tiling as similar as possible. If we naïvely apply the rules for third-order recurrences, we would get , , since the coefficient $b = 0$, we would get no tiles of size 1×2. This tiling runs into a problem, however, since the board we would be tiling is $\boxed{0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 6 \mid 9 \mid \cdots}$. Then, observe that there would be no way to place a tiling to represent the number 1, for example, within any decomposition, because the only tile that is non-opaque at any point is the tromino, which is of length 3, so the lowest space it can cover is the space labeled with a 2. We now show two equivalent workarounds: adding additional initial zeros to the beginning of the board, or using phased tilings.

Additional Initial Zeros

With this tiling interpretation, we increase the number of initial zeros on the board from 1 to $1 + m$, where m is the maximum number of consecutive zeros in the coefficients used to generate the PLRS. Additionally, recall that we have always had a restriction that no tile can be placed with a transparent portion (equivalently a nonzero multiplier) over the initial 0. Here, we extend that restriction to be that no tile can be placed with a transparent portion (equivalently a nonzero multiplier) over any 0 on the board. This is necessary to retain uniqueness of the tilings.

Consider the PLRS generated by $[1, 0, 1]$, which is $\{1, 2, 3, 4, 6, 9, 13, 19, \ldots\}$. The maximum number of consecutive zeros is 1, so we begin the boards with two zeros instead of the usual one. Based on the coefficients, we use the tiles:  and . See Figure 3.22 for examples.

Consider the PLRS generated by $[1, 0, 1, 0, 0, 2]$, which is $\{1, 2, 3, 5, 8, 12, 19, \ldots\}$. The maximum number of consecutive zeros is 2, so we begin the boards with three zeros. Based on the coefficients, we use the tiles: , , , and . See Figure 3.23 for examples.

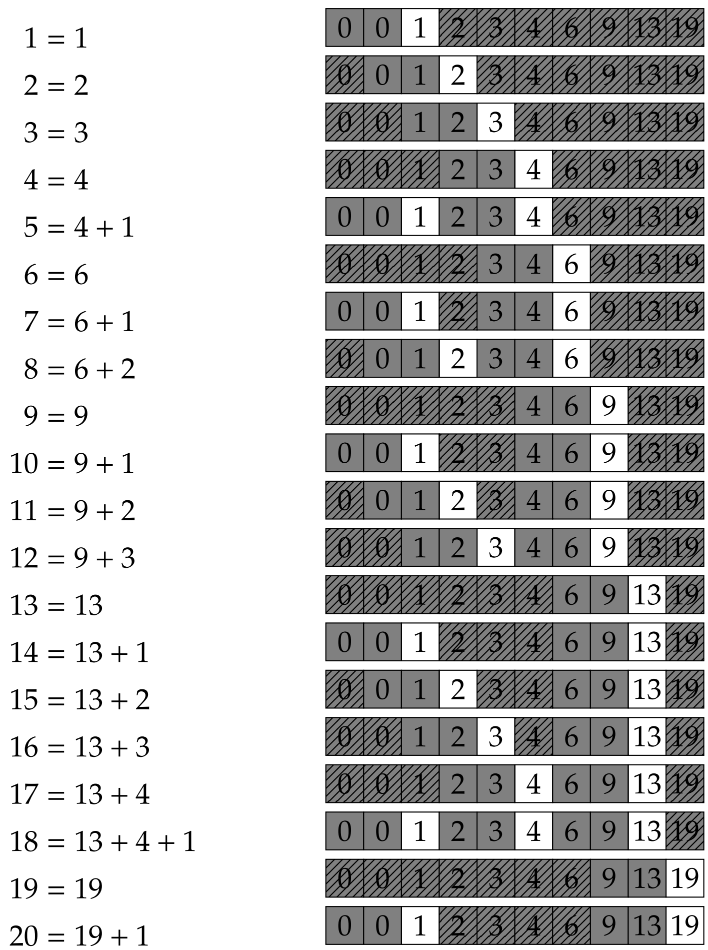


Figure 3.22 Examples of decomposition tilings for the PLRS generated by $[1, 0, 1]$, using the “additional zeros” interpretation.

$1 = 1$
$2 = 2$
$3 = 3$
$4 = 3 + 1$
$5 = 5$
$6 = 5 + 1$
$7 = 5 + 2$
$8 = 8$
$9 = 8 + 1$
$10 = 8 + 2$
$11 = 8 + 3$
$12 = 12$
$13 = 12 + 1$
$14 = 12 + 2$
$15 = 12 + 3$
$16 = 12 + 3 + 1$
$17 = 12 + 5$
$18 = 12 + 5 + 1$
$19 = 19$
$20 = 19 + 1$

