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# A Look at Biseparating Maps from an Algebraic Point of View

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ABSTRACT. In **[ABN]**, Araujo, Beckenstein, and Narici add the capstone to a series of papers by several groups of authors by showing that if $\varphi$ is a biseparating map between two algebras of all real or complex-valued functions on realcompact spaces, then it is a continuous multiple of an isomorphism between these rings. Their proof uses relatively powerful analytic and topological techniques. In what follows, the extent to which such a result can be generalized to a wider class of algebras using algebraic techniques is investigated. We are unable, however to obtain the main result of **[ABN]** using these techniques.

## 1. Introduction

Throughout, $A$ and $B$ will denote unital, commutative rings or algebras over a field whose identity elements are denoted by $\mathbf{1}_A$ and $\mathbf{1}_B$, respectively (or just $\mathbf{1}$ if which ring is meant is clear from context). Unless the contrary is stated explicitly, the rings considered are assumed to be *reduced* (i.e., their only nilpotent element is 0). A mapping $\varphi : A \to B$ is called *separating* if $ab = 0$ in $A$ implies $\varphi(a)\varphi(b) = 0$ in $B$, and $\varphi$ is called *biseparating* if it is a bijection and both $\varphi$ and $\varphi^{\leftarrow}$ are separating. If $S \subset A$, let $S^d = \{ a \in A : aS = \{0\} \}$, and if $s \in S$, abbreviate $\{s\}^d$ by $s^d$. $S^d$ is called the *annihilator of* $S$, and is clearly an ideal of $A$.

A (group) homomorphism of $(A, +)$ into $(B, +)$ is called a *linear map* of the ring $A$ into the ring $B$. If F is a field and $A$ is an algebra over F, then a mapping $\varsigma$ of $A$ into an F-algebra is said to be F-*linear* if for all $\alpha, \beta \in$ F and $x, y \in A$, $\varsigma(\alpha x + \beta y) = \alpha\varsigma(x) + \beta\varsigma(y)$. That is, $\varsigma$ is F-linear if it is a linear map such that $\varsigma(\alpha a) = \alpha\varsigma(a)$ for all $\alpha \in$ F and $a \in A$. Note that in an F-algebra, an ideal $I$ is assumed to be closed under multiplication by elements of F. An ideal (respectively, subspace) $M$ of an F-algebra $A$ is called an F-*ideal* (respectively, F-*subspace*) if $A/M$ and F are isomorphic as algebras (respectively, vector spaces). Clearly every F-ideal is an F-subspace. It is an exercise to verify that:

(∗) *A subspace of an F-algebra is an F-subspace if and only if it has co-dimension 1.*

While no use of (∗) is made below, it does show that a linear biseparating map sends F-ideals into F-subspaces.

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Throughout, $X$ and $Y$ will denote completely regular (Hausdorff) spaces, unless additional restrictions are placed on them explicitly.

The rational, real, and complex fields are abbreviated by Q, R, and C, respectively. In [**GJ**], an R-ideal of the algebra $C(X)$ of all continuous functions into the field of real numbers is called a *real maximal ideal.* We call $X$ *realcompact* if for each homomorphism $e$ of $C(X)$ into R, there is an $x \in X$ such that $e(f) = f(x)$ for all $f \in C(X)$ (that is, $e$ is a point evaluation). In [**ABN**], Araujo, Beckenstein, and Narici show that if $X$ and $Y$ are realcompact topological spaces, and $\varphi : C(X) \to C(Y)$ is an R-linear biseparating map, then there is a homeomorphism $\psi : Y \to X$ and an $h \in C(Y)$ such that $\varphi(f) = (f \circ \psi)h$ (that is, $\varphi(f)(y) = f(\psi(y))h(y)$ for all $f \in C(X)$ and $y \in Y$). Such a mapping $\varphi$ is called a *weighted homomorphism.* [**ABN**] is the capstone of a series of papers establishing this in special cases. They show also that the corresponding result also holds for algebras of complex-valued continuous functions; see [**ABN**], where references to variations on this result are also given. Their proofs make use of function space topologies.

In this paper, biseparating maps are regarded as mappings between rings or algebras that send annihilator ideals to annihilator ideals, and the Araujo-Beckenstein-Narici theorem is generalized to some classes of R-algebras and established with techniques from ordered algebraic systems.

Throughout, Z denotes the set of integers considered either as a ring or as an additive group, depending on the context.

