2022

Algebraic Invariants of Knot Diagrams on Surfaces

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May, 2022

Abstract

In this thesis we first give an introduction to knots, knot diagrams, and algebraic structures defined on them accessible to anyone with knowledge of very basic abstract algebra and topology. Of particular interest in this thesis is the concept of the *quandle* which "colors" knot diagrams. Usually, quandles are only used to color knot diagrams in the plane or on a sphere, so this thesis extends quandles to knot diagrams on any surface and begins to classify the fundamental quandles of knot diagrams on the torus.

This thesis also breifly looks into Niebrzydowski tribrackets which are a different algebraic structure which, in future work, may have interesting behavior on knot diagrams in arbitrary surfaces.

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Acknowledgments

I'd like to thank Sam Nelson for helping me push through all this cool math and Alfonso Castro for being my second reader and an all around supportive person.

Further, I'd like to thank Jon Jacobsen, Melissa Hernandez-Alvarez, DruAnn Thomas, and all the 2022 HMC Math Thesis students for forming and supporting the Math Thesis community this year - life is more fun with friends!

Chapter 1

Introduction

In this work we ask the following question: suppose we have a loop (or many loops) of wire tangled up in space. For instance, consider the thickened loops in figure 1.1.

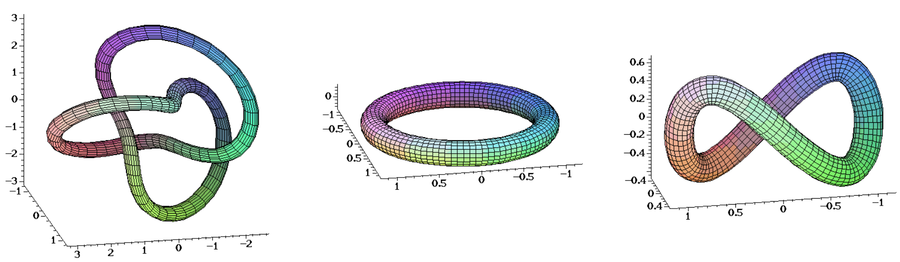


Figure 1.1 The Trefoil Knot (left), the Unknot (middle), and a bent Unknot (right)

It is a natural question to ask if these are the same or different. On one hand, they are differently embedded in space, so we might say that they are different for that reason. However, there's a natural way in which the middle and the right loops are the same: the right is simply the middle loop physically twisted in space (this is why we call both of them the "unknot.") If we allow the loop to phase through itself while we bend, then we can bend the left (and in fact any loop made this way) into the unknot as well. However, this does not match what happens physically. In particular, wire does not phase through itself, and if we want to study the properties of these wires we should disallow such phasing.

It's natural to wonder if the left loop, which I've labeled "Trefoil" is the unknot or not. We may get an answer in one of the following to ways. On one hand, the left loop may be bendable without crossing such that it can be made to look identical to the unknot. But on the other, if there is no such bending we are left with a problem: how can we prove that there is *no* way to bend the Trefoil into the unknot.

The answer to this question lives in Knot Theory, which is the subject of this thesis.

Chapter 2

Preliminaries: Knots and Their Invariants

2.1 Knots

For the remainder of this thesis, we will formalize the notion of knots in the following way.

Definition 2.1.1 (Nosaka (2017))**.** A *knot* K is a smooth one-to-one embedding of the circle $\mathbb{S}^1$ into space $\mathbb{R}^3$ (or sometimes the 3-sphere $\mathbb{S}^3$). That is,

$$K : \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$$

We say two embeddings K_1, K_2 are *ambient isotopic* if there is a smooth map

$$H : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$$

such that for each $t \in [0, 1]$, the map $x \mapsto H(x, t)$ is a diffeomorphism ($\mathbb{R}^3 \cong H(\mathbb{R}^3, t)$ as smooth manifolds), and for all $x \in \mathbb{R}^3$ and $\theta \in \mathbb{S}^1$,

$$H(x, 0) = x, \qquad H(K_1(\theta), 1) = K_2(\theta)$$

We think of two knots as being the same if and only if they are ambient isotopic.

Definition 2.1.1 agrees with our intuition in the following way. First, it defines a knot as a smooth one-to-one embedding of a circle, which represents our wire. We use a circle because we want the loop to be closed.

We choose a smooth embedding because we want to avoid any kinks or other odd non-differentiable elements in our wire, which would only confuse the underlying questions we have.

Secondly, this definition encodes the bending as a smooth family of diffeomorphisms H. Let K_1 and K_2 be embeddings of the circle into space and let H be an ambient isotopy between them as in the definition. Since differential geometry is not the focus of this thesis, we leave out some detail here, but the intuition behind this definition is as follows. The function H at each time t represents a smooth transformation of space into itself. Since H is smooth as a function of t as well, we can think of H as smoothly deforming $\mathbb{R}^3$ as t goes from 0 to 1.

Now consider $H(K_1(\theta), t)$ as t goes from 0 to 1. Since the composition of smooth maps is smooth and the composition of injections is injective, we have that for each $t \in [0, 1]$, $\theta \mapsto H(K_1(\theta), t)$ is a smooth, injective embedding of the circle into $\mathbb{R}^3$. In other words, each $t \in [0, 1]$ gives us a knotted loop in space, such that at $t = 0$, we have K_1 and at $t = 1$ we have K_2. Since H is smooth in all variables, we have that $H(K_1(\theta), t)$ literally traces out the bending of K_1 into K_2 as t goes from 0 to 1!

While this definition matches our physical intuition very well, it does not help us determine if two knots are the same or different. This is because, in general, it is a non-trivial problem to write down an ambient isotopy given two embeddings of the same knot. Further, we do not have any tools with which to prove that two embeddings are definitely *not* isotopic. We develop the tools formulated by Knot Theorists previously in the remainder of this section.

2.2 Reidemeister Moves

In the 1930's, Kurt Reidemeister significantly simplified the issue by giving a small set of intermediate moves that completely categorize all ambient isotopies of knots (the history is described by Colberg (2017)).

We build up the details to give the theorem.

First we will want to give a lower dimensional representation of our knots. In particular, consider a projection a knot K into $\mathbb{R}^2$. In general, this projection fails to be injective. But, we can do some small ambient isotopies as well as rotations of the projection plane to guarantee that there are only finitely many non-injectivities and that each represents a crossing of only two strands. In order, to recover an ambient isotopic copy of our original knot it turns out we just need to know which arc of the knot is "closer" to the viewer. Rigorously, this can be computed by a selection of a normal on the plane. A more general and complete analysis of this kind of process is given in 4.1. For now, here is a formal definition of the type of object we just produced:

Definition 2.2.1. A *knot diagram* in the plane is a smooth directed closed curve in $\mathbb{R}^2$ with finitely many self-intersections, each of which "looks" like an X, and for each self intersection data about which arc is "over" the other.

More formally we may say that a knot diagram is a smooth function

$$f : \mathbb{S}^1 \to \mathbb{R}^2$$

with the following properties

- There are only finitely many pairs $\theta_1 \neq \theta_2 \in \mathbb{S}^1$ for which $f(\theta_1) = f(\theta_2)$. (Finitely many self intersections).
- $\forall \theta_1, \theta_2, \theta_3 \in \mathbb{S}^1$ if $f(\theta_1) = f(\theta_2) = f(\theta_3)$ then two of θ_1, θ_2, and θ_3 are the same. (No points of triple intersection).
- If $\theta_1 \neq \theta_2 \in \mathbb{S}^1$ with $f(\theta_1) = f(\theta_2)$ then the tangent vectors are unequal: $f'(\theta_1) \neq f'(\theta_2)$. (No intersections with tangency).

along with information at each intersection $\theta_1 \neq \theta_2 \in \mathbb{S}^1$ with $f(\theta_1) = f(\theta_2)$ about which of θ_1 or θ_2 is "over" the other.

Now we can draw our knots as *knot diagrams* in the plane as in figure 2.1 by representing the "under" arc by having a gap in it.

In figure 2.1, we have also included a direction of the knot. If we choose a direction of the circle $\mathbb{S}^1$, this induces an orientation in our knots, which is preserved by ambient isotopy. In particular, if two knots have the same image but opposite directions, they need not be ambient isotopic. This follows from the fact that $\mathbb{R}^3$ is orientable.

Figure 2.1 Two copies of the Trefoil knot

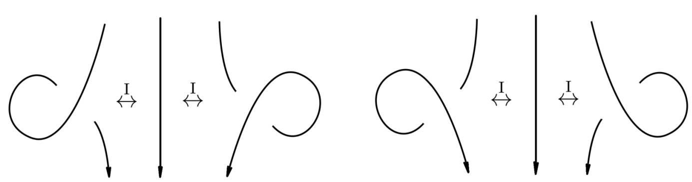
> **Proposition 2.2.2.** *A knot* (*as defined in Definition* 2.1.1) *is determined uniquely, up to ambient isotopy, by it's projection in the plane along with crossing information and direction.*

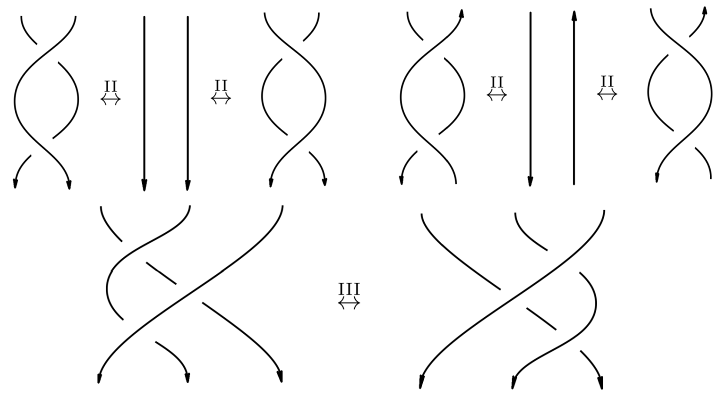
Remark. We can use the same ambient isotopy as before in the plane. It is fairly reasonable that an ambient isotopies in the plane will induce ambient isotopies of corresponding knots back in $\mathbb{R}^3$; that is if the projections of two knots are ambient isotopic in $\mathbb{R}^2$ then the knots themselves are ambient isotopic in $\mathbb{R}^3$.