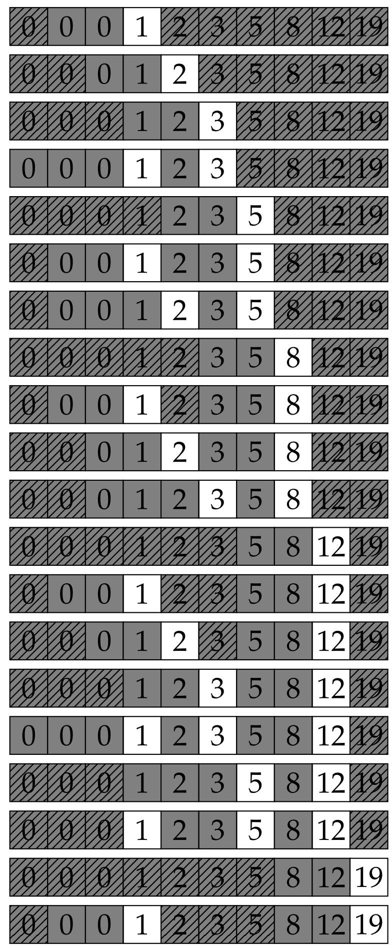
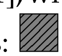
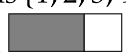
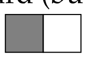
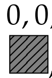
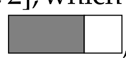
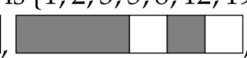
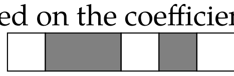
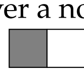
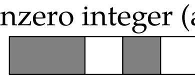
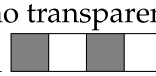


Figure 3.23 Examples of decomposition tilings for the PLRS generated by $[1, 0, 1, 0, 0, 2]$, using the “additional zeros” interpretation.

Phased Tilings

One benefit of additional zeros is that all tiles can fit entirely on the board. However, if there is a large number of consecutive zeros, this can occupy a lot of space at the beginning fo each tiling without providing much information on any decompositions, since there must always be an opaque tile covering a 0. So, the *phased tilings* method allows us to keep the tilings more compact, by not using any additional zeros. In this case, a phased tiling is exactly the same as its equivalent *additional initial zeros* tiling, with all but the final zero removed. This means that an initial tile may be cut off in part at the beginning. The word "phased" just refers to the fact that there are different rules for the first tile (in this case, that it can be a shortened version of another tile). In Benjamin and Quinn (2003), phased tilings are discussed in more detail, but what we need to know here is just that phases affect the initial conditions of a recurrence relation, but do not cause any further affects once the recurrence relation takes over. Now, we will see the same examples as before, and note how just all but one of the initial zeros are removed from the beginning of the board.

Consider the PLRS generated by $[1, 0, 1]$, which is $\{1, 2, 3, 4, 6, 9, 13, 19, \ldots\}$. Based on the coefficients, we use the tiles:  and  in any position. As the tile  could be placed such that the transparent portion is over the 1 on the board (but it wouldn't fit), we cut off the leftmost part, and create a phased tile  that can only be used in the first position. See Figure 3.24 for examples.

Consider the PLRS generated by $[1, 0, 1, 0, 0, 2]$, which is $\{1, 2, 3, 5, 8, 12, 19, \ldots\}$. Based on the coefficients, we use the tiles: , , , and  in any position. We can also use the following phased tiles in the first position only, that we calculate by just removing one unit from the leftmost portion at a time, as long as there remains a transparent portion that will cover a nonzero integer (and no transparent portion will cover a zero either): , , and .[11] See Figure 3.25 for examples.

[11]Note that the a phased tile for the first position may arise during the cropping process from multiple of the original tiles.

$1 = 1$
$2 = 2$
$3 = 3$
$4 = 4$
$5 = 4 + 1$
$6 = 6$
$7 = 6 + 1$
$8 = 6 + 2$
$9 = 9$
$10 = 9 + 1$
$11 = 9 + 2$
$12 = 9 + 3$
$13 = 13$
$14 = 13 + 1$
$15 = 13 + 2$
$16 = 13 + 3$
$17 = 13 + 4$
$18 = 13 + 4 + 1$
$19 = 19$
$20 = 19 + 1$

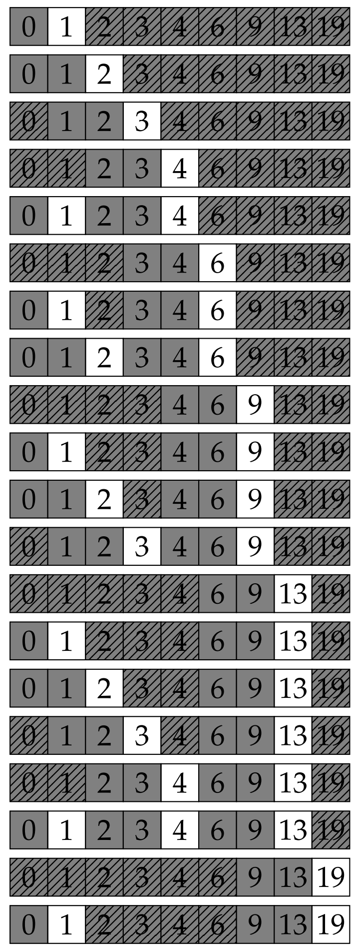


Figure 3.24 Examples of decomposition tilings for the PLRS generated by $[1, 0, 1]$, using the "phased tiling" interpretation.

