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1. INTRODUCTION

Throughout A denotes an *f-ring*; that is, a lattice-ordered ring that is a subdirect union of totally ordered rings. We let $\mathcal{D}(A)$ denote the set of derivations $D : A \longrightarrow A$ such that $a \geq 0$ implies $Da \geq 0$, and we call such derivations *positive*. In [CDK], P. Coleville, G. Davis, and K. Keimel initiated a study of positive derivations on f-rings. Their main results are (i) $D \in \mathcal{D}(A)$ and A archimedean imply $D = 0$, and (ii) if A has an identity element 1 and a is the supremum of a set of integral multiples of 1, then $Da = 0$. Their proof of (i) relies heavily on the theory of positive orthomorphisms on archimedean f-rings and gives no insight into the general case. Below, in Theorem 4 and its corollary, we give a direct proof of (i), and in Theorem 10, we generalize (ii). Throughout, we improve on results in [CDK], and we study a variety of topics not considered therein.

2. THE RESULTS

In the sequel, A will always denote an f-ring, and $A^+ = \{a \in A : a \geq 0\}$ its positive cone. If $a \in A$, let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. Then $a = a^+ - a^-$, $|a| = a^+ + a^-$, and $a^+a^- = a^-a^+ = a^+ \wedge a^- = 0$. A subset I of A that is a ring ideal and such that $|b| \leq |a|$, and $a \in I$ imply $b \in I$ is called an *ℓ-ideal*. The ℓ-ideals are the kernels of homomorphisms that preserve lattice as well as ring operations [BKW, Chap. 8].

A *derivation* on A is a linear map $D : A \longrightarrow A$ such that if $a,b \in A$, then $D(ab) = aDb + (Da)b$. A derivation D is called *positive* if $D(A^+) \subset A^+$. The family of all positive derivations on A will be denoted by $\mathcal{D}(A)$.

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In any f-ring *rad* A, the set of all nilpotent elements of A, coincides with the intersection of all the prime ℓ-ideals of A, and hence is an ℓ-ideal [BKW, 9.2.6]. If rad A = {0}, then A is said to be *reduced*. In [CDK], it is shown that if A is commutative and $a^n = 0$, then $[Da]^{2n-1} = 0$. We improve this result next. We begin by observing that if $a, b, \in A^+$ then

(1) $$ab = 0 \quad \text{implies} \quad aDb = (Da)b = 0.$$

1. PROPOSITION. *Suppose* $a \in A$ *and* $D \in \mathcal{D}(A)$. *Then* $a^n = 0$ *implies* $(Da)^n = 0$. *In particular,* $D[\text{rad } A] \subset \text{rad } A$.

PROOF. Since $a^n = 0$ if and only if $|a|^n = 0$, we may assume $a \in A^+$ and $n > 1$. By (1), $a^{n-1}Da = 0$. So $a^{n-2}(aDa) = 0$. Using (1) again yields $0 = a^{n-2}D(aDa) = a^{n-1}D^2a + a^{n-2}(Da)^2$. Since $a \in A^+$, $a^{n-2}(Da)^2 = 0$. Continuing this process yields $(Da)^n = 0$ and hence that $D[\text{rad } A] \subset \text{rad } A$.

The next example will show that the index of nilpotency of Da need not be less than that of a. We note first that if $D \in \mathcal{D}(A)$ and I is an ℓ-ideal of A such that $D(I) \subset I$, then $D_I \in \mathcal{D}(A/I)$, where

(2) $$D_I(a+I) = Da+I,$$

2. EXAMPLE. Let R denote all rational functions with real coefficients of negative degree. If $r(x) = \frac{p(x)}{q(x)} \in R$, we may assume that $q(x) = x^m + a_1x^{m-1} + \dots$ has leading coefficient 1, and we let $r(x)$ be positive if the leading coefficient of $p(x)$ is positive. With this order, R is a totally ordered ring. If $r(x) \in R$, let $Dr(x) = -r'(x)$ be the negative of the usual derivative. Then $D \in \mathcal{D}(R)$, as is $(xD) : R \longrightarrow R$, where $(xD)r(x) = xDr(x) = -xr'(x)$. If n is a positive integer, let I_n denote the set of all $r(x)$ in R of degree $\leq -n$. Clearly I_n is an ℓ-ideal of R, and $(xD)(I_n) \subset I_n$. If $R_n = R/I_n$, and $(xD)_n(r(x)+I_n) = xDr(x) + I_n$, then $(xD)_n \in \mathcal{D}(R_n)$, and $(xD)_n(\frac{1}{x} + I_n) = \frac{1}{x} + I_n$ is nilpotent of index n.