However, doing a little mental gymnastics reveals that the two projections in figure 2.1 come from ambient isotopic knots in $\mathbb{R}^3$! However, an ambient isotopy of the plane will never change the number of crossings in a projection. In particular, ambient isotopy in the plane is not a complete picture of ambient isotopy back in $\mathbb{R}^3$.

Luckily, the following theorem closes the gap.

> **Definition 2.2.3** (Reidemeister Moves, Nelson (2018))**.** Consider the following set of diagrams.





We interpret each as a transition of a local piece of a projection of a knot. We say that two projections of knots P_1 and P_2 are *related by a Reidemeister move* if there is some simply connected open set of the plane such that P_1 is ambient isotopic to one of these diagrams in that region, and making a substitution along one of the arrows in P_1 gives a projection that is ambient isotopic to P_2.

Example 2.2.4. To make this clear, consider the set of knot diagrams related by Reidemeister moves in figure 2.2.

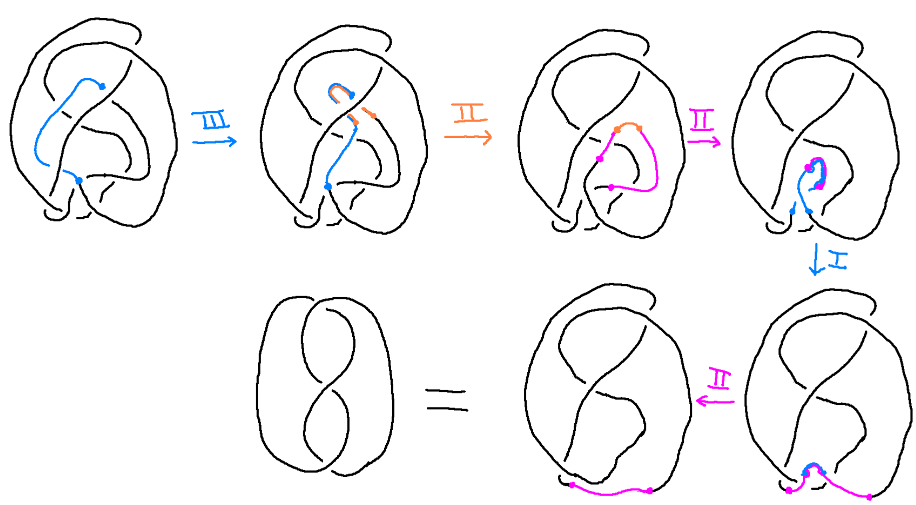


Figure 2.2 Knot diagrams related by Reidemeister moves

Remark. Note that in this last step we have applied an ambient isotopy. In the plane, it is fairly easy to tell if two knots are ambient isotopic because the crossings are unaffected (up to sliding around) by ambient isotopy. We will still say that ambient isotopic knots are related by (the empty set of) Reidemeister moves, but it is important to note that ambient isotopy in the plane does not give us much trouble (unlike in space).

Theorem 2.2.5 (Reidemeister)**.** *Let K_1, K_2 be knots as usual and let P_1 and P_2 be projections of K_1 and K_2 into (potentially different) planes. Then K_1 and K_2 are ambient isotopic if and only if P_1 and P_2 are related by a finite set of Reidemeister moves.*

Because of this theorem, we may refer to a knot by its projection in some plane without losing any information. Thus, for the remainder of this chapter (before we start delving into different projections in 4), we will refer to the knots by their embeddings in space and projections in the plane interchangibly.

2.3 Invariants

Consider the following situation.

Suppose we have two knots in space and we want to know if they are ambient isotopic. We pick a arbitrary plane and project the knots onto said plane in the way described above. There are two things that can happen.

- If the knots truly are the same, then we should be able to find a finite set of Reidemeister moves that relate the two projections, and this will suffice as a proof that the knots are the same.
- Otherwise, if the knots are truly different, then we will not be able to find a finite set of Reidemeister moves to relate the two knots.

The issue is this: how will we be able to distinguish the case where the knots are the same and we aren't good at using the Reidemeister moves, and the case where the knots are different and the knots are not related by Reidemeister moves?

Thus, we would benefit from tools that help us in this second case, which leads us to the study of knot Invariants. We use the tools of Category Theory.

Definition 2.3.1 (Category, Hatcher (2002)). A *category* $\mathcal{C}$ consists of the following three things:

- A collection $\mathrm{Ob}(\mathcal{C})$ of *objects*.
- Set $\mathrm{Hom}(X, Y)$ of *morphisms* for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$, including a distinguished 'identity' morphism $\mathbb{1}_X \in \mathrm{Hom}(X, X)$ for each $X \in \mathrm{Ob}(\mathcal{C})$.
- A 'composition' map $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, satisfying for each $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$ and $h \in \mathrm{Hom}(X, Z)$,

$$f \circ \mathbb{1}_X = f, \qquad \mathbb{1}_Y \circ f = f, \qquad (f \circ g) \circ h = f \circ (g \circ h)$$

We call $X, Y \in \mathrm{Ob}(\mathcal{C})$ *isomorphic* if there exists $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ such that $g \circ f = \mathbb{1}_X$ and $f \circ g = \mathbb{1}_Y$. In this situation we write $g = f^{-1}$ or $f = g^{-1}$ and call f and g *inverse morphisms*.

Some familiar examples of categories are

- Groups with group homomorphisms as morphisms.
- Rings with ring homomorphisms as morphisms.
- Topological spaces with continuous functions as morphisms.

Example 2.3.2. We will consider the collection of knot diagrams, $\mathcal{K}$ to be category in the following way.

- Let $\mathrm{Ob}(\mathcal{K})$ be the collection of all knot diagrams in the plane.
- For each pair of knot diagrams $k_1, k_2 \in \mathrm{Ob}(\mathcal{K})$, let $\mathrm{Hom}(k_1, k_2)$ be the set of all finite lists of Reidemeister moves that send k_1 to k_2 up to redundancies in the list. That is we identify a list of Reidemeister moves with the same list but with all redundant Reidemeister moves (a move immediately followed by its inverse) removed.

  Note that $\mathrm{Hom}(k_1, k_2)$ is empty if k_1 and k_2 are not equivalent knot diagrams. Further set $\mathbb{1}_k$ to be the empty list of Reidemeister moves applied to the knot diagram k.

- Let $\circ : \mathrm{Hom}(k_1, k_2) \times \mathrm{Hom}(k_2, k_3) \to \mathrm{Hom}(k_1, k_3)$ be defined by appending lists in the order they appear, again up to redundancies. We see that this is associative because appending lists is associative, and respects the identity since appending the empty list does not change a list.

Note that every morphism in $\mathcal{K}$ has an inverse generated by spelling the list of Reidemeister moves backwards and inverting each. In particular, two knot diagrams are isomorphic in $\mathcal{P}$ if and only if they are related by Reidemeister moves! By Theorem 2.2.5, two knot diagrams are isomorphic in $\mathcal{K}$ if and only if the knots they represent are ambient isotopic in space. Thus, in this language, our problem becomes determining if knot diagrams are isomorphic.

Definition 2.3.3 ((Covariant) Functor Hatcher (2002))**.** A *(covariant) functor F* from a category $\mathcal{C}$ to $\mathcal{D}$ assigns to each object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$ and assigns to each morphism $f \in \mathrm{Hom}(X, Y)$ in $\mathcal{C}$ a morphism $F(f) \in \mathrm{Hom}(F(X), F(Y))$ in $\mathcal{D}$ such that for each $X \in \mathrm{Ob}(\mathcal{C})$, $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$ and for each $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$ in $\mathcal{C}$

$$F(f \circ g) = F(f) \circ F(g).$$

Remark. Note that we could also consider the collection of knots in space a category with ambient isotopy as morphism. Then projecting onto the plane is a covariant functor from knots to knot diagrams.

What is important to us about functors is the following result:

Proposition 2.3.4. *Let F be a* (*covariant*) *functor between categories $\mathcal{C}$ and $\mathcal{D}$ and let $X, Y \in Ob(\mathcal{C})$. If X is isomorphic to Y, then $F(X)$ is isomorphic to $F(Y)$.*

Usually, we use the contrapositive of this statement: *If $F(X)$ is not isomorphic to $F(Y)$ then X is not isomorphic to Y.*

Proof. Suppose X and Y are isomorphic. Then there exists morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ with $g \circ f = \mathbb{1}_X$ and $f \circ g = \mathbb{1}_Y$. Note that since F is a functor we have morphisms $F(f) \in \mathrm{Hom}(F(X), F(Y))$ and $F(g) \in \mathrm{Hom}(F(Y), F(X))$. Futher, since F is a covariant functor we have

$$F(f) \circ F(g) = F(f \circ g) = F(\mathbb{1}_Y) = \mathbb{1}_{F(Y)}$$

and similarly

$$F(g) \circ F(f) = F(g \circ f) = F(\mathbb{1}_X) = \mathbb{1}_{F(X)}.$$

In particular, $F(X)$ and $F(Y)$ are isomorphic. □

Remark. We use this to help us distinguish non-isomorphic knots in the following way. Suppose k_1 and k_2 are arbitrary knot diagrams, and we have some functor F from the category of knot diagrams, to another category C where isomophism is easy to compute. If $F(k_1)$ is not isomorphic to $F(k_2)$, then we know that k_1 and k_2 are non-isomorphic knot diagrams! In particular, this gives us an easier way to compute when two knots are different.

However, it is important to know that if $F(k_1)$ and $F(k_2)$ are isomorphic in whichever category they live in, this *does not* tell us that k_1 and k_2 are the same!

We call functors from the category of knot diagrams to another category *knot invariants* because the output of the functor does not vary over isomorphic knot diagrams.

In the next sections we discuss two particular invariants.

2.4 The Knot Quandle

In this section we give a definition for the algebraic structure of *quandles*, which we will see form a category, and which we can easily create a functor to from the category of knot diagrams. Quandles were first introduced in Joyce (1982) for studying knots in exactly the way we will here.

The axiomatic definition is given in Definition 2.4.1. However, the quandle can be thought of as coming from the Reidemeister moves in the following way.

Say we have a knot presented in the plane. We wish to generate a category, which we will call the category of *Quandles*, and a (hopefully simple) functor from the category of knot diagrams to the category of quandles. That is, to each knot diagram, we want to associate a quandle, such that any knot diagram morphism induces a quandle morphism.