$1 = 1$
$2 = 2$
$3 = 3$
$4 = 3 + 1$
$5 = 5$
$6 = 5 + 1$
$7 = 5 + 2$
$8 = 8$
$9 = 8 + 1$
$10 = 8 + 2$
$11 = 8 + 3$
$12 = 12$
$13 = 12 + 1$
$14 = 12 + 2$
$15 = 12 + 3$
$16 = 12 + 3 + 1$
$17 = 12 + 5$
$18 = 12 + 5 + 1$
$19 = 19$
$20 = 19 + 1$

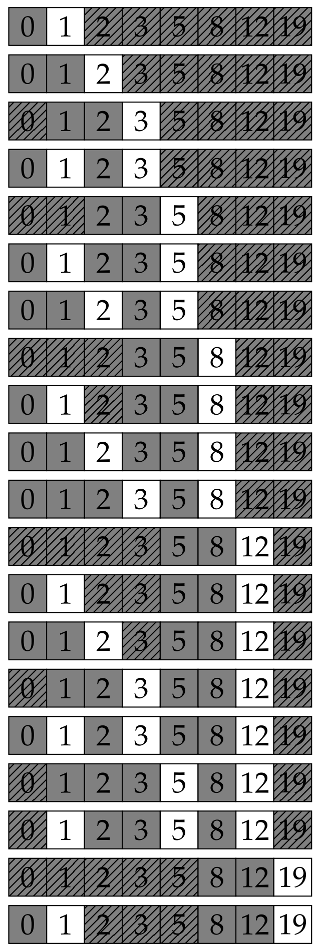


Figure 3.25 Examples of decomposition tilings for the PLRS generated by $[1, 0, 1, 0, 0, 2]$, using the "phased tiling" interpretation.

Chapter 4

Number of Summands in an Interval

Zeckendorf's theorem gives us unique decompositions. A follow-up question is: how many summands are needed in the decomposition of a given number? There has been interest in studying the distribution of the number of summands within the interval $[h_n, h_{n+1})$. Lekkerkerker (1951) found that the average number of such summands needed for integers in $[f_n, f_{n+1})$ is $n/(\varphi^2 + 1) + O(1)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Kopp et al. (2011) showed that this converges to a Gaussian as $n \to \infty$, and that for PLRS's in general, the average number of such summands needed for integers in $[h_n, h_{n+1})$ also converges to a Gaussian as $n \to \infty$. These results are shown combinatorially in Kopp et al. (2011) in the special case of the Fibonacci numbers, and shown in the general case in Miller and Wang (2012).

In this chapter, we begin an investigation of tiling interpretations of this result. We combinatorially derive formulas for the number of summands in the interval $[h_n, h_{n+1})$ for all L-bonacci sequences; however, as L increases, the formulas become significantly more complex.

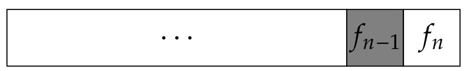
4.1 Combinatorial Formulas for the Number of Summands in L-bonacci Sequences

This argument is well known in the case of the Fibonacci numbers, for example, see Kopp et al. (2011). We can use our tiling argument in the case of Fibonacci, Tribonacci, and in general, L-bonacci numbers. (We use Tiling Interpretation 3.5.) However, we only get a closed form solution in the case of the Fibonacci numbers. Each time we increase L by 1, we'll see how this introduces another type of tile, which introduces a new variable; finally we have to sum over all valid values for each of those additional variables. We begin with the simplest example–the Fibonacci case.

Proposition 4.1 (The Fibonacci Case)**.** *The number of integers in the interval* $[f_n, f_{n+1})$ *with* $k+1$ *summands*[1] *is*

$$\binom{n-1-k}{k}.$$

Proof. For any $x \in [f_n, f_{n+1})$, note that f_n must be used in the Zeckendorf decomposition of x. Then, as $f_{n+1} = f_n + f_{n-1}$ by the recurrence relation, it is not possible for f_{n-1} to be used as well in the decomposition of x, since that would force $x \geq f_n + f_{n-1} = f_{n+1}$. We tile a board labeled with $0, f_1, f_2, \ldots, f_{n-1}, f_n$, of length $n+1$. This implies that the decomposition tiling for x ends in a domino, i.e., is



Now define $d = \#\{\text{dominoes used in the decomposition of } x\}$. The total number of summands in the decomposition of x is then d, as Fibonacci numbers will only be used once in a decomposition when there is a domino. As we have used one domino already, the number of dominoes left to place is $d-1$. The number of opaque squares left to place is $n+1-2d$. Then, out of the total number of tiles left to place choose which tiles are to be dominoes, which can be done in

$$\binom{n+1-2d}{d-1} \tag{4.1}$$

ways. Lastly, to count the number of $x \in [f_n, f_{n+1})$ that use $k+1$ summands, let $k+1 = d$. This substitution gives

$$\binom{n-1-2k}{k}. \tag{4.2}$$

□

[1] $k+1$ is used instead of perhaps k as in Kopp et al. (2011), and because within each interval $[f_n, f_{n+1})$, there is always a summand of f_n used, so k then counts those additional terms.