If G is an abelian ℓ-group, and $T : G \longrightarrow$ is an order preserving endomorphism of G such that $x \wedge y = 0$ implies $x \wedge Ty = 0$ for x,y in G^+, then T is called a *positive orthomorphism* of G. If A is reduced, then $x \wedge y = 0$ if and only if $xy = 0$ [BKW, 9.3.1].

So each positive derivation on an f-ring is an orthomorphism by (1). The next result appears implicitly in [CDK]. We include a proof for the sake of completeness.

3. PROPOSITION. *If* P *is a minimal prime* ℓ*-ideal of* A, *and* $D \in \mathcal{D}(A)$, *then* $D(P) \subset P$. *In particular,* $D_P \in \mathcal{D}(A/P)$.

PROOF. As is noted in [BKW, 9.3.2 and 12.1.1], if A is reduced, then each positive orthomorphism of $A(+)$ maps a minimal prime subgroup into itself, and P is a minimal prime ℓ-ideal of A if and only if it is a minimal prime subgroup. So $D(P) \subset P$ if A is reduced. In the general case, if we let $I = \text{rad } A$ in (2), we obtain $D(P) \subset P$.

We do not know if $D(P) \subset P$ for any prime ℓ-ideal of P.

Recall that A is said to be *archimedean* if $a \in A^+$ and $\{na : n=1,2,\dots\}$ bounded above imply $a = 0$. The next theorem is the key to an alternate proof of the fact that a reduced archimedean f-ring admits no nontrivial derivations [CDK].

4. THEOREM. *Suppose* A *is reduced,* $D \in \mathcal{D}(A)$, $a \in A^+$, *and* n *is a positive integer. Then*

(a) $n(a \wedge a^2)Da \le (a \vee a^2)Da$,

(b) $nDa(a \wedge a^2) \le Da(a \vee a^2)$, and

(c) $nD(a^2) \le (a^2Da + (Da)a^2) \vee Da$.

PROOF. Since A is reduced, $\{0\}$ is an intersection of minimal prime ideals and A is a subdirect sum of totally ordered rings A/P such that P is a minimal prime ℓ-ideal. Thus, by Proposition 3, it suffices to verify these identities in case A is totally ordered and has no proper divisors of 0 [BKW, 9.2.5].

Let $x = (na-a^2)^+Da$. Then $x \in A^+$. We consider two cases:

(i) Suppose $x = 0$. Then $Da = 0$ or $na \le a^2$. In either case we obtain

$$naDa \le a^2Da \quad \text{and} \quad n(Da)a \le (Da)a^2. \tag{3}$$

(ii) Suppose $x > 0$. Then $Da > 0$ and $a^2 < na$. Hence $aDa + (Da)a \le nDa$. Since A is totally ordered, $aDa \le (Da)a$ or $(Da)a \le a(Da)$.

Suppose the former holds. Then

$$2aDa \le nDa \quad \text{and hence} \quad (na-2a^2)Da \ge 0.$$

But $Da > 0$, so $2a^2 \leq na$. By induction, we get $2^k a^2 \leq na$ for $k = 0,1,2,\ldots$. If we choose k so large that $n^2 \leq 2^k$, we get

$$na^2 \leq a. \tag{4}$$

If, instead, $(Da)a \leq aDa$, an obvious modification of this latter argument also yields (4). Pre or post multiplying by Da yields

$$na^2Da \leq aDa \quad \text{and} \quad n(Da)a^2 \leq (Da)a. \tag{5}$$

Since either (3) or (5) must hold in A/P for any minimal prime ideal P, the conclusions of (a) and (b) hold.

By (4), if $x > 0$, then $nD(a^2) \leq D(a)$. If $x = 0$, then adding the inequalities in (3) yields $nD(a^2) \leq (a^2Da + (Da)a^2)$. Hence (c) holds as well.

5. COROLLARY. [CDK] *If* A *is archimedean and* $D \in \mathcal{D}(A)$, *then* $D(A) \subset \text{rad } A$ *and* $D(A^2) = 0$.

PROOF. By (c) of the last theorem and Proposition 3, if $a \in A$, then $D(a^2) \in \text{rad } A$. Since $aDa \leq D(a^2)$, $(Da)^2 \leq D(aDa) \leq D^2(a^2) \in D(\text{rad } A) \subset \text{rad } A$ by Proposition 1. Since each element of rad A is nilpotent, so is Da.