In particular, since all knot diagram morphisms are finite lists of Reidemeister moves and are all invertible, we want each Reidemeister move on an arbitrary knot to induce an isomophism on the category of quandles.

If we take a quandle to be a set X, with some binary operation $\triangleright :$ $X \times X \to X$ (as is a usual playground for categories), we can think of the elements of X as coloring the diagram of a knot in the plane, such that at each crossing we apply the operation according to figure 2.3. That is, when an arc of the diagram goes under another arc oriented down, we apply the operation on the under strand's left color to get the under strand's right color.

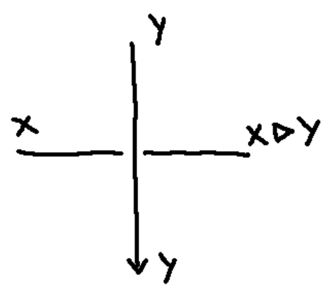


Figure 2.3 The quandle rule.

Now, given this set up, the Reidemeister moves induce rules on our quandle operation in the following way. Suppose we have a quandle that colors a given knot in the plane according to the rule given in figure 2.3. Now, suppose there is an open disk of the plane that contains arcs where we can apply a Reidemeister move. Since applying the Reidemeister move will give us an isomorphic knot, then the new knot should be colorable by an isomorphic quandle (and in particular, given our morphisms, the same quandle).

We can color the part of the knot not involved in the move the same and everything there will work the same way as it did before the move. But in the region of the move, we need a unique element of the quandle to color each arc such that our rule is satisfied. It needs to be unique, or else the Reidemeister moves will not induce well defined isomophisms on quandles. How this applies to each Reidemeister move is given in the following figures.

We see that the first Reidemeister move, in figure 2.4, induces the rule: for all $x \in X$, $x \triangleright x = x$. This is because the outside of the region must still be labeled x since it hasn't changed. Further, we see that this must apply to all

x, because any arc may be labeled x and each arc can have the Reidemeister 1 applied to it.

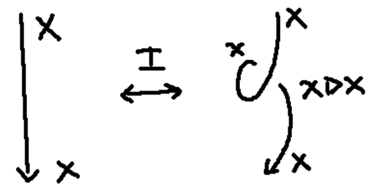


Figure 2.4 Reidemeister 1 Rule

We see that the second Reidemeister move, in figure 2.5 induces the rule: for all $y \in X$, the function $x \mapsto x \triangleright y$ is a bijection. This is because we required uniqueness. So the two ways of doing Reidemeister 2 must each give unique results, and since they give inverse applications of $\triangleright$, we see that $\triangleright$ must be bijective on the left.

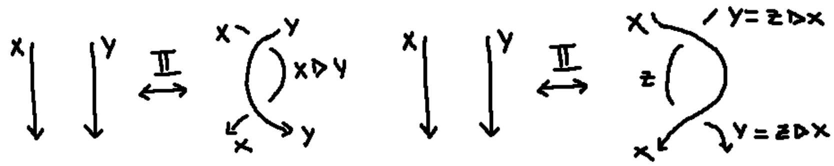


Figure 2.5 Reidemeister 2 Rule

Finally, the third Reidemeister move in figure 2.6 gives us: for all $x, y, z \in X$,

$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$$

since the ends must agree. This is again because the remainder of the knot we intend to be colored the same as before the move.

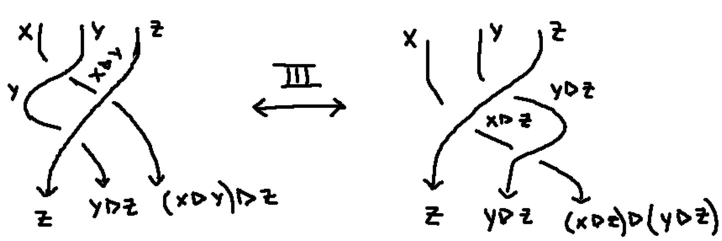


Figure 2.6 Reidemeister 3 Rule

We collect these rules into the following definition.

Definition 2.4.1 (Quandle, Nosaka (2017))**.** A *quandle* is a set X with a binary operation $\triangleright : X \times X \to X$ satisfying the following three conditions:

- For all $a \in X$, $a \triangleright a = a$,
- For all b in X the map $a \mapsto a \triangleright b$ is a bijection.
- For all $a, b, c \in X$, $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

Further, for quandles X, Y with operations $\triangleright_X$ and $\triangleright_Y$ respectively, a map $f : X \to Y$ is called a *quandle homomorphism* if for all $a, b \in X$,

$$f(a \triangleright_X b) = f(a) \triangleright_Y f(b)$$

Here are some familiar examples of quandles.

Example 2.4.2. (Trivial Quandles) A set X with $x \triangleright y = x$ for all $x, y \in X$.

This immediately gives us that there is a quandle of every cardinality. Note that in this thesis we call the trivial quandle with one element the *singleton quandle*.

Example 2.4.3. (Group Conjugation) A group G with $x \triangleright y = yxy^{-1}$.

Example 2.4.4. (Alexander Quandles) A ring R with unit u and $x \triangleright y = y + u(x - y)$

Setting $R = \mathbb{Z}/n\mathbb{Z}$ and $u = -1$, we have $x \triangleright y = 2y - x$ which gives a non-trivial quandle on each $n \geq 3$ elements.

Remark. Currently, there is no definitive classification of all finite quandles.

Proposition 2.4.5. *Quandles form a category with quandle homomorphisms as morphisms.*

To continue, we need a way to assign a specific quandle to each knot diagram. We get this through the following definitions.

Definition 2.4.6. We say a quandle X *colors* a diagram of a knot K if there is a map from the set of arcs of K (that is, the set of unbroken lines in K), to X such that at each crossing of K, the relation in figure 2.3 holds.

Now we may give the functor from the category of knots to the category of quandles.

Definition 2.4.7. For a knot K presented in the plane, label each arc of K with a unique arbitrary symbol, and at each crossing note the relation given by the quandle rule in figure 2.3. Then, the quandle associated to this knot is the free quandle generated by these symbols mod the given relations. That is, the quandle associated to K is the set of formal strings of arc labels separated by $\triangleright$ and $\triangleright^{-1}$ using parentheses to indicate association. with the relations given by the crossings of K according to the quandle rule in figure 2.3. For instance

$$(w \triangleright (x \triangleright^{-1} y)) \triangleright z$$

Further, to each isomorphism of knots we assign the unique isomorphism of quandles induced by the Reidemeister moves in the preceding discussion.

In this thesis we call this pairing the *fundamental quandle* of a knot, but in the literature you may also see *knot quandle*.

Finally, note that the fundamental quandle of K colors K by construction and is in fact the "most free" coloring of K.

Remark. We include the symbol $\triangleright^{-1}$ to indicate the inverse guaranteed by the bijectivity of $a \mapsto a \triangleright b$. In particular, if $c = a \triangleright b$ then $a = c \triangleright^{-1} b$. Note that homomorphisms between quandles that respect $\triangleright$ will also respect $\triangleright^{-1}$ so that respecting $\triangleright$ is all we need to worry about.

Example 2.4.8. We can best see this in figure 2.7.

We write the fundamental quandle using the same generator notation for groups. That is something of the form

$$\left\langle a_1, \ldots, a_n \mid a_i = a_j \triangleright a_k \ldots \right\rangle .$$

where the a_i before the bar are the list of generators and the equalities on the right tell us how we can simplify formal words and come from the relations at the crossings given by the quandle rule in figure 2.3.

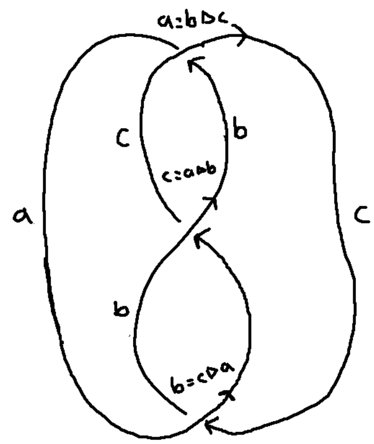


Figure 2.7 The Fundamental Quandle associated to the Overhand knot

The fundamental quandle of the knot in figure 2.7 is

$$\langle a, b, c \mid a = b \triangleright c, b = c \triangleright a, c = a \triangleright b \rangle .$$

Proposition 2.4.9 (Theorem 15.1, Joyce (1982)). *The assignment given by Definition* 2.4.7 *is a covariant functor from the category of knot diagrams to the category of quandles.*

Proof. This follows by the assigment of Reidemeister moves to quandle isomorphism in the discussion above. This is the same argument Joyce used originally. □

It follows that the fundamental quandle is an invariant of knots. It turns out that the situation is better which is given in the next theorem.

Theorem 2.4.10 (Corollary 16.3, Joyce (1982)). *Let K and K' be two oriented knots in the* 3*-sphere,* $\mathbb{S}^3$*, and let k and k' be their associated knot diagrams.*

Then K' is ambient isotopic to either K or the mirror image of K with direction reversed if and only if the fundamental quandles of K and K' are isomorphic.

Now, we build up some intuition about quandles to better decide when two quandles are isomorphic or not.

Example 2.4.11. First, consider the free quandle on one element $\langle a|\rangle$ which is the fundamental quandle of a the unknot (which has one arc and no crossings). By the first property of quandles (coming from Reidemeister 1), we have $a \triangleright a = a$. Thus, every formal string using $\triangleright$ collapses down to a. In particular,

$$\langle a|\rangle = \{a\}.$$

There is clearly only one quandle on 1 element, and it is the trivial quandle on 1 element. We call this quandle the singleton quandle.

WARNING: This notion of "freeness" for quandles *does not* match up with the usual definition of free for modules. In particular, the rules of quandles induced by the Reidemeister moves prevent us from having a notion of basis or anything like this.