Next, we can extend this to the Tribonacci case by splitting it up into two cases, by whether the tiling ends in a domino, or a tromino.

Proposition 4.2 (The Tribonacci Case)**.** *The number of integers in the interval* $[t_n, t_{n+1})$ *with* $k+1$ *summands is*

$$\sum_{t=0}^{\lfloor \frac{k+1}{2} \rfloor} S(t, n, k),$$

where

$$S(t, n, k) := \binom{n-k-1}{t, k-2t, n-1-2k+t} + \binom{n-k-1}{t-1, k-2t+1, n-1-2k+t}.$$

Proof. For any $x \in [t_n, t_{n+1})$, note that t_n must be used in the Zeckendorf decomposition of x. Then, as $t_{n+1} = t_n + t_{n-1} + t_{n-2}$ by the recurrence relation, it is possible for t_{n-1} to be used as well in the decomposition of x, but not both t_{n-1} and t_{n-2}. We use our decomposition tiling for the Tribonacci numbers for x, on a board of length $n+1$, where the positions are labelled $0, t_1, t_2, \ldots, t_{n-2}, t_{n-1}, t_n$. This gives us two cases:

Case 1: The tiling ends in a domino if t_n is used and t_{n-1} is not.

$\cdots$	t_{n-1}	t_n

Case 2: The tiling ends in a tromino if t_n and t_{n-1} are used.

$\cdots$	t_{n-2}	t_{n-1}	t_n

Now define $t = \#\{\text{trominoes used in the decomposition of } x\}$ and $d = \#\{\text{dominoes used in the decomposition of } x\}$. The total number of summands in the decomposition of x is then $2t + d$.

Case 1: The number of trominoes left to place is t. The number of dominoes left to place is $d-1$. The number of squares left to place is $n+1-2d-3t$, since the length of the board is $n+1$. Then, out of the total number of tiles left to place, choose what order to assign them using a multinomial coefficient

$$\binom{t + (d-1) + (n+1-2d-3t)}{t, d-1, n+1-2d-3t} = \binom{n-d-2t}{t, d-1, n+1-2d-3t}. \tag{4.3}$$

Case 2: The number of trominoes left to place is $t-1$. The number of dominoes left to place is d. The number of squares left to place is $n+1-2d-3t$, since the length of the board is $n+1$. Then, out of the total number of tiles left to place, choose what order to assign them using a multinomial coefficient

$$\binom{(t-1)+d+(n+1-2d-3t)}{t-1, d, n+1-2d-3t} = \binom{n-d-2t}{t-1, d, n+1-2d-3t}. \tag{4.4}$$

Summing equations 4.3 and 4.4 gives the total number of ways to tile all $x \in [t_n, t_{n+1})$. To count the number of decompositions in $[t_n, t_{n+1})$, let $k+1 = 2t+d$, so $k+1$ is the number of terms in the decomposition. As k is expressed in terms of two variables, t and d, we can only fully replace one, for which we choose d. This gives us

$$S(t,n,k) := \binom{n-k-1}{t, k-2t, n-1-2k+t} + \binom{n-k-1}{t-1, k-2t+1, n-1-2k+t}. \tag{4.5}$$

We then need to sum this over all possible t, which is for $0 \le t \le \lfloor (k+1)/2 \rfloor$. Hence, we get that the number of integers in the interval $[h_n, h_{n+1})$ with $k+1$ summands is

$$\sum_{t=0}^{\lfloor \frac{k+1}{2} \rfloor} S(t,n,k).$$

□

Using the convention that if any $k_i < 0$, then

$$\binom{n}{k_1, \ldots, k_m} = 0,$$

we could also write the sum as

$$\sum_{t \ge 0} S(t,n,k).$$

Lastly, we can generalize this fully to the L-bonacci case by just adding more cases.

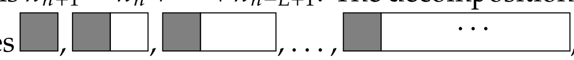
Proposition 4.3 (The L-bonacci Case)**.** *Let $\{h_n\}_n$ be a PLRS defined by $h_{n+1} = h_n + \cdots + h_{n-L+1}$. The number of integers in the interval $[h_n, h_{n+1})$ with $k+1$ summands is*

$$\sum_{t_3, \ldots, t_L} S(t_3, \ldots, t_L, n, k),$$

where

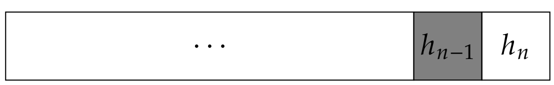
$$S(t_3,\ldots,t_L,n,k) := \binom{n-k}{k+1-\sum_i (i-1)t_i,\ t_3,\ldots,t_L,\ n+1-2k+\sum_i (i-2)t_i} - \binom{n-k-1}{k+1-\sum_i (i-1)t_i,\ t_3,\ldots,t_L,\ n-2-2k-\sum_i (i-2)t_i},$$