If $a,b \in A$, then $D(ab) = aDb + (Da)b = 0$, since $(\text{rad } A)A = A(\text{rad } A) = 0$ in an archimedean f-ring [BKW, 12.3.11]. Hence $D(A^2) = 0$.

6. PROPOSITION. *Suppose* $e^2 = e \in A$ *and* $D \in \mathcal{D}(A)$.

(a) $(De)^2 = e(De)e = (De)e(De) = 0$.

(b) *If* A *is reduced or has an identity element or* e *is in the center of* A, *then* $De = 0$.

PROOF. Since $e^2 = e$, we have

$$eDe + (De)e = De \tag{6}$$

Multiplying (6) on the left by e yields

$$e(De)e = 0. \tag{7}$$

Applying D to (7), we obtain

$$eD[(De)e] + (De)^2e = 0 = e(De)^2 + D(eDe)e.$$

Hence

(8) $$e(De)^2 = (De)^2e = 0.$$

Multiplying both sides of (6) on the left by (De) and using (8) yields

(9) $$(De)e(De) = (De)^2.$$

By (7), (8), and (9), we obtain

$$[eDe - (De)e]^2 + (De)e(De) = 0.$$

Hence $(De)^2 = (De)e(De) = 0$, which together with (7), completes the proof of (a).

Clearly $De = 0$ if $\mathrm{rad}\, A = \{0\}$. If $eDe = (De)e$, then by (6) and (7), $De = 2eDe = 0$. If A has an identity element, then each of its idempotents is in the center of A by [BKW, 9.4.20]. This completes the proof of (b).

The next example shows that the hypotheses of (b) above cannot be omitted.

7. EXAMPLE. *A totally ordered ring with an idempotent* e *and a positive derivation* D *such that* $De \neq 0$.

Let S denote the algebra over the real field $\mathbb{R}$ (with the usual order) with basis $\{e,z\}$, where $e^2 = e$, $ez = z^2 = 0$, and $ze = z$. If $x = \alpha e + \beta z \in S$, let $x > 0$ if $\alpha > 0$ or $\alpha = 0$ and $\beta > 0$. If we let $Dx = zx - xz = \alpha z$, then $D \in \mathcal{D}(S)$, and $De = z \neq 0$.

If $D \in \mathcal{D}(A)$, let $\ker D = \{a \in A : Da = 0\}$. If G is an abelian ℓ-group and $H \subset G$, let $H^{\perp} = \{g \in G : |g| \wedge |h| = 0$ for all $h \in H\}$, and let $H^{\perp\perp} = (H^{\perp})^{\perp}$. Note that $H^{\perp}$ is an ℓ-subgroup of G (that is, H is a subgroup and $|a| \leq |b|$, and $b \in H^{\perp}$ implies $a \in H$). A *band* in G is an ℓ-subgroup H of G such that if $K \subset H$ and $\sup K \in G$, then $\sup K \in H$. If H is a subset of G, the intersection $B(H)$ of all the bands in G containing H is also a band. Moreover, $B(H) \subset H^{\perp\perp}$. See [LZ, Theorem 19.2]. An element e of G such that $\{e\}^{\perp} = 0$ is called a *weak order unit* of G. An element e of an f-ring A such that $ex = 0$ or $xe = 0$ implies $x = 0$ is called *regular*. Note that if $e \in A$ is regular, then e is a weak order unit, and the converse holds if A is reduced.

The following lemma will be useful in what follows.

8. LEMMA. *Suppose* A *is an* f-*ring and* $D \in \mathcal{D}(A)$.

(a) $xDx \wedge (Dx)x \geq 0$ *for every* $x \in A$.

(b) *If* A *is reduced, then* D *is an* ℓ-*endomorphism.*

(c) *If* A *has an identity element* 1, *and* n *is a positive integer, then* $nDx \leq xDx \wedge (Dx)x$ *for every* $x \in A^+$ *and* $D(I) \subset I$ *for every* ℓ-*ideal* I *of* A.

PROOF. (a) holds since this inequality holds whenever A is totally ordered.

(b) holds since if A is reduced, then D is a positive orthomorphism and hence an ℓ-endomorphism [BKW, 12.1].

(c) by Proposition 6(b), $1 \in \ker D$, and by (a) $(x-n1)D(x-n1) \geq 0$. Hence $nDx \leq xDx$. Similarly, $nDx \leq (Dx)x$. Hence $x \in I$ implies $Dx \in I$ since I is an ℓ-ideal.