Example 2.4.12. Now let's consider the free quandle on two elements $\langle a, b|\rangle$, which doesn't correspond to a single knot, but a pair of disjoint, unconnected unknots (groups of knots are called links). It turns out that the quandle axioms make no assertion about what $a \triangleright b$ should be. In particular, $a \triangleright b \neq a$ and $a \triangleright b \neq b$. This continues to give unique elements of the form

$$a \triangleright (b \triangleright (a \triangleright (\cdots))).$$

The reason that this does not collapse down is because the first and third rules don't apply (we have no $x \triangleright x$ or situation to right distribute), and the second Reidemeister move doesn't give us any ground to collapse the situation. Thus, the free quandle on 2 generators is infinite.

Example 2.4.12 hints at the following issue: *Most fundamental quandles are infinite.* This is a problem since it is in general a non-trivial task to distinguish infinite quandles from their presentation (much like how distinguishing group presentations is undecidable), and we chose this approach to make things easier!

The solution comes from an application of the following proposition:

Proposition 2.4.13. *Let K be a knot presented in the plane and let X be its fundamental quandle. Then for any quandle Y, Y colors K if and only if there is a quandle homomorphism from X to Y.*

Proof. ($\Longrightarrow$) We construct the quandle homomorphism in the following way. Since both X and Y color K, there are maps φ, ψ from the arcs of K to X and Y respectively. In the case of the fundamental quandle X, we know that each arc of K is mapped to a unique generator of X. That is φ is invertible on the set of generators for X. Let $f : X \to Y$ be defined on the generators of X by

$$f = \varphi^{-1} \circ \psi$$

and extend "linearly." That is, for any formal string of operations in X written in terms of the generators of X, we simply take f of each generator. For example,

$$f((a \triangleright b) \triangleright ((c \triangleright b) \triangleright a)) = (f(a) \triangleright f(b)) \triangleright ((f(c) \triangleright f(b)) \triangleright f(a)).$$

Unlike in the case of vector spaces or free modules, this is not trivially well defined. However, we have that both X and Y obey the same relations at the crossings.

With some work, it follows that f is well defined by induction.

($\Longleftarrow$) If there is a homomorphism $f : X \to Y$, then since we know K is colored by X, we have a map φ from the arcs of K to X and thus we have a map $\varphi \circ f$ from the arcs of K to Y. Now, at each crossing we have to satisfy a relation in Y of the form $f(a) \triangleright f(b) = f(c)$. But since X satisfies the relation at the crossing we have $a \triangleright b = c$ and so

$$f(c) = f(a \triangleright b) = f(a) \triangleright f(b).$$

$\square$

It follows from this proposition that in fact *any* quandle is an invariant of knots. In particular, say we pick two isomorphic knots K_1 and K_2 with fundamental quandles X_1 and X_2. Then for any quandle Y, we have a quandle homomorphism from X_1 to Y if and only if we have a quandle homomorphism from X_2 to Y (since X_1 and X_2 are isomorphic), and thus Y colors K_1 if and only if Y colors K_2 as well.

But in fact, the relationship is stronger. Since each coloring induces a unique homomorphism and each homomorphism induces a unique coloring, the set of colorings of Y onto K is also an invariant of knots.

That is we can fix any quandle Y we want, and the set of ways it colors K is fixed over all isomorphs of K.

Remark. Thus, we may freely look for finite quandles when trying to distinguish knots. However, it should be noted that it is an open problem whether every pair of knots is distinguished by a finite quandle.

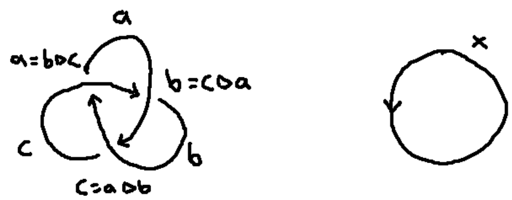


Figure 2.8 Trefoil (left) and unknot (right) with arc labels and relations.

Example 2.4.14. We can distinguish the unknot and the trefoil knot using the Alexander Quandle $Q = \mathbb{Z}/3\mathbb{Z}$ with unit -1.

We see that the fundamental quandles are

$$L = \langle a, b, c \mid a = b \triangleright c, b = c \triangleright a, c = a \triangleright b \rangle$$

for the trefoil and $R = \langle x \mid \rangle$ for the unknot. We look for homomorphisms from these into $Q = \mathbb{Z}/3\mathbb{Z}$.

First we consider the Trefoil. Let $f : L \longrightarrow Q = \mathbb{Z}/3\mathbb{Z}$ be a homomorphism. Every homomorphism is determined exactly by its action on the generators, so if we know $f(a), f(b), f(c) \in Q$, then we know f exactly. However, these are not free! We must have (using the ring structure of $\mathbb{Z}/3\mathbb{Z}$)

$$f(a) = f(b \triangleright_L c) = f(b) \triangleright_Q f(c) = 2f(c) - f(b)$$

$$f(b) = f(c \triangleright_L a) = f(c) \triangleright_Q f(a) = 2f(a) - f(c)$$

$$f(c) = f(a \triangleright_L b) = f(a) \triangleright_Q f(b) = 2f(b) - f(a)$$

We can solve this system using Linear Algebra (which is what makes Alexander Quandles so nice), and we get that $f(b)$ and $f(c)$ are free and that

$$f(a) = 2f(b) + 2f(c).$$

Since $f(b)$ and $f(c)$ can be anything in $\mathbb{Z}/3\mathbb{Z}$ we see that there are exactly nine unique homomorphisms $f : L \longrightarrow \mathbb{Z}/3\mathbb{Z}$.

On the other hand, the fundamental quandle of the unknot has no relations and one generator. Thus, we have three homomorphisms $f(x) = 0$, $f(x) = 1$ and $f(x) = 2$.

Since nine is not the same number as three, we see that these two knots are not isomorphic.

We also see that not every finite quandle distinguishes different knots. For example the number of colorings of the Alexander quandle $\mathbb{Z}/3\mathbb{Z}$ on the square and granny knots in figure 2.9 are both 27 (this is calculated using similar linear algebra as in example 2.4.14), even though they can be distinguished by a larger quandle (of size 24).

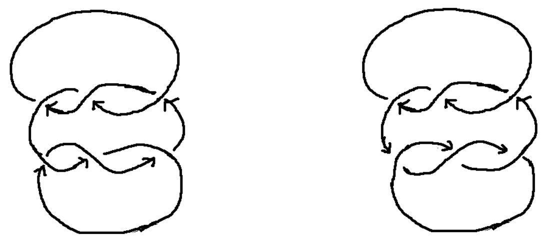


Figure 2.9 Square and Granny Knots

Thus, in the next section we strengthen the finite quandle coloring invariant by introducing the notion of a state sum.

2.5 Quandle Cocycle State-Sums

We can strengthen the quandle coloring invariant in the following way. Suppose, we already have a quandle that colors a presentation of a knot in the plane. We are motivated by the fact that since one quandle can color a knot in different ways, then maybe knowing *how* a quandle colors a knot will help us further distinguish different knots.

In particular, we may look at what happens at the crossings, and assign to each crossing a member of an abelian group A, such that the sum over the crossings of a knot is an invariant of the particular coloring. Similarly to the construction of the quandle, we give a rule and appeal to the Reidemeister moves.

WARNING: Here we use additive notation for the abelian group A so that the commutativity is clear, but Carter et al. (2008) uses multiplicative notation!

Let K be a knot, X a quandle which colors K, and A an abelian group. To start we decide on a rule, which is given in figure 2.10. That is, we choose functions $\phi, \psi : X \times X \to A$ with the intent of summing up the values ϕ

and ψ for each crossing of the knot diagram of K.

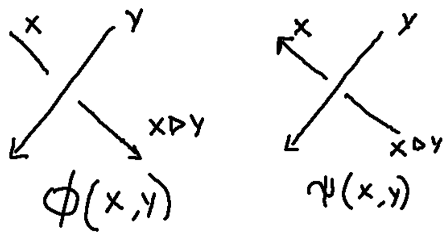


Figure 2.10 State Sum Rule

We want to make sure that this sum is the same for isomorphic knots with the coloring by X induced by the isomophism. Thus, we make sure that each Reidemeister move preserves our sum. Note that if there are no crossings, then the sum should be 0, the identity of A. The work of this is given in figures 2.11, 2.12, and 2.13.

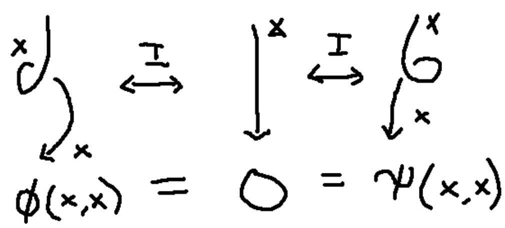


Figure 2.11 State Sum Rule induced by Reidemeister 1

Note that we left out the Reidemeister moves that do not give us any new information. Also note that the second Reidemeister move asserts that ϕ and ψ are inverses, so we really only need to consider one function. Thus our coloring rule can be reduced to that of figure 2.14.

It turns out that these condition are identical to the condition for a function ϕ to be a 2-cocycle in a certain cohomology theory on Quandles. Thus, we call such a ϕ a 2-cocycle. We leave out the details because cohomology theory is quite vast. Although, for the sake of completeness note that the boundary map of this homology theory is given on a basis for $A[X^n]/\{(\ldots, x_i, x_{i+1}, \ldots) : x_i = x_{i+1}\}$ by

$$\partial_n(x_1, \ldots, x_n) = \sum_{k=2}^{n} \left[(-1)^k (x_1, \ldots, \hat{x}_k, \ldots, x_n) - (x_1 \triangleright x_k, x_2 \triangleright x_k, \ldots, x_k \hat{\triangleright} x_k, x_{k+1}, \ldots, x_n)\right]$$