as i ranges from 3 *to L.*

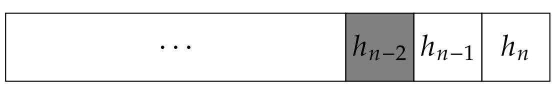
Proof. The recurrence relation is $h_{n+1} = h_n + \cdots + h_{n-L+1}$. The decomposition tilings of this sequence uses tiles 

which are $1\times 1, 1\times 2, 1\times 3, \ldots, 1\times L$ tiles with the first 1×1 part opaque and the remaining part transparent. For any $x \in [h_n, h_{n+1})$, note that h_n must be used in the Zeckendorf decomposition of x. Then, it is possible for h_{n-1} to be used or not. In the case that h_{n-1} is used, we then consider whether h_{n-2} is used or not. We continue this, looking to see if the next term h_{n-j} is used only when the term h_{n-j+1} is used. The last possible consecutive term that could be used is h_{n-L+2}, because if h_{n-L+1} were used as well, then $x \geq h_{n+1}$, which is not the case. We use our decomposition tiling for the L-bonacci numbers for x, on a board of length $n+1$, where the positions are labelled $0, h_1, h_2, \ldots, h_{n-2}, h_{n-1}, h_n$. This gives us $L-1$ cases:

Case 1: The tiling ends in a domino if h_n is used and h_{n-1} is not.

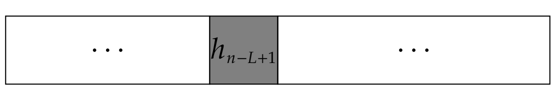


Case 2: The tiling ends in a tromino if h_n and h_{n-1} are used, while h_{n-2} is not.



Each case is constructed analogously, and we skip to the final case next.

Case $L-1$: The tiling ends in an L-omino if $h_n, h_{n-1}, \ldots, h_{n-L+2}$ are used, while h_{n-L+1} is not.



Now define $d = \#\{\text{dominoes used in the decomposition of } x\}$. Let $t_i = \#\{i\text{-ominoes used in the decomposition of } x\}$ for $3 \leq i \leq L$. Since the region we are tiling is of length $n+1$, that leaves the number of squares used in the decomposition of x as $n+1-2d-\sum_i it_i$.

Case 1: The number of dominoes left to place is $d-1$. The number of all i-ominoes for $3 \le i \le L$ left to place is t_i. The number of squares left to place is $n+1-2d-\sum_i it_i$. Then, out of the total number of tiles left to place, choose what order to assign them using a multinomial coefficient

$$\binom{n-d-\sum_i(i-1)t_i}{d-1, t_3, \dots, t_L, n+1-2d-\sum_i it_i}. \tag{4.6}$$

Case 2: The number of dominoes left to place is d. The number of trominoes left to place is t_3-1. The number of all i-ominoes for $4 \le i \le L$ left to place is t_i.The number of squares left to place is $n+1-2d-\sum_i it_i$. Then, out of the total number of tiles left to place, choose what order to assign them using a multinomial coefficient

$$\binom{n-d-\sum_i(i-1)t_i}{d, t_3-1, t_4 \dots, t_L, n+1-2d-\sum_i it_i}. \tag{4.7}$$

Each case is constructed analogously, and we skip to the final case next.

Case $L-1$: The number of dominoes left to place is d. The number of L-ominoes left to place is t_L-1. The number of all i-ominoes for $3 \le i \le L-1$ left to place is t_i.The number of squares left to place is $n+1-2d-\sum_i it_i$. Then, out of the total number of tiles left to place, choose what order to assign them using a multinomial coefficient

$$\binom{n-d-\sum_i(i-1)t_i}{d, t_3, \dots, t_{L-1}, t_L-1, n+1-2d-\sum_i it_i}. \tag{4.8}$$

Note that the expression for an arbitrary $3 \le j \le L$ (i.e., Case $j-1$) is

$$\binom{n-d-\sum_i(i-1)t_i}{d, t_3, \dots, t_j-1, \dots, t_L, n+1-2d-\sum_i it_i}. \tag{4.9}$$

Summing over all cases gives

$$\binom{n-d-\sum_i(i-1)t_i}{d-1, t_3, \dots, t_L, n+1-2d-\sum_i it_i}+\sum_{j=3}^{L}\binom{n-d-\sum_i(i-1)t_i}{d, t_3, \dots, t_j-1, \dots, t_L, n+1-2d-\sum_i it_i}. \tag{4.10}$$

We will use Pascal's formula for multinomial coefficients[2], which says

$$\binom{n}{k_1,\ldots,k_m} = \binom{n-1}{k_1-1,\ldots,k_m} + \cdots + \binom{n-1}{k_1,\ldots,k_m-1},$$

so equation 4.10 can be simplified to

$$\binom{n+1-d-\sum_i (i-1)t_i}{d,t_3,\ldots,t_L,n+1-2d-\sum_i it_i} - \binom{n-d-\sum_i (i-1)t_i}{d,t_3,\ldots,t_L,n-2d-\sum_i it_i}. \quad (4.11)$$