Let Spec($A$) denote the space of prime ideals of ideals of $A$ in the hull-kernel topology, and let Max($A$) (respectively, Min($A$)) denote the subspaces of maximal ideals (respectively, minimal prime ideals) of $A$.

Several kinds of questions are pursued in what follows. In particular:

Can one find a class of rings or algebras containing all rings or algebras of continuous functions on realcompact spaces such that whenever $A$ and $B$ are in that class:

**I** If there is a linear biseparating map $\varphi$ of $A$ onto $B$, then Min($A$) and Min($B$) are homeomorphic?

**II** If there is a linear biseparating map $\varphi$ of $A$ onto $B$, then Max($A$) and Max($B$) are homeomorphic?

**III** If $\varphi$ is a linear biseparating map of $A$ onto $B$, then $\varphi$ is a weighted homomorphism?

Below, reasonable sufficient conditions are obtained that yield affirmative answers to **I** and **II.**

The next example shows that an affirmative answer to **I** and **II** need not imply one to **III** even for F-linear maps unless the field F is chosen carefully.

EXAMPLE 1.1. Suppose $A = B = \mathrm{R}$ (which may be regarded as $C(X)$ for $X$ a one-point topological space), regard R as a Q-algebra, and let $\mathcal{B}$ denote a (Hamel) basis for R over Q as a vector space. Any permutation of the set $\mathcal{B}$ (which has cardinality $\mathfrak{c}$) induces a one-to-one Q-linear map $\varphi$ of R onto itself. There are $2^{\mathfrak{c}}$ such linear maps; but because the only automorphism of the field R is the identity, there are at most $\mathfrak{c}$ weighted homomorphisms of R onto itself. So **III** has a negative answer even though there is a positive one to **I** and **II**. It is an exercise to construct a similar example in case $A = B = \mathrm{C}$.

EXAMPLE 1.2. Suppose $A = B = \mathrm{R}[x]$ is the algebra of real polynomials, let $\pi$ denote a permutation of the set of nonnegative integers, and define $\varphi_\pi : A \to A$ by letting $\varphi_\pi(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n a_i x^{\pi(i)}$. If $\pi$ is not the identity, then this cannot be a weighted homomorphism, because the only units of $\mathrm{R}[x]$ are the (nonzero) constant polynomials. So, even for real algebras, **III** may have a negative answer even though there is a positive one to **I** and **II**.

## 2. Biseparating maps and spaces of minimal prime ideals

As in [**D**], a proper ideal $I$ of the ring $A$ is called *pure* if $I = \bigcup_{i \in I}(\mathbf{1} - i)^d$. It is shown in [**JM**] and [**He1**] that if $I$ is an ideal of $A$, then $mI := \{a \in A : I + a^d = A\}$ is the largest pure ideal of $A$ that is contained in $I$ and that $mI = \bigcap\{P \in \mathrm{Min}(A) : I + P \neq A\}$. This latter ideal is also discussed in [**M**]. An ideal $I$ is called a *d-ideal* if $a^{dd} \subset I$ whenever $a \in I$, and is called *pseudoprime* if $ab = 0$ implies $a \in I$ or $b \in I$. More generally, if $n$ is a positive integer, call an ideal $I$ of $A$ an *$n$-pseudoprime* ideal if whenever a product of $n$ elements of $A$ is 0, one of them is in $I$. A ring is called *reduced* if 0 is its only nilpotent element. As is well known, $A$ is reduced if and only if the intersection of all of its prime ideals is $\{0\}$.

($\alpha$) Pseudoprime ideals are studied thoroughly in [**GK**] where it is shown that an ideal $I$ contains a prime ideal if and only if it is $n$-pseudoprime for every positive integer $n$.

It is easy to see, using Zorn's lemma, that every prime ideal of $A$ contains a minimal prime ideal, and that:

($\beta$) A prime ideal $P$ of a reduced commutative ring is minimal if and only if $a \in P$ implies $a^d \not\subset P$.

It follows that $a^{dd} \subset P$. Thus

($\gamma$) Every minimal prime ideal of a reduced commutative ring is a $d$-ideal.

Let $\mathcal{S}(A)$ denote a collection of ideals of $A$. If $S \subset A$, let $h(S) = \{\, P \in \mathcal{S}(A) : S \subset P \,\}$, and let $h^c(S) = \mathcal{S}(A) \setminus h(S)$. We regard $\mathcal{S}(A)$ as a topological space whose closed sets are generated by $\{\, h^c(a) : a \in A \,\}$. This topology is usually called the *hull-kernel* or *Zariski* topology. $\mathrm{Min}(A)$ with this topology is usually called *the space of minimal prime ideals of* $A$ and carries the relative topology induced by the Zariski typology.