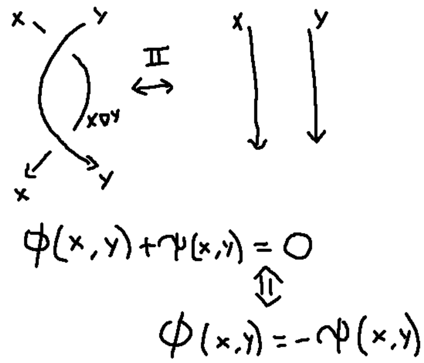


Figure 2.12 State Sum Rule induced by Reidemeister 2

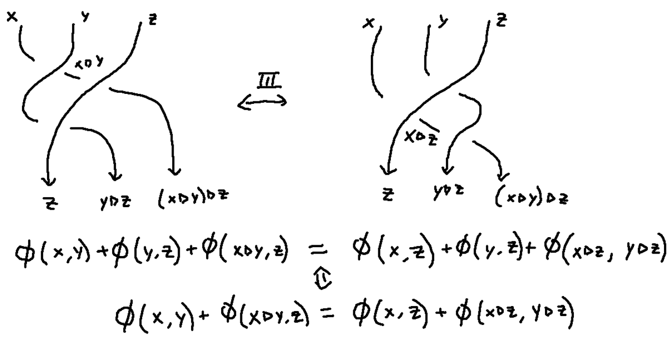


Figure 2.13 State Sum Rule induced by Reidemeister 3

Definition 2.5.1. Let X be a quandle and A an abelian group. Then a function $\phi : X \times X \to A$ is a *2 cocycle* if and only if

- for all $x \in X$, $\phi(x, x) = 0$ and
- for all $x, y, z \in X$,

$$\phi(x, y) + \phi(x \triangleright y, z) = \phi(x, z) + \phi(x \triangleright z, y \triangleright z).$$

One issue of using the state sum is that we are only guaranteed the same sum for the *same way* of coloring a knot by a quandle. In particular, if we have isomorphic knots K_1 and K_2 and a quandle that colors each 2 ways, we may get two distinct sums, one from each coloring, but this is still invariant because the multiset of sums are identical. In particular, we have

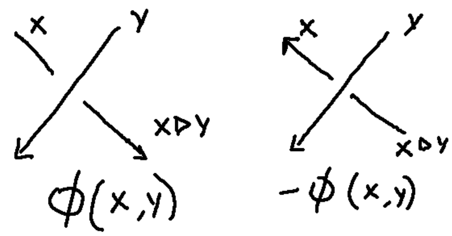


Figure 2.14 State Sum Rule

the following proposition.

Proposition 2.5.2. *Let K_1 and K_2 be isomorphic knots, X an arbitrary quandle, A an arbitrary abelian group and $\phi : X \times X \to A$ satisfying. Then the multiset of state sums for each coloring of K_1 by X is the same as the multiset of state sums for each coloring of K_2 by X.*

In particular, the multiset of state sums is an invariant of knots.

Finally, if we go back to cohomology theory, it turns out that if two co-cycles differ by a coboundary, then they give the same multiset of state sums.

Proposition 2.5.3 (Prop 4.5 Carter et al. (2008))**.** *Let K be a knot, X be a quandle, and A be an abelian group. Let ϕ, ϕ' be 2-cocycles. If Φ_ϕ and $\Phi_{\phi'}$ denote the multisets of state sums and $\phi = \phi' + \delta\psi = \phi' + \psi \circ \partial_2$ (where $\psi : X \to A$), then $\Phi_\phi = \Phi_{\phi'}$. In particular, for any $\psi : X \to A$, $\Phi_{\psi\circ\partial_2}$ has only the element 0, with multiplicity of the number of colorings of X on K.*

2.6 Niebrzydowski Tribrackets

Now, we switch gears and discuss another category which we can use to distinguish knots.

The intuition for this category comes from coloring the regions of the plane that the knot cuts out instead of coloring the arcs of the knot itself. We give this category the same treatment as quandles. Since there are four regions next to each crossing (as opposed to three arcs) we want our category to be composed of a set X along with a ternary operation $\langle\rangle : X^3 \to X$. Again we generate rules by appealing to the Reidemeister moves.

First we start of we an arbitrary rule relating one of the regions to the other three in figure 2.15. We choose what is now called the vertical Niebrzydowski tribracket Nelson et al. (2019). We only fix one crossing in our initial rule, because the rule at the other crossing falls out of Reidemeister 2. Further, it turns out that Reidemeister 1 does not put any further restriction on the rules of the tribracket, which is odd and potentially means that we aren't capturing a full set of information.

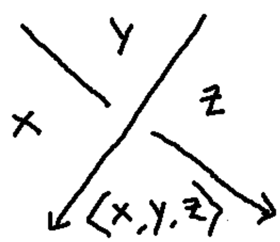


Figure 2.15 The vertical Niebrzydowski tribracket rule.

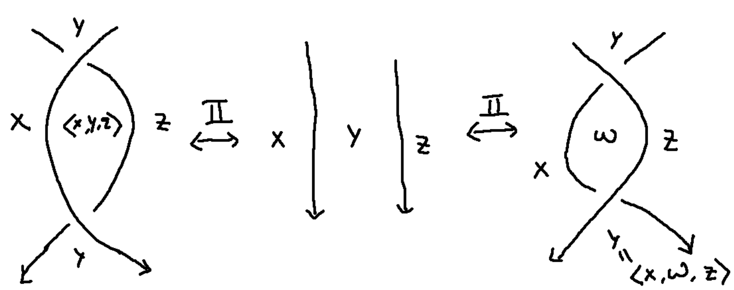


Figure 2.16 Tribracket Rule induced from parallel Reidemeister 2

Figure 2.16 shows two things. First on the left we see we have a positive crossing on top and have labeled it according to our rule. However, since doing Reidemeister 2 connects the top and bottom middle regions, we see they must be the same (in order to preserve isomophism of knots). Thus, the lower left crossing gives us our rule for the other type of crossing shown in figure 2.19. Second, we have invertibility in the middle position coming from uniqueness on the right side.

The antiparallel versions of Reidemeister 2 are given in figure 2.17.

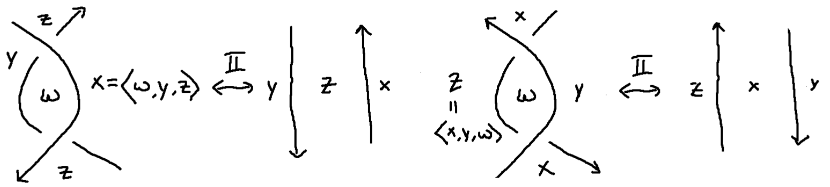


Figure 2.17 Tribracket Rule induced from antiparallel Reidemeister 2

Similarly give us invertibility at the first and third positions. Putting all this together gives that for fixed $x, y \in X$ the maps

$$a \mapsto \langle a, x, y \rangle, \qquad a \mapsto \langle x, a, y \rangle, \qquad a \mapsto \langle x, y, a \rangle$$

are all bijective.

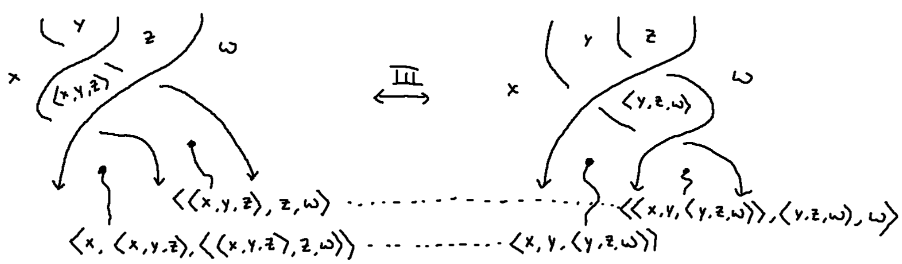


Figure 2.18 Tribracket Rule induced from Reidemeister 3

Finally, figure 2.18 gives some rather complicated relationships induced from Reidemeister 3. They are for all $x, y, z, w \in X$ we have

$$\langle x, y, \langle y, z, w \rangle \rangle = \langle x, \langle x, y, z \rangle, \langle \langle x, y, z \rangle, z, w \rangle \rangle$$

and

$$\langle \langle x, y, z \rangle, z, w \rangle = \langle \langle x, y, \langle y, z, w \rangle \rangle, \langle y, z, w \rangle, w \rangle$$

We collect all these rules in a definition.

Definition 2.6.1. A *vertical Niebrzydowski tribracket* (or *tribracket* for short in this work) is a set X along with a ternary operation $\langle\rangle : X^3 \to X$ satisfying

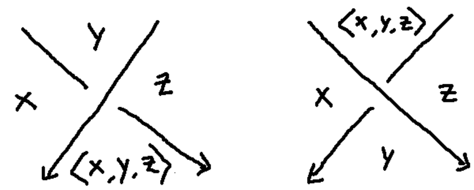


Figure 2.19 The Full Vertical Niebrzydowski tribracket rule.

- Invertibility at all positions. That is the maps

$$a \mapsto \langle a, x, y \rangle, \qquad a \mapsto \langle x, a, y \rangle, \qquad a \mapsto \langle x, y, a \rangle$$

  are all bijective.

- For all $x, y, z, w \in X$ we have

$$\langle x, y, \langle y, z, w \rangle \rangle = \langle x, \langle x, y, z \rangle, \langle \langle x, y, z \rangle, z, w \rangle \rangle$$

  and

$$\langle \langle x, y, z \rangle, z, w \rangle = \langle \langle x, y, \langle y, z, w \rangle \rangle, \langle y, z, w \rangle, w \rangle$$

  Further, a tribracket homomorphism between tribrackets X and Y is a function $f : X \to Y$ with

$$f(\langle x, y, z \rangle) = \langle f(x), f(y), f(z) \rangle$$

Proposition 2.6.2. *Tribrackets form a category with tribracket homomorphisms as morphisms.*

Now everything falls into place just as in the quandle case. We may speak of the "free tribracket" for a given knot diagram, but these are again infinite and rather hard to work with.

Remark. In fact, even less is known about the classification of Niebrzydowski tribrackets than of quandles.

Thus, we can resort to colorings by finite tribrackets.

Definition 2.6.3. We say a tribracket X *colors* a knot diagram if there is a map from the regions of the knot's complement to X.

Proposition 2.6.4. *Let K_1 and K_2 be knots, and X a tribracket. Then X colors K_1 if and only if X colors K_2.*

Proof. This follows in exactly the same way as in the quandle case: a tribracket will color a knot diagram if and only if there is a morphism from the "free tribracket" to the given tribracket. □

Remark. Thus, colorings by tribrackets are an invariant of knots! This gives us another way to distinguish knots, which may give different results than using quandles. There is some work to unify the ideas in Nelson et al. (2019).

2.7 Tribracket Cocycle State-Sums

Niebrzydowski (2017) gives a homology theory for tribrackets which is quite messy. We give it here for completeness but it is quite hard to manage.