By substituting in $k+1 = d + \sum_i (i-1)t_i$, we can eliminate d. This gives us

$$\begin{aligned} S(t_3,\ldots,t_L,n,k) :=& \binom{n-k}{k+1-\sum_i (i-1)t_i, t_3,\ldots,t_L, n+1-2k+\sum_i (i-2)t_i} \\ &- \binom{n-k-1}{k+1-\sum_i (i-1)t_i, t_3,\ldots,t_L, n-2-2k-\sum_i (i-2)t_i}. \end{aligned} \quad (4.12)$$

We then need to sum this over all $t_3,\ldots,t_L$. We use the convention that

$$\binom{n}{k_1,\ldots,k_m} = 0$$

in the case that any $k_i < 0$. Hence, we get that the number of integers in the interval $[h_n, h_{n+1})$ with $k+1$ summands is

$$\sum_{t_3,\ldots,t_L} S(t_3,\ldots,t_L,n,k).$$

□

Unfortunately, this expression gets very complicated as L increases.

[2]Pascal's formula for multinomial coefficients can be proved by counting in two ways. We wish to assign n people each to one of m teams, where the teams are of size $k_1,\ldots,k_m$. This is $\binom{n}{k_1,\ldots,k_m}$. Pick one of the n people, and suppose that her name is Michelle. She must be on one of the m teams. Suppose Michelle is on team 1; then there are $\binom{n-1}{k_1-1,k_2,\ldots,k_m}$ ways to assign all other people. In general, suppose Michelle is on team i; then there are $\binom{n-1}{k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_m}$ ways to assign all other people. Summing over all possible i from 1 to m gives $\binom{n-1}{k_1-1,\ldots,k_m} + \cdots + \binom{n-1}{k_1,\ldots,k_m-1}$. It can also be shown with the algebraic definition of multinomial coefficients; for example see Brualdi (1992).

Chapter 5

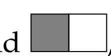
Future Work

This chapter contains some ideas for potential future exploration.

5.1 Nonstandard Initial Conditions

As discussed in Section 2, there is a trade off between the number of terms in the recurrence relation, restrictions on coefficients, and the initial conditions. This thesis has focused on the restrictive initial conditions given by the definition of a PLRS, where for each recurrence relation, there is one set of initial conditions. However, these initial conditions are not the only interesting ones. For example, the Lucas numbers and the Pell numbers both have recurrence relations that satisfy the definition of a PLRS, except that they have different initial conditions.

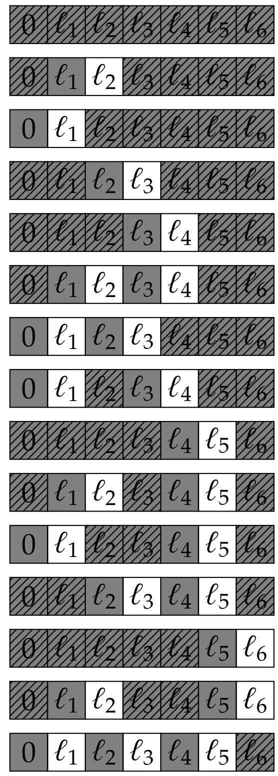
5.1.1 Lucas Numbers

Consider the Lucas numbers, which are defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ and the initial conditions of $L_0 = 2$ and $L_1 = 1$. We will use the Lucas numbers with the indices shifted by one, denoted by $\ell_n = \ell_{n-1} + \ell_{n-2}$ with initial conditions $\ell_1 = 2$ and $\ell_2 = 1$. This gives the sequence $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$. We can tile a board that is labelled with 0 and then the sequence $\boxed{0 \,|\, \ell_1 \,|\, \ell_2 \,|\, \ell_3 \,|\, \ell_4 \,|\, \cdots}$ with the tiles  and , like with the Fibonacci numbers, to represent decompositions of positive integers, due to the following theorem of Zeckendorf.

Theorem 5.1. *Every natural number can be represented by a sum of distinct, nonconsecutive Lucas numbers. The representation is unique, except for the* *numbers* $L_{2v+1} + 1 = \ell_{2v} + 1$ *for* $v = 2, 3, \ldots$.

Now, let's look at examples for the integers 0 through 12. Recall that we're using the set $\{\ell_1 = 2, \ell_2 = 1, \ell_3 = 3, \ell_4 = 4, \ell_5 = 7, \ell_6 = 11, \ldots\}$. The first two numbers that have non-unique decompositions are 5 and 12 (which occur when $v = 2, 3$ in Theorem 5.1), which are written in red below.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= \ell_2 \\
2 &= \ell_1 \\
3 &= \ell_3 \\
4 &= \ell_4 \\
5 &= \ell_4 + \ell_2 \\
&= \ell_3 + \ell_1 \\
6 &= \ell_4 + \ell_1 \\
7 &= \ell_5 \\
8 &= \ell_5 + \ell_2 \\
9 &= \ell_5 + \ell_1 \\
10 &= \ell_5 + \ell_3 \\
11 &= \ell_6 \\
12 &= \ell_6 + \ell_2 \\
&= \ell_5 + \ell_3 + \ell_1
\end{aligned}
$$



This raises the question: are there other simple or informative decomposition tilings of PLRS's where we change the initial conditions?