An element $a \in A$ such that $a^d = \{0\}$ is called *regular.*

If $\phi : A \to B$ is linear, and if $A$ is an integral domain, then $\phi$ is separating; this is immediate from the definition at the beginning of Section 1. If, in addition, $\phi$ is bijective and $B$, too, is a domain, then $\phi$ is biseparating. So, linear biseparating maps between domains need not preserve much of the multiplicative structure. In particular, such maps need not send units to units, as the next example shows.

EXAMPLE 2.1. Some linear biseparating maps between integral domains.

Suppose $A$ and $B$ are integral domains whose additive subgroups are free modules over a principal ideal domain $D$ with bases $\mathrm{B}(A)$ and $\mathrm{B}(B)$ respectively of the same cardinality. Any bijection of $\mathrm{B}(A)$ onto $\mathrm{B}(B)$ has a bijective extension $\varphi$ from $(A, +)$ onto $(B, +)$ that is biseparating, since neither $A$ nor $B$ have proper divisors of 0. Applying this in case $A = \mathrm{Z}(\sqrt{2})$ and $B = \mathrm{Z}(\sqrt{3})$ or $\mathrm{Z}(i)$, or when $A = \mathrm{Q}(\sqrt{2})$ and $B = \mathrm{Q}(\sqrt{3})$ or $\mathrm{Q}(i)$, illustrates that such linear biseparating maps can exist between nonisomorphic rings. Note in particular that if $A = \mathrm{Z}(\sqrt{2})$ and

$B = \mathrm{Z}(\sqrt{3})$ and $\varphi(a + b\sqrt{2}) = a + b\sqrt{3}$ whenever $a, b \in \mathrm{Z}$, then $-1 + \sqrt{2}$ is a unit of $A$, while its image $-1 + \sqrt{3}$ under $\varphi$ is not.

The next proposition illustrates, however that linear biseparating maps do preserve ideals determined in some sense by their zero divisors.

PROPOSITION 2.2. *Suppose $\varphi : A \to B$ is a linear biseparating map. Then:*
*(a) $A$ is reduced if and only if $B$ is.*
*(b) If $S \subset A$, then $\varphi(S)^d = \varphi(S^d)$. Thus if $S = \{s\}$ is a singleton, then $\varphi(s^d)^d = \varphi(s^{dd}) = \varphi(s)^{dd}$. In particular, $\varphi$ sends regular elements to regular elements.*
*(c) If $I$ is a union of annihilator ideals, then $\varphi(I)$ is an ideal of $B$ that is a sum of annihilator ideals. Moreover:*
  *(i) The image under $\varphi$ of a d-ideal of $A$ is a d-ideal of $B$, and*
  *(ii) if $I$ is an ideal of $A$ and $\varphi(I)$ is an ideal of $B$, then $\varphi(mI) = m\varphi(I)$. So the image under $\varphi$ of a pure ideal is a pure ideal. Moreover:*
*(d) If $A$ is reduced and $P \in \mathrm{Min}(A)$, then $\varphi(P)$ is a pseudoprime d-ideal such that $\varphi(a) \in \varphi(P)$ implies $\varphi(a)^d$ is not contained in $\varphi(P)$.*

PROOF. Since $\varphi$ is a bijection, each element of $B$ may be written as $\varphi(a)$ for a unique $a \in A$.

(a) If $\varphi(a)^2 = 0$, then $\varphi(a) = 0$, whence $a = 0$.

(b) Since $\varphi^{\leftarrow}$ is separating, if $\varphi(a)\varphi(S) = 0$, then $aS = \{0\}$, i.e., $a \in S^d$. It follows that $\varphi(a) \in \varphi(S^d)$, so $\varphi(S)^d \subset \varphi(S^d)$. The reverse inequality holds since $\varphi$ is also separating. In particular, $\varphi(s^d)^d = \varphi(s^{dd})$.

(c) The first part of (c) and (i) follow immediately from (b).

(ii) By assumption $\varphi(I)$ is an ideal of $B$. Because $mI = \{\, a \in A : I + a^d = A \,\}$, $\varphi(I)$ is an ideal of $B$, and $\varphi(mI) = \{\, \varphi(a) \in B : \varphi(I) + \varphi(a)^d = B \,\} = m\varphi(I)$. The last assertion is immediate.