Next, we provide some examples to show that the hypotheses of (b) and (c) above cannot be omitted.

9. EXAMPLES. (i) Let E denote the direct sum of two copies of the real line $\mathbb{R}$ with trivial multiplication, and let $(r,s) \geq 0$ mean $r \geq s \geq 0$. As is noted in [GJ, 5B], the map $D : E \longrightarrow E$ such that $D(r,s) = (r,0)$ is a positive endomorphism that is not an ℓ-homomorphism. To see the latter, note that $(1,2)^+ = (2,2)$. So $D[(1,2)^+]$ $= (2,0) \neq (1,0) = [D(1,2]^+$.

(ii) Let R and (xD) be as in Example 2, and let $y = \frac{1}{x}$. Then $n(xD)y = \frac{n}{x}$, while $y(xD)y = x^{-2}$, so the conclusion of (c) fails.

The next theorem summarizes most of what we know about kernels of positive derivations.

10. THEOREM. *Suppose* $D \in \mathcal{D}(A)$, $x \in A$, *and* n *is a positive integer.*

(a) *If* e *is regular, and* $ex \in \ker D$, *then* $x \in \ker D$.

(b) *If* A *is reduced then:*

(i) $x \in \ker D$ *implies* $\{x\}^{\perp\perp} \in \ker D$,

(ii) $x^n \in \ker D$ *implies* $x \in \ker D$,

(iii) $\ker D$ *is a band,*

(iv) $D^n = 0$ *implies* $D = 0$, *and*

(v) $e^2 = e \in A$ *implies* $e \in \ker D$.

(c) *If* A *has an identity element and* $U(A)$ *is the smallest band containing the units of* A, *then* $U(A) \subset \ker D$. *In particular,* $\mathrm{rad}\, A \subset \ker D$. *Also, if* $x^2 \leq x$, *then* $x \in \ker D$.

PROOF. (a) By (1), $D(ex) = 0$ implies $eDx = 0$, which, in turn implies $Dx = 0$.

(b) (i) By Lemma 8(b), and [BKW, 3.2.2], $D(\{x\}^{\perp\perp}) \subset D(\{x\}^{\perp})^{\perp} \subset \{(Dx)\}^{\perp\perp} = \{0\}$ since $x \in \ker D$ and A is reduced.

(ii) follows from (i) and the fact that $\{x\}^{\perp\perp}$ is the intersection of all the minimal prime ℓ-ideals that contains x [BKW, 3.4.12].

(iii) As was noted above, the smallest band containing $\ker D$ is contained in $\{(\ker D)\}^{\perp\perp}$ and the latter is contained in $\ker D$ by (i).

(iv) Since x is a difference of positive elements, it suffices to show that $Dx = 0$ whenever $x \in A^+$. The proof will proceed by induction on n. It is obvious when $n = 1$. Assume that $D^n(A) = 0$ implies $D(A) = 0$ whenever A is a reduced f-ring and $n \geq 1$ is an integer. If $0 = D^{n+1}(A) = D^n(D(A))$, then $D^n(D(A)^{\perp\perp}) = 0$ by (i). So $D(D(A)^{\perp\perp}) = 0$ by the induction hypothesis. In particular, $D^2(x^2) = 0$. Since $xDx \leq D(x^2)$, $0 = D(xDx) = xD^2x + (Dx)^2$. So $(Dx)^2 = 0 = Dx$ since A is reduced.

(v) is a restatement of Proposition 16(b).

(c) That $U(A) \subset \ker D$ follows directly from (a) and (b) (iii) above. If $x^n = 0$, then $(1-x)(1+x+\cdots+x^{n-1}) = 1$, so $1-x$ is a unit and $x = 1 - (1-x) \in U(A) \subset \ker D$. Finally, if $x^2 \leq x$, then $D(x^2) = xDx + (Dx)x \leq Dx \leq xDx \wedge (Dx)x$ by Lemma 8(c). Hence $xDx = (Dx)x = 0$. Thus $Dx = 0$. This completes the proof of Theorem 10.

11. EXAMPLES AND REMARKS. The assumption that A is reduced in Theorem 10(b) cannot be dropped. For example, if $A = C[0,1]$, the ℓ-group of continuous real-valued functions on $[0,1]$, with trivial multiplication for all $f \in C[0,1]$, we let $Df = f(\frac{1}{2})$, then $D \in \mathcal{D}(A)$, and $\ker D$ fails to be a band [DV, p. 12]. Also, the plane E^2 with the usual coordinatewise addition and trivial multiplication admits positive endomorphisms that are nilpotent. (For example, let $T(a,b) = (0,a)$ for all $(a,b) \in E^2$).