5.2 Connections to Other Combinatorial Objects

Additionally, it would be interesting to connect the tiling interpretation of PLRS's with other combinatorial objects, such as the Wythoff array, or continued fractions. The Wythoff array,[1] is defined as follows. Let $A_{m,n}$ be

[1]For more on its many interesting properties, see Kimberling (1995); Morrison (1980).

the entry in row m and column n. Then,

$$\begin{aligned} A_{m,1} &= \lfloor \lfloor m\varphi \rfloor \varphi \rfloor, \\ A_{m,2} &= \lfloor \lfloor m\varphi \rfloor \varphi^2 \rfloor, \\ A_{m,n} &= A_{m,n-2} + A_{m,n-1} \text{ for } n > 2. \end{aligned}$$

Expanding this gives the Wythoff array.

1	2	3	5	8	13	21	...
4	7	11	18	29	47	76	...
6	10	16	26	42	68	110	...
9	15	24	39	63	102	165	...
12	20	32	52	84	136	220	...
14	23	37	60	97	157	254	...
17	28	45	73	118	191	309	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 5.1 The Wythoff Array

See Dekking (2021) for recent work that classifies Zeckendorf decompositions (slightly reformulated as "Zeckendorf expansions") using digit blocks (which are like decomposition block coefficients). These digit blocks are used to label a tree which branches according to the Fibonacci sequence. The numbers ending with a certain digit block in their Zeckendorf expansion appear as compound Wythoff sequences in the tree. Since there is a connection between digit blocks and decomposition blocks, is there a tiling interpretation of this result?

5.3 Complete Sequences

A sequence of positive integers is *complete* if every positive integer is a sum of distinct terms of the sequence. The classification of PLRS's as complete or not was begun by Bołdyriew et al. (2020). Could the tiling interpretation of PLRS's discussed in Chapter 3 be used to explain or extend these results on completeness?

5.4 Tilings and Probability

In Chapter 4, combinatorial formulas were provides for the number of summands in the interval $[h_n, h_{n+1})$ for all L-bonacci sequences, using tilings. Perhaps this could be extended to all PLRS's. However, one drawback is that the formulas are already very complex as L grows in the L-bonacci case. One question that remains is: can the tiling interpretations for these formulas be used to get the probabilistic results shown in Kopp et al. (2011)?

Appendix A

Source Code

In this appendix are some Python functions that can be used to generate a PLRS and decompose a number using a PLRS into a sum of PLRS terms via decomposition blocks. These functions are available to download as `PLRS_functions_Thesis.py`.

Here is a Python function `generatePLRS(n, coeffs)` that can be used to generate a PLRS.

```
def generatePLRS(n, coeffs):
    """
    Generates a PLRS.

    Parameters:
        n (int): Number of PLRS terms to generate.
        coeffs (lst): Coefficients used to define the PLRS.

    Returns:
        lst: List of n terms of the PLRS defined by coeffs.
    """
    L = len(coeffs)
    terms = [1]
    # generate the first L terms
    for _ in range(min(n, L-1)):
        terms.append(sum(terms[-1-i]*coeffs[i] for i in \
        range(len(terms)))+1)
    # generate any additional terms
```

```
    for _ in range(n-L):
        terms.append(sum(terms[-1-i]*coeffs[i] for i in \
        range(L)))
    return terms
```

Here are some examples, where we find the first 10 terms in the PLRS's defined by $[1,1]$ and $[1,4,9]$.

```
>>> generatePLRS(10,[1,1])
[1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
>>> generatePLRS(10,[1,4,9])
[1, 2, 7, 24, 70, 229, 725, 2271, 7232, 22841]
```

Here is a Python function `PLRSdecomposition(coeffs, n)` that can be used to decompose a number into terms of a PLRS such that the decomposition block structure can be seen. It uses the helper function `decomp(l, plrs, coeffs, n)` to process each decomposition block recursively.

```
def PLRSdecomposition(coeffs, n):
    """
    Represents the decomposition of a number into terms of
    a PLRS.

    Parameters:
        coeffs (lst): The list of coefficients used to generate
        the PLRS.
        n (int): A number to decompose.

    Returns:
        lst: A list where each entry contains a decomposition
        block and the location of the rightmost element in the
        decomposition block. The indexing begins with 0 for "0"
        and is i for "h_i", the i-th term in the PLRS.
    """
    # generate enough PLRS terms
    a = 10
    plrs = generatePLRS(a, coeffs)[:]
    while plrs[-1] < n:
        a += 1
```

```
        plrs = generatePLRS(a, coeffs)[:]
    # begin making the decomposition
    l = []
    while n > 0:
        (l, n) = decomp(l, plrs, coeffs, n)
    location_list = []
    for r in range(len(l)):
        location = l[r][1]
        length = len(l[r][0])
        location_list.append([length, location])
    # check for gaps
    for t in (range (len(location_list)-1)):
        ending = location_list[t][1] - location_list[t][0]
        gap = ending - location_list[t+1][1]
        shift = 0
        while gap > 0:
            l.insert(t+1+shift, [[0], ending - shift])
            gap -= 1
            shift += 1
    # check for an initial gap
    ending = location_list[-1][1] - location_list[-1][0]
    gap = ending + 2
    shift = 0
    while gap > 0:
        l.append([[0], ending - shift])
        gap -= 1
        shift += 1
    # adjust the indexing
    l = [[x[0],x[1]+1] for x in l]
    return l

def decomp(l, plrs, coeffs, n):
    """
    Helper function that assists in the decomposition of a
    number.

    Parameters:
        l (lst): The decomposition blocks that have already
        been determined.
```