(d) By ($\gamma$) and (c), $\varphi(P)$ is a $d$-ideal. Suppose $\varphi(a)\varphi(b) = 0$, in which case $ab = 0$. Because $P$ is prime, $a \in P$ or $b \in P$; say $a \in P$. So $\varphi(a) \in \varphi(P)$ and we conclude that $\varphi(P)$ is a pseudoprime ideal. If $\varphi(a)^d$ is contained in $\varphi(P)$ for some $\varphi(a) \in \varphi(P)$, then by (b), $\varphi(a^d) \subset \varphi(P)$. So both $a$ and $a^d$ are in the minimal prime ideal $P$, contrary to ($\beta$). □

A group $G$ that is also a lattice (with lattice-operations $\vee$ and $\wedge$) in which the sum of nonnegative elements is nonnegative is called a *lattice-ordered group* or an *$\ell$-group.* If $G$ is also a vector space over R, then it is called a *vector lattice* or a *Riesz space*; see [**LZ**]. An abelian $\ell$-group that is a ring in which a product of positive elements is positive is called an *$\ell$-ring,* and an $\ell$-ring in which $a \wedge b = 0$ and $c \geq 0$ imply $a \wedge bc = a \wedge cb = 0$ is called an *$f$-ring.* It is easily seen that a subdirect product of totally ordered rings is an $f$-ring, and the converse holds in the presence of the prime ideal theorem for boolean algebras; see [**FH**]. Many authors (including [**BKW**]) use the property of being a subdirect product of totally ordered rings as the definition of $f$-ring and make the axiom of choice a blanket assumption.

The kernel of a homomorphism of one $\ell$-ring into another that preserves the lattice as well as the ring operations is called an *$\ell$-ideal.* An $\ell$-ideal is a ring ideal $I$ such that $|a| \leq |b|$ and $b \in I$ imply $a \in I$. It is well-known and easily seen that:

($\delta$) Every $d$-ideal of a commutative reduced $f$-ring is a semiprime $\ell$-ideal.

(For background, see Section 4 of [**HdeP3**].)

A commutative $f$-ring $A$ with identity is said to *closed under bounded inversion* if each $a \geq 1$ in $A$ is invertible, or equivalently if each maximal ideal of $A$ is an $\ell$-ideal. See [**HIJ**].

DEFINITION 2.3. An ideal $I$ of a commutative semiprime ring $A$ is said to be *close to minimal prime* or *cmp* provided that $a \in I$ if and only if $a^d \not\subset I$. The space of cmp-ideals of $A$ with the hull-kernel topology is denoted by $\mathrm{Cmp}(A)$.

PROPOSITION 2.4. *Suppose $A$ is a unital reduced commutative ring.*
*(a) If $P \in \mathrm{Min}(A)$, then $P$ is cmp.*
*(b) An ideal $I$ is cmp if and only if (i) it is pseudoprime, (ii) it is a d-ideal, and (iii) $a \in I$ implies $a^d \not\subset I$.*

PROOF. (a) If $P \in \mathrm{Min}(A)$ and $a \in P$, then $a^d \not\subset I$ by ($\beta$). Because $P$ is prime and $a(a^d) = \{0\}$, if $a^d \not\subset I$, then $a \in I$.

(b) If $I$ is cmp, then (iii) holds, by definition. Suppose $ab = 0$ and $a \notin I$. Then $a^d \subset I$, so $b \in I$ and hence (i) holds. If $a \in I$ and $c \in a^{dd}$, then since $a^d a^{dd} = \{0\}$, $a^d \not\subset I$, and $I$ is pseudoprime, (ii) holds as well.

Conversely, suppose (i), (ii), and (iii) hold. If $a \in I$, then $a^d \not\subset I$ by (iii). If $a^d \not\subset I$, then $a \in I$ by (i). □

We are indebted to Suzanne Larson for the next example. It is a modification of Example 21 in [**He1**] and exhibits a unital commutative reduced algebra with a cmp-ideal that fails to be prime.