Definition 2.7.1 (Niebrzydowski (2017))**.** Fix an abelian group A and a tribracket X, and set $C_n = A[X^{n+2}]$ the free module generated by elements of X over A. We define functions $\partial_n : C_n \to C_{n-1}$ by

$$\partial_n = \partial_n^L - \partial_n^R$$

These functions are given by

$$\partial_n^L(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n} (-1)^i d_i^{n,L}(x_0, \ldots, x_{n+1})$$

where

$$d_0^{n,L}(x_0, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1})$$

$$d_i^{n,L}(x_0, \ldots, x_{n+1}) = d_{i-1}^{n,L}(x_0, \ldots x_{i-1}, \langle x_{i-1}, x_i, x_{i+1} \rangle, x_{i+1}, \ldots, x_{n+1})$$

and
$$\partial_n^R(x_0,\dots,x_{n+1}) = \sum_{i=0}^{n}(-1)^i d_i^{n,L}(x_0,\dots,x_{n+1})$$
where
$$d_0^{n,R}(x_0,\dots,x_{n+1}) = (z_0,z_1,\dots,z_n)$$
where $z_0 = x_0, z_i = \langle z_{i-1}, x_i, x_{i+1}\rangle$ and
$$d_i^{n,R}(x_0,\dots,x_{n+1}) = d_{i-1}^{n,R}(x_0,\dots,x_{n+1})[\langle x_{i-1},x_i,x_{i+1}\rangle \mapsto x_i]$$

That ism the formula for $d_i^{n,R}$ is obtained from $d_{i-1}^{n,R}$ by replacing $\langle x_{i-1},x_i,x_{i+1}\rangle$ with x_i.

Example 2.7.2. In low dimensions Niebrzydowski (2017) gives

$$\begin{aligned}
\partial_1(a,b,c) &= (b,c) - (a,\langle a,b,c\rangle) \\
&\quad - (\langle a,b,c\rangle, c) + (a,b) \\
\partial_2(a,b,c,d) &= (b,c,d) - (a,\langle a,b,c\rangle, \langle\langle a,b,c\rangle, c, d\rangle) \\
&\quad - (\langle a,b,c\rangle, c, d) + (a,b,\langle b,c,d\rangle) \\
&\quad + (\langle a,b,\langle b,c,d\rangle\rangle, \langle b,c,d\rangle, d) - (a,b,c)
\end{aligned}$$

Niebrzydowski (2017) further claims that 2-cocycles in this homology form a state sum invariant for knots, still summing up over the crossings. That is, any function $\phi : X^3 \to A$ for some abelian group A which satisfies $\phi \circ \partial_2 = 0$ (which comes from Reidemeister 3) as well as a non-degeneracy condition (which comes from Reidemeister 1):

The function ϕ must be zero on any $(a,b,c) \in X^3$ with $b = \langle a,b,c\rangle$. Note that, by invertibility at each position, we have that, for fixed a,b there is a unique c that satisfies the above condition, and similarly for fixed b,c there is a unique a that satisfies the above condition. However, for fixed a,c there need not be a b that has $\langle a,b,c\rangle = b$! In particular we can only guarantee a unique d with $\langle a,d,c\rangle = b$.

One odd thing about this homology theory is that it is off by one. In particular, what are usually the n-tuples in a homology theory, here have 1 more entry!

This is because, we need one more piece of information to nail a crossing down than in the quandle case: there are four regions near a crossing but only 3 arcs. Thus, the homology theories are naturally one off from each other.

Chapter 3

State Sums of Lower Dimension

Since state sums can strengthen coloring invariants for both quandles and tribrackets, we want them to be easy to calculate. However, constructing a function that satisfies the cocycle condition can be challenging. In general, the conditions for cocycles of one lower dimension are easier to satisfy, however, we don't know if one cocycles will even be an invariant for knots. Thus, we look for the conditions a function of one fewer input (2 in the tribracket case and 1 in the Quandle case) needs to be an invariant.

3.1 1 Cochains of Tribrackets

Let X be a tribracket, A an abelian group, and consider a function $\phi : X^2 \to A$. The most natural way to apply a function of two region colors to a knot diagram is to apply it whenever we have adjacent regions. That is for each arc in the knot diagram we have the rule in figure 3.1.

We ask what conditions on ϕ make ϕ an invariant under Reidemeister moves.

Reidemeister 1 gives us the equality in figure 3.2, which is similar to the non-degeneracy rule from Niebrzydowski (2017). Stated formally, for all $a, b, c \in X$ with $b = \langle a, b, c \rangle$ we have

$$\phi(a, b) + \phi(b, c) = 0.$$

It turns out this immediately presents us with the following problem. Since the arcs are global in the diagram, but the Reidemeister moves are

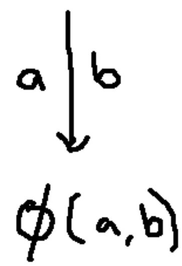


Figure 3.1 1 Cochain coloring rule for tribrackets

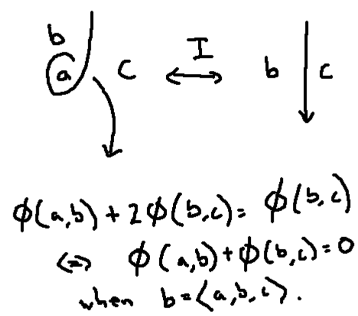


Figure 3.2 1 Cochain Rule induced by Reidemeister 1

local to the crossings we have disagreements. For example, consider figure 3.3

Since any tribracket with two elements colors the circle, we have that for all $b, c \in X$ $\phi(b, c) = 0$. The reason we are unconstrained is because every tribracket will also have a unique a with $b = \langle a, b, c \rangle$. That is, the only invariant attainable this way is the constant 0 invariant, which clearly is not very useful.

Remark. There are two ways to address this issue. The first is to move to knotoids which we can think of as knots with a cut in them. (They become useful in section 4.4, but not enough to warrant a deep explanation). Because they have a cut they are curves with endpoints instead of closed curves and lose their global arc structure. This would solve the problem by adding in a double count, but this is no longer a knot invariant as it is particular to knotoids.

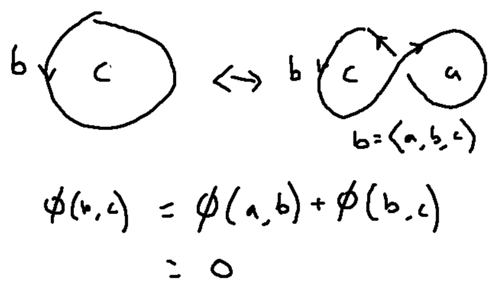


Figure 3.3 ϕ is 0

There may be a way to resolve this while remaining in the space of knots. We can make the summing local to the crossings (and thus more compatible with the Reidemeister moves) by letting the arcs contribute at each crossing rather than globally. (This is very similar to the work of Nelson et al. (2019).)

Going to knotoids (see Turaev (2010)) simply removes this problem. Briefly, knotoids are knot diagrams with a cut in them, so that instead of closed loops, knot diagrams are open arcs. In order to prevent ourselves from being able to simply untie knotoids by pushing the endpoints, we restrict ourselves to manipulations away from the endpoints which are just the normal Reidemeister moves.

We can consider the effect of the other two Reidemeister moves on knotoids:

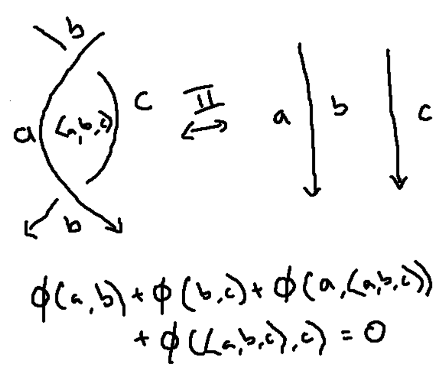


Figure 3.4 1 Cochain Rule induced by Reidemeister 2

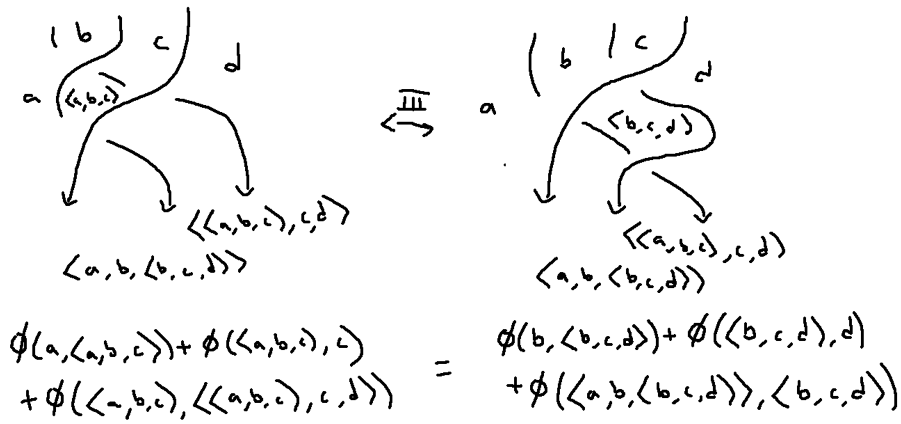


Figure 3.5 1 Cochain Rule induced by Reidemeister 3

We see that these rules will create an invariant of knotoids.

Proposition 3.1.1. *Let X be Niebrzydowski tribracket. Any $\phi : X^2 \to A$ that satisfies the following is an invariant of knotoids when applied in the way discussed above.*

- *For all $a, b, c \in X$ with $b = \langle a, b, c\rangle$,*

$$\phi(a, b) + \phi(b, c) = 0$$

- *For all $a, b, c \in X$*

$$\phi(a, b) + \phi(b, c) + \phi(a, \langle a, b, c\rangle) + \phi(\langle a, b, c\rangle, c) = 0$$

- *For all $a, b, c, d \in X$*

$$\begin{aligned}\phi(c, d) &+ \phi(\langle a, b, c\rangle, \langle\langle a, b, c\rangle, c, d\rangle) \\ &= \phi(a, b) + \phi(\langle a, b, \langle b, c, d\rangle\rangle, \langle b, c, d\rangle).\end{aligned}$$

We see that the second condition is a couple negatives away from being the 1 cocycle condition. But it's not yet clear to me what this means.