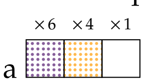
```
        plrs (lst): The PLRS generated by coeffs.
        coeffs (lst): The coefficients used to generate the
        PLRS.
        n (int): The remaining amount left to decompose.

    Returns:
        l (lst): The previously determined decomposition
        blocks, along with one more new decomposition block.
        n (int): The new remaining amount left to decompose.
    """
    # locate the index i of the larget term in the PLRS
    # that will be used
    i = 0
    while plrs[i] <= n:
        i += 1
    i -=1
    # locate the largest multiple of the largest term
    # that will be used; if that is the maximum, consider
    # the subsequent coefficients
    coeffs_maxed = 0
    temp_l = []
    while True:
        # j is the multiple of the sequence term
        j = 0
        while (j*plrs[i-coeffs_maxed] <= n) and \
        (j <= coeffs[coeffs_maxed]):
            j += 1
        j -= 1
        # j has now been maximized
        temp_l.append(j)
        n -= j*plrs[i-coeffs_maxed]
        if j==coeffs[coeffs_maxed]:
            coeffs_maxed += 1
        else:
            break
    l.append([temp_l, i])
    return (l, n)
```

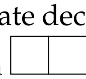
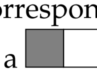
Here are some examples, where we decompose the numbers 21 and 32 into a sum of terms of the PLRS defined by $[1, 1]$ (the Fibonacci numbers). We interpret the first output as $21 = [1 \cdot f_7 + 0 \cdot f_6] + [0 \cdot f_5] + [0 \cdot f_4] + [0 \cdot f_3] + [0 \cdot f_2] + [0 \cdot f_1] + [0 \cdot 0]$, where the brackets indicate decomposition blocks[1], which correspond to tiles. Thus we know there is a  domino on positions $(6, 7)$, and opaque squares everywhere else. We can also simplify the output to just get the decomposition without additional knowledge about the decomposition blocks/tilings, which is $21 = f_7$.

We interpret the second output as $32 = [1 \cdot f_7 + 0 \cdot f_6] + [1 \cdot f_5 + 0 \cdot f_4] + [1 \cdot f_3 + 0 \cdot f_2] + [0 \cdot f_1] + [0 \cdot 0]$, where the brackets indicate decomposition blocks, which correspond to tiles. Thus we know there is a  domino on positions $(6, 7)$, $(4, 5)$, and $(2, 3)$ and opaque squares everywhere else. Also, the simplified output is $32 = f_7 + f_5 + f_3$.

```
>>> PLRSdecomposition([1,1],21)
[[[1, 0], 7], [[0], 5], [[0], 4], [[0], 3], [[0], 2], [[0], 1], [[0], 0]]
>>> PLRSdecomposition([1,1],32)
[[[1, 0], 7], [[1, 0], 5], [[1, 0], 3], [[0], 1], [[0], 0]]
```

Next, we decompose the numbers 21 and 32 into a sum of terms of the PLRS defined by $[1, 4, 9]$. We interpret the first output as $21 = [1 \cdot h_3 + 4 \cdot h_2 + 6 \cdot h_1] + [0 \cdot 0]$, where the brackets indicate decomposition blocks, which correspond to tiles. Thus we know there is a 

tromino on positions $(1, 2, 3)$, and opaque squares everywhere else. Also, the simplified output is $21 = h_3 + 4 \cdot h_2 + 6 \cdot h_1$.

We interpret the second output as $32 = [1 \cdot h_4 + 1 \cdot h_3] + [0 \cdot h_2] + [1 \cdot h_1 + 0 \cdot 0]$, where the brackets indicate decomposition blocks, which correspond to tiles. Thus we know there is a  domino on positions $(3, 4)$, a  domino on positions $(0, 1)$ and opaque squares everywhere else. The simplified output is $32 = h_4 + h_3 + h_1$.

```
>>> PLRSdecomposition([1,4,9],21)
[[[1, 4, 6], 3], [[0], 0]]
>>> PLRSdecomposition([1,4,9],32)
[[[1, 1], 4], [[0], 2], [[1, 0], 1]]
```

[1]However, the $[0 \cdot 0]$ is technically not a decomposition block, since 0 is not a term in the PLRS.

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