Theorem 10(c) generalizes [CDK, Theorem 7] where it is shown that $\ker D$ contains the supremum of any set of elements bounded above by some integral multiple of the identity element.

As in [P], we let $I_0(A) = \{a \in A : n|a| \le x$ for some $x \in A^+$ and $n = 1,2,\ldots\}$. Clearly $I_0(A)$ is an ℓ-ideal and $I_0(A) = \{0\}$ if and only if A is archimedian.

12. THEOREM. *Suppose* $D \in \mathcal{D}(A)$.

(a) *If* A *is reduced, then* $D(A^2) \subset I_0(A)$.

(b) *If* A *has an identity element, then* $D(A) \subset I_0(A)$. *If, moreover,* A *is reduced and* $I_0(A) \subset \mathcal{U}(A)$, *then* $D = 0$.

PROOF. (a) follows immediately from Theorem 4 and the fact that $ab \le (a \vee b)^2$ whenever $a,b \in A^+$.

(b) That $D(A) \subset I_0(A)$ is a restatement of Lemma 10(c). If $I_0(A) \subset \mathcal{U}(A)$, then by Theorem 10(c), $D^2(A) \subset D(\mathcal{U}(A)) = \{0\}$. Hence if A is reduced, then $D = 0$ by Theorem 10(b).

13. EXAMPLES AND REMARKS.

(a) The reader may easily verify for the f-ring R of Example 2, $I_0(R) = I_2$, while $(xD)(R) = R$. So the hypothesis in Theorem 12(b) that A has an identity element may not be dropped if we wish to have $D(A) \subset I_0(A)$.

(b) Let S denote the ring of all functions of the form

$$\sum_{i=0}^n a_i x^{r_i}$$

where a_i is an integer and r_i is a nonnegative rational number, ordered lexicographically, with the coefficient of the largest power of x dominating. Then $I_0(S) = S$, and $\mathcal{U}(A)$ is the set of constant polynomials. So, the condition of Theorem 12(b) fails. Despite this, $D \in \mathcal{D}(S)$ implies $D = 0$.

For if $D \in \mathcal{D}(S)$, then $D(x) = D((x^{1/2})^2 = 2x^{1/2}D((x^{1/4})^2) = 4x^{3/4}D((x^{1/8})^2) = \cdots = 2^n x^{1-1/2n} D(x^{1/2n})$. Hence $2^n|D(x)$ for $n = 0,1,2,\ldots$. Since the coefficients of any element of S are integers, it follows that $D(x) = 0$. A similar argument will show that $x^r \in \ker D$ whenever r is a nonnegative rational number. It follows that $D = 0$.

We do not, however, know of any such example that is an algebraa over an ordered field. If S^* is the result of allowing the coefficients of the elements of S to be arbitrary rational numbers, and we let $D(x^r) = rx^r$ for any positive rational number r, then D is a positive derivation. To see why, map x^r to e^{rx} and note that S^* is isomorphic as an ordered ring to a subring of the ring of exponential polynomials, and the usual derivative on the latter maps the image of S^* into itself.

Our last result applies more general theorems and techniques of Herstein [H_1] [H_2] to the context of positive derivations.

14. THEOREM. *Suppose* A *is reduced and* $D \in \mathcal{D}(A)$.

(a) *If* $D \neq 0$, *then the ring* S *generated by* $\{Da : a \in A\}$ *contains a nonzero ideal of* A.

(b) *If* S *is commutative, then* S *is contained in the center of* A.

(c) *If* $z \in A$ *commutes with every element of* S, $(az-za) \in \ker D$ *for every* $a \in A$. *If, in addition,* A *is totally ordered and* $D \neq 0$, *then* z *is in the center of* A.

PROOF. (a) It is shown in [H_1] that the conclusion holds for any derivation on any ring if $D^3 \neq 0$. Since A is reduced, $D^3 \neq 0$ implies $D \neq 0$ by Theorem 10(b).

(b) Suppose $a \in S$ and $x \in A$. Then

$$0 = (Da)D(ax) - D(ax)(Da) = Da[aDx + (Da)x] - [aDx + (Da)x]Da = Da[(Da)x - x(Da)].$$

By [H_3, Lemma 1.1.4], Da is in the center of A.

(c) The second statement is shown in [H_2], and the first follows immediately from the second and Theorem 10(b).

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