Further, it's not clear if there are any non-trivial functions that satisfy these axioms.

Chapter 4

Projections and Realizations

4.1 Definitions

At this point we will switch gears a bit to generalize the process that we used in 2.2 to project a knot in space into a *knot diagram* in the plane.

The intuition behind this generalization is that at the end of the day we ended up with a curve in the plane with equivalence between curves given by Reidemeister moves. However, the Reidemeister moves are *local* moves. That is, each Reidemeister move only operates on a small section of the diagram. Thus, the Reidemeister moves will work perfectly well on knot diagrams on other surfaces which locally look like a section of the plane.

This type of object is called a 2-dimensional (smooth) real Manifold which you can read more about in Hatcher (2002) (we may want smooth so that we don't have weird kinks in our surfaces). For our purposes, we just want to think of surfaces or sheets living inside of $\mathbb{R}^3$. Finally, we will need our surfaces to have an orientation (or smooth choice of non-zero normal vector) everywhere, we will see why when we define realizations below.

Definition 4.1.1 (Knot Diagram)**.** Let Σ be an orientable surface with choice of orientation. A *knot diagram* in Σ is a smooth directed closed curve in Σ with finitely many self-intersections, each of which "looks" like an X, and for each self intersection data about which arc is "over" the other.

More formally we may say that a knot diagram is a smooth function

$$f : \mathbb{S}^1 \to \Sigma$$

with the following properties

- There are only finitely many pairs $\theta_1 \neq \theta_2 \in \mathbb{S}^1$ for which $f(\theta_1) = f(\theta_2)$. (Finitely many self intersections).
- $\forall \theta_1, \theta_2, \theta_3 \in \mathbb{S}^1$ if $f(\theta_1) = f(\theta_2) = f(\theta_3) \in \Sigma$ then two of θ_1, θ_2, and θ_3 are the same. (No points of triple intersection).
- If $\theta_1 \neq \theta_2 \in \mathbb{S}^1$ with $f(\theta_1) = f(\theta_2)$ then the tangent vectors are linearly independent: $af'(\theta_1) + bf'(\theta_2) = 0 \implies a = b = 0$. (No intersections with tangency or only order 1 intersections).

along with information at each intersection $\theta_1 \neq \theta_2 \in \mathbb{S}^1$ with $f(\theta_1) = f(\theta_2)$ about which of θ_1 or θ_2 is "over" the other.

By the discussion above the Reidemeister moves in 2.2 work on knot diagrams. If k_1 and k_2 are knot diagrams in Σ with choice of orientation that are related by Reidemeister moves we say k_1 and k_2 are the "same" knot diagram (or related by Reidemeister moves).

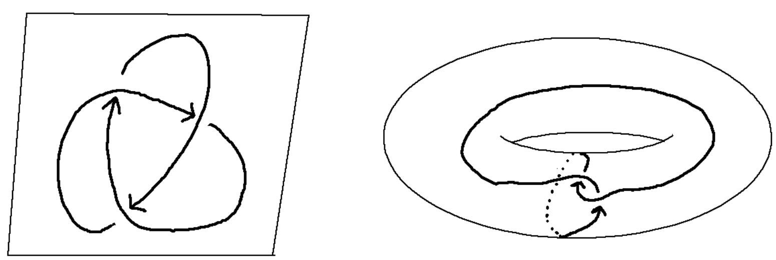


Figure 4.1 Examples of knot diagrams on a subsurface of the plane (left) and on the torus (right)

We next can describe in this more general setting the relationship between knots in space and knot diagrams. We will use the following terminology to be very clear about whether we are considering a knot or knot diagram.

Definition 4.1.2 (Realizations)**.** Let k be a knot diagram in a surface Σ with a choice of orientation. Choose an embedding (or smooth injective map) $\phi : \Sigma \to \mathbb{R}^3$. We construct a knot K from the knot diagram k in the following way. Following ϕ embeds the knot diagram k into $\mathbb{R}^3$ with finitely many self intersections. For each self intersection we perturb the curve by nudging the "over" strand in the direction of Σ's normal at the self-intersection. The resulting smooth embedding of $\mathbb{S}^1$ into $\mathbb{R}^3$ is a knot which we call the *realization of* k with respect to ϕ.

More formally, let $k : \mathbb{S}^1 \to \Sigma$ be a knot diagram function and let

$$K' = \phi \circ k : \mathbb{S}^1 \to \mathbb{R}^3.$$

Since K' has well behaved self-intersections, we may resolve each smoothly by nudging the "over" strand with a small smooth bump function pointing in the direction of the normal of Σ to generate a smooth injective function $K : \mathbb{S}^1 \to \mathbb{R}^3$. In other words K is a knot.

The intuition here is given by figure 4.2. Note that switching the orientation on Σ switches the direction that the nudging happens which mirrors the knot.

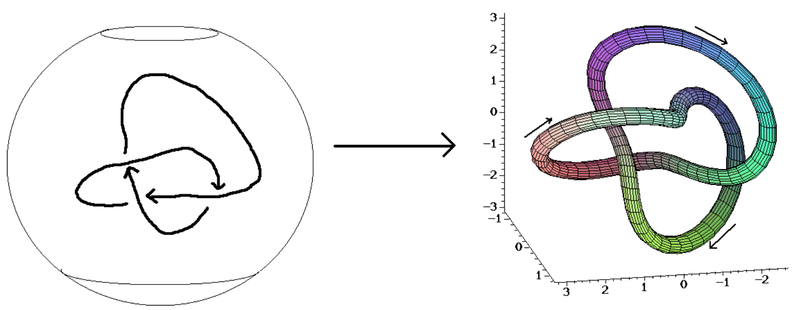


Figure 4.2 The realization of a knot diagram on the sphere with outward oriented normal

We have the following partial analog of Reidemeister's Theorem which gives tells us realizations are well defined up to Reidemeister moves of knot diagrams and ambient isotopy of knots. (We do not attempt to prove this here).

Theorem 4.1.3 (Reidemeister). *Let k_1 and k_2 be knot diagrams in an oriented surface Σ. Choose an embedding of Σ into $\mathbb{R}^3$. If k_1 and k_2 are related by*

Reidemeister moves then their respective realizations are ambient isotopic.

We also have the following proposition concerning different embeddings of the same surface.

Proposition 4.1.4. *Let k be a knot diagram in an oriented surface Σ and let $\phi, \psi : \Sigma \to \mathbb{R}^3$ be embeddings of Σ into space. If ϕ and ψ are ambient isotopic then the realizations of k with respect to ϕ and ψ are ambient isotopic as well.*

Proof. The ambient isotopy between ϕ and ψ is an ambient isotopy between the realizations of k. This follows from the fact that, before nudging, the ambient isotopy between ϕ and ψ moves the curve $\phi(k)$ exactly into the curve $\psi(k)$. Since ambient isotopy are orientation preserving, it follows that the nudges will be in the same direction and thus accounting for how the nudging happens we get an ambient isotopy. □

Most surfaces have embeddings which are not ambient isotopic. In this case we may have non-isotopic realizations of the same knot diagram (by using these non-isotopic embeddings). Because of this ambiguity of which embedding we use, we will refer to *realizations* (plural) of a knot diagram k in an oriented surface Σ. That is even though for each embedding of Σ we have a unique realization, over the set of all embeddings we may have multiple distinct realizations.

We also give ourselves the terminology to describe the inverse process.

Definition 4.1.5 (Projections)**.** Let k be a knot diagram in an oriented surface Σ and let K be a realization of k for some embedding of Σ. Then we call k a *projection* of the knot K.

We see that every knot K has at least one projection onto Σ using the following process: First choose an embedding of Σ in space and use ambient isotopy to shrink K to an equivalent smaller knot that is small enough so that Σ looks "flat" on the scale of K. Next project K's shadow onto Σ and record "over" information.

This process is identical to the process used to in 2.2 to make knot diagrams on the plane.

When Σ is the plane (or the sphere) two very special things happen. First, up to ambient isotopy there is only one embedding of Σ since every embedding of the plane (or sphere) is related by a rotation and translation. In this setting realizations truly are unique since there is no ambiguity about which embedding we should be using!

Second, the full power of the Reidemeister moves kicks in 2.2.5, and gives us that projections are unique as well!

However, when Σ is not the plane or sphere, in general we have different embeddings and projections. We can see examples of this when Σ is the torus, $\mathbb{T}^2$. In figure 4.3 we have in the upper left a knot diagram on the abstract torus which is defined as a square whose top edge is glued to its bottom edge and left edge is glued to its right edge. The torus has many distinct embeddings into space but the two easiest depend on which order you choose to fold the square to glue the two sides together. Gluing the left and first right follows the right path in figure 4.3 whereas gluing the top and bottom first follows the bottom path. The bottom path realizes the trefoil knot whereas the right path realizes the unknot.

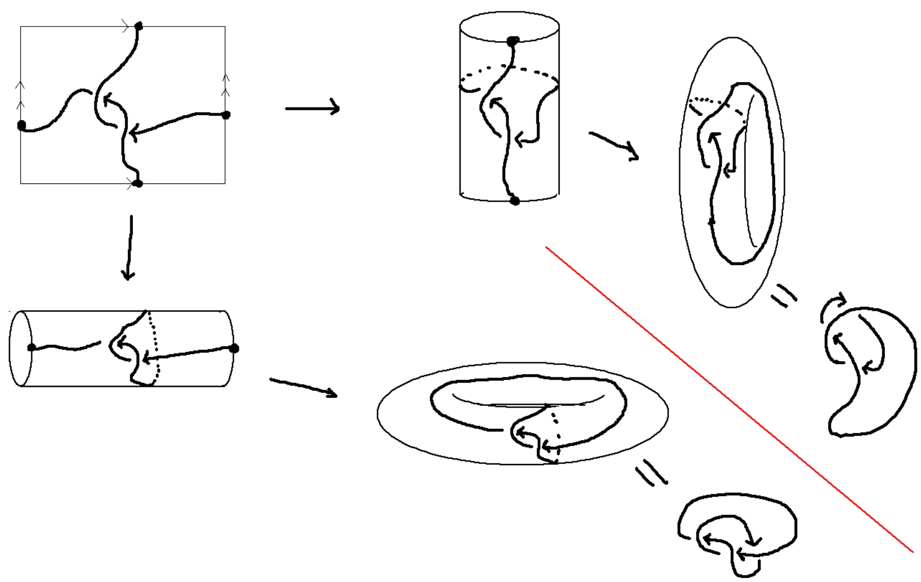


Figure 4.3 The same knot diagram on the abstract torus (upper left) is realized as 2 non-isotopic knots for different torus embeddings

Conversely we do not in general have a unique projection for a given knot. Figure 4.4 shows the unknot in space projected onto the Torus 3 unique ways. We can prove that these are not the "same" (that is not related by

Reidemeister moves) because the Reidemeister moves are local moves, but these unknots differ *globally*. In particular, none of the Reidemeister moves (nor ambient isotopy) will be able to move an arc across the hole of the torus. In particular, as closed curves they are not homotopic (see Hatcher (2002)) and neither the Reidemeister moves nor ambient isotopy can change homotopy classes.