EXAMPLE 2.5 (S. Larson). Let $A = F[x_0, x_1, \cdots, x_n, \cdots]$ denote the ring of polynomials in a countable infinity of (commuting) variables $\{x_n\}_{n<\omega}$ over a field $F$, let $K$ denote the smallest ideal of $A$ containing all $x_i x_j x_k$, where $i < j < k$, and let $B = A/K$. For each $p \in A$, abbreviate the coset $p + K$ by $K(p)$, and let $I$ denote the smallest ideal of $B$ containing all $K(x_i x_j)$, where $i < j$. It will be shown that $I$ is a cmp-ideal that is not prime.

To see this, note first that every zero divisor in $B$ is a finite sum of elements that lie in $K(x_j)B$ for some $j < \omega$. If $K(a) \in I$, then there is an $n < \omega$ such that $K(a)$ is a finite sum of elements in $K(x_i x_j)B$, where $i < j \leq n$. Thus, $K(x_{n+1}) \in K(a)^d \setminus I$. Conversely, if $K(a) \not\subset I$, then $K(a)$ is regular, or there is a $j < \omega$ such that $K(a) \in K(x_j)B \setminus K(x_i)B$ for any $i \neq j$, in which case $K(a)^d \subset I$. Hence $I$ is a cmp-ideal. Because $K(x_1 x_2) \in I$ and neither $K(x_1)$ nor $K(x_2)$ is in $I$, the ideal $I$ fails to be prime.

The next result follows immediately from 2.3 and 2.4.

COROLLARY 2.6. *If $A$ and $B$ are reduced and $\varphi : A \to B$ is a linear biseparating map, then $I$ is a cmp-ideal of $A$ if and only if $\varphi(I)$ is a cmp-ideal of $B$.*

DEFINITION 2.7. $A$ is called a *cmp-ring* if each of its cmp-ideals is prime.

It is shown in [**HdeP1**, 4.2] and [**S**, 2.5] that:

($\epsilon$) Every pseudoprime semiprime ideal of a commutative $f$-ring with identity is a prime ideal. Hence every commutative reduced $f$-ring is a cmp-ring.

The next theorem provides an answer to Question **I** of the introduction.

THEOREM 2.8. *If $A$ and $B$ are reduced, and $\varphi$ is a linear biseparating map of $A$ onto $B$, then:*

into another that preserves scalar multiplication as well as the lattice and group operations is called a *Riesz homomorphism.* Its kernel $\ker\psi$ is a Riesz subspace such that $|h| \leq |g|$ and $g \in \ker\psi$ imply $h \in \ker\psi$. It is noted in Section 18 of [**LZ**] that:

(**) *An R-linear map $\psi$ of one Riesz space into another is a Riesz homomorphism if and only if $a \wedge b = 0$ implies $\psi a \wedge \psi b = 0$.*

An element $e \in A^+$ such that $a \wedge e = 0$ implies $e = 0$ is called a *weak order unit.* An archimedean $f$-algebra is called a *$\Phi$-algebra.* Equivalently, a $\Phi$-algebra is an archimedean lattice-ordered algebra over R with an identity element that is a weak order unit. See [**BKW**] or [**HR**].

EXAMPLE 4.1. $R[x]$ becomes an archimedean lattice-ordered algebra if we let a polynomial be nonnegative if each of its coefficients is nonnegative. The linear biseparating maps described in Example 1.2 are positive while failing to be weighted homomorphisms, because the only units of this ring are the nonzero constant functions.

In the remainder of this section, our efforts are concentrated on $\Phi$-algebras.

It is well known and easily seen that in a reduced $f$-ring $A$, if $a, b \in A^+$, then $a \wedge b = 0$ if and only if $ab = 0$. It follows from (**) that:

LEMMA 4.2. *If $A$ and $B$ are reduced $f$-algebras, then an R-linear positive bijection of $A$ onto $B$ is biseparating if and only if it is a Riesz homomorphism.*

The question of when a Riesz homomorphism is a $\Phi$-algebra homomorphism is considered by A. Hager and L. Robertson in [**HR**], where they show in Corollary 4.3 that:

LEMMA 4.3 (Hager-Robertson). *A Riesz homomorphism between $\Phi$-algebras that sends the identity element to the identity element is a $\Phi$-algebra homomorphism.*

Their proof makes use of the fact that a $\Phi$-algebra may be represented (using AC) as an algebra of extended real-valued functions on its compact (in the hull-kernel topology) space of maximal $\ell$-ideals. See [**HR**] for details. This result was obtained later independently by C. Huijsmans and B. de Pagter in [**HdeP2**] using the weaker definition of $f$-ring. Indeed, they show that it is enough to assume that the image of the identity element of $A$ is idempotent.

To make use of Lemma 2.2, the following definition is introduced which appears in [**HJ**].

DEFINITION 4.4. If $A$ is an R-algebra and $a \in A$ is invertible only if $a$ is in no R-ideal, then $A$ is said to be *closed under inversion.*

It is easy to verify that if $A$ is closed under inversion, then $J(A)$ is the intersection of all the R-ideals of $A$. Hence:

PROPOSITION 4.5. *A semiprimitive* R*-algebra $A$ that is closed under inversion is a subdirect product of copies of R. Thus $A$ is an algebra of real-valued functions.*

The main result of this section follows.

THEOREM 4.6. *If $A$ and $B$ are $\Phi$-algebras closed under* R*-inversion, and $\varphi : A \to B$ is a positive linear biseparating map, then $\varphi$ is a weighted homomorphism.*

PROOF. By Lemma 4.2, $\varphi$ is a Riesz homomorphism. We show first that $\varphi(1_A)$ is a unit of $B$. Otherwise, $\varphi(1_A)$ is in a real maximal ideal $M$ of $B$. In his Leiden doctoral dissertation B. de Pagter showed that because $A$ is archimedean,

$$(\dagger) \qquad x - x \wedge n1_A \;\leq\; n^{-1}x^2 \text{ for each } n < \omega \text{ and } x \in A^+.$$

(For a proof, see 8.22 in [**AB**].) So, $\varphi(x) - \varphi(x) \wedge n\varphi(1_A) \;\leq\; n^{-1}\varphi(x^2)$ and hence $n\varphi(x) \leq \varphi(x^2) \mod M$ whenever $n < \omega$. Because R is archimedean, this implies that $\varphi[A^+] = 0$ and hence that $\varphi(1_A) = 0$, contrary to the fact that $\varphi$ is one-one.

Let $\psi = [\varphi(1_A)]^{-1}\varphi$. Then $\psi$ is a Riesz homomorphism such that $\psi(1_A) = 1_B$. So, by Lemma 4.3, $\psi$ is a $\Phi$-algebra homomorphism, and hence $\varphi = [\varphi(1_A)]\psi$ is a weighted homomorphism. □

Note that the hypothesis of Theorem 4.6 is satisfied by $\Phi$-algebras that need not be uniformly closed; e.g., by the algebra of continuous real-valued functions on $[0, \infty)$ that are eventually rational functions with no poles on $[0, \infty)$.

The last theorem is the best positive result we have been able to obtain using our techniques. We close with a number of questions that illustrate their limitations.

QUESTIONS AND REMARKS 4.7. Suppose $A$ and $B$ are soft rings and $\varphi : A \to B$ is a linear biseparating map.

**A.** Suppose in addition $A$ and $B$ are $f$-algebras and $\varphi$ is positive. Must $\varphi$ be a weighted homomorphism?

Our guess is that this has a negative answer, but we have been unable to find an example to support it. Note that if the requirement that $\varphi$ is positive were dropped, Example 1.2 provides a negative answer to the modified question. Observe that R$[x]$ can be made into a (totally ordered) $f$-algebra in a number of ways (e.g., by letting $\sum_{k=0}^{n} a_k x^k > 0$ if $a_k > 0$).

**B.** Suppose in addition $A$ and $B$ are $\Phi$-algebras. Must $\varphi$ be a unit multiple of a Riesz homomorphism?

The Araujo-Beckenstein-Narici theorem yields an affirmative answer to this question in case $A$ and $B$ are each the ring of all continuous real-valued functions on a realcompact space. We seem neither to be able to establish this with the techniques developed above or to find examples to show that this conclusion need not hold without assuming such strong hypotheses. For example, does the conclusion of **B** hold if $A$ and $B$ are algebras of real-valued functions that fail to be uniformly closed?

**C.** In Sections 91 and 92 of [**Z**], linear spaces and algebras of the form $A + iA$ are considered, where $A$ is a Riesz space or $f$-algebra, and are called complex Riesz spaces or complex $f$-algebras. Note that the Riesz space or algebra of all complex valued continuous functions on a topological space $X$ takes this form. An absolute value function may be defined on the algebras $A + iA$ with the usual properties in case $A$ is uniformly complete.

Can the theory developed above be extended to cover complex $f$-algebras $A+iA$ in case $A$ is uniformly complete and $A + iA$ is closed under some appropriate kind of inversion?

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