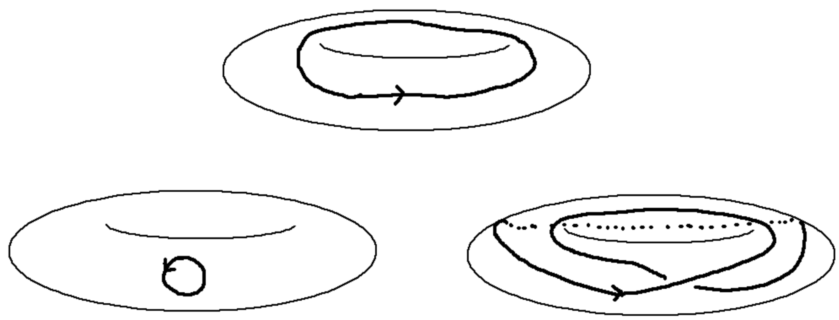


Figure 4.4 The unknot projected onto the Torus 3 distinct ways

In this more general setting things are much less well behaved because neither realizations nor projections are unique. However, if we can understand the ways in which these realizations and projections fail to be unique it may give us a deeper understanding of knot diagrams and the surfaces they live on.

4.2 Quandles On Surfaces

In section 2.4 we built up quandles as the category which respects the Reidemeister moves and we saw how quandles naturally color knot diagrams in the plane. Since Reidemeister moves are defined locally, it follows exactly the same way as before that the fundamental quandle 2.4.7 of a knot is a covariant functor from the category of knots in space (with ambient isotopy as isomophism) to the category of knot diagrams in a fixed oriented surface Σ. In particular, we know that the fundamental quandle (and also its morphisms into other quandles) will be an invariant of knot diagrams in any fixed oriented surface Σ.

However, we lose theorem 2.4.10 which told us that two knot diagrams were related by Reidemeister moves (up to a mirror and a reversal) if and only if their fundamental quandles were isomorphic. In fact we have a counterexample for a knot diagram on the torus.

Proposition 4.2.1. *For a general oriented surface Σ, the fundamental quandle is not a complete invariant up to mirror reversal. In particular, let Σ be the torus. Then there exist two knot diagrams k_1 and k_2 which are not related by Reidemeister moves and mirror reversals whose fundamental quandles are isomorphic.*

Proof. One proof strategy is to choose two distinctly embedded unknots from figure 4.4. We already argued that they were unrelated by Reidemeister moves in the discussion following the figure, but this required some abstract argument with local and global structures. So we give a more concrete argument which will also help remind us how to work with quandles. Consider figure 4.5 which depicts two knot diagrams in the abstract torus.

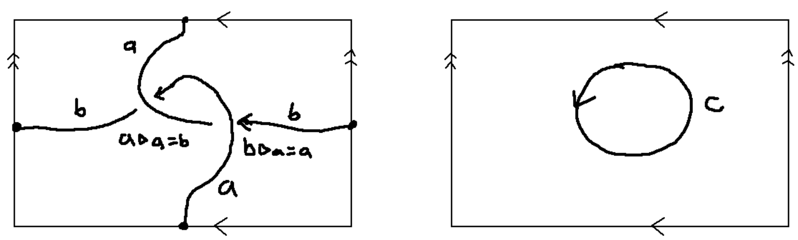


Figure 4.5 Generators and relations for the fundamental quandle of the knot diagram in 4.3 (left) and for an unknot diagram (right).

We see that the fundamental quandle of the left knot diagram is

$$L = \langle a, b \mid a \triangleright a = b, b \triangleright a = a \rangle .$$

By the first quandle rule (from Reidemeister 1) we have that

$$b = a \triangleright a = a$$

and we may rewrite

$$L = \langle a \mid a \triangleright a = a \rangle = \langle a \mid \rangle = \{a\}.$$

This is the fundamental quandle we saw in example 2.4.11.

We can also see that the fundamental quandle of the right knot diagram is

$$R = \langle c \mid \rangle = \{c\}.$$

These quandles are isomorphic using the morphism $f : L \to R$ with $f(a) = c$. Thus, we have two knot diagrams on the Torus with the same fundamental quandle. We wish to show that these knot diagrams are not related by Reidemeister moves and mirror reversals. First, note that the right diagram is related by Reidemeister moves to its mirror (since it has no crossings) and to its reversal by applying Reidemeister 1 twice as in figure 4.6.

Figure 4.6 The unknot diagram is related to its reversal by Reidemeister 1 moves.

Thus, if the left knot diagram is not related by Reidemeister moves to the right knot diagram it is also not related to its mirror reversal. Finally, suppose for the sake of contradiction that our knot diagrams are related by Reidemeister moves. Then, by Theorem 4.1.3 we have that for any fixed embedding of the torus into space, the realizations of our knots will be ambient isotopic. However, consider the embedding of the torus created by folding the top to bottom first (as in the bottom path of figure 4.3). Then the left knot diagram is realized as the Trefoil knot but the right knot is realized as the unknot. We saw in 2.4.14 that these knots are not ambient isotopic. This gives us our contradiction. $\square$

The previous proposition leads us to ask the following question:

Question 4.2.2. *Fix an abstract surface Σ with orientation. Can we describe the extent to which the fundamental quandle fails to distinguish different knot* *diagrams on Σ?*

Perhaps a simpler question:

Question 4.2.3. *Fix an abstract surface Σ with orientation. Which knot diagrams have fundamental quandle isomorphic to the fundamental quandle of the unknot diagram*: $\langle x \mid \rangle$.

These questions will be the focus of the remainder of the chapter.

4.3 Canonical Projections

If all of a knot diagram in surface Σ is inside of an open disk $U \subseteq \Sigma$ (for instance the unknot diagram in figure 4.5) then any embedding of Σ will embed U only one way up to ambient isotopy (since U is a disk). Since there is only one way to embed U, and the knot diagram we are considering lives in U, there is exactly one realization.

Further, this defines a unique projection for simply connected surfaces, since all disks are ambient isotopic to each other in such surfaces.

This is the first type of projection we will see. Let's give it a name.

Definition 4.3.1 (Trivial Projection)**.** Let Σ be an oriented simply connected surface and let K be a knot in space. Any projection of K onto Σ which lives strictly in a disk of Σ will be related by Reidemeister moves. Thus, it makes sense to name this projection the *trivial projection* of K onto Σ.

Remark. We won't prove it rigorously here, but it is worth noting that the fundamental quandle of the Trivial Projection of a knot K onto any surface Σ is isomorphic to the fundamental quandle of the unique projection of K onto the plane (or sphere). The intuition behind this is that the knot diagram in Σ has the same arc and crossing information as the knot diagram in the plane. Since the fundamental quandle is determined exactly by the arcs and crossings of a diagram, both diagrams will result in isomorphic fundamental quandles.

This gives us another piece of intuition as to why the case of the plane (and sphere) gives an exact correspondence between knots and knot diagrams: every smooth curve in the plane or sphere lives in an open disk.

To get some more intuition on what kinds of other projections a surface may have, we construct a specific type of projection onto the torus.

Definition 4.3.2 (Canonical Projections)**.** Let K be a knot in space. First project K onto the sphere (trivially) and fix a particular knot diagram k which is equivalent to this projection which has at least n crossings. A *canonical projection* of K onto the n-holed torus is given by "removing" the crossings of k by extruding toruses out of the sphere as in figure 4.7. We know that the toruses can be made small enough so that none of the extrusions interfere with each other. Further the resulting surface is indeed an embedding of the n-holed torus.

To get a smooth surface we may smooth out the intersections.

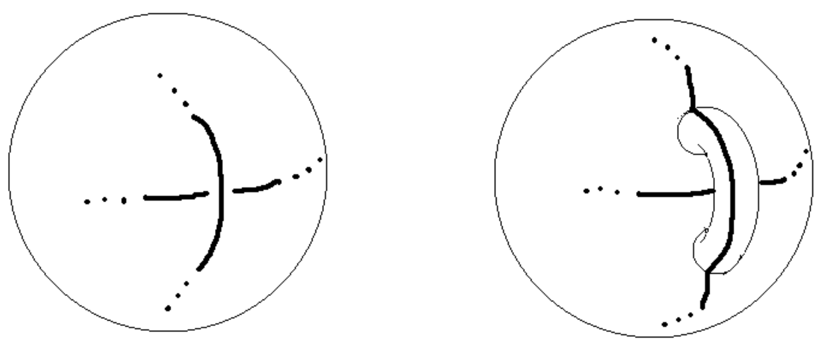


Figure 4.7 Removing a crossing by extruding a torus.

Remark. Note that this process is not unique! In particular, we may get distinct knot diagrams by choosing different crossings to remove. Even worse, if we add crossings to the knot diagram in the sphere using Reidemeister 1 and Reidemeister 2 moves, then removing these may result in canonical projections that we could not get otherwise. In particular, the set of canonical projections of a knot *is not* an invariant.

Example 4.3.3. When we remove exactly one crossing to get a canonical projection onto the torus, there is a convenient way of drawing the resulting knot diagram on the abstract torus. First, draw the diagram of the knot we want to project and isolate one of the crossings and surround the rest of the knot with a disk. Since crossings involve four semi-arcs, exactly 4 arcs will leave the disk. Gluing together opposing arcs on the abstract torus gives the canonical projection. This process is shown in figure 4.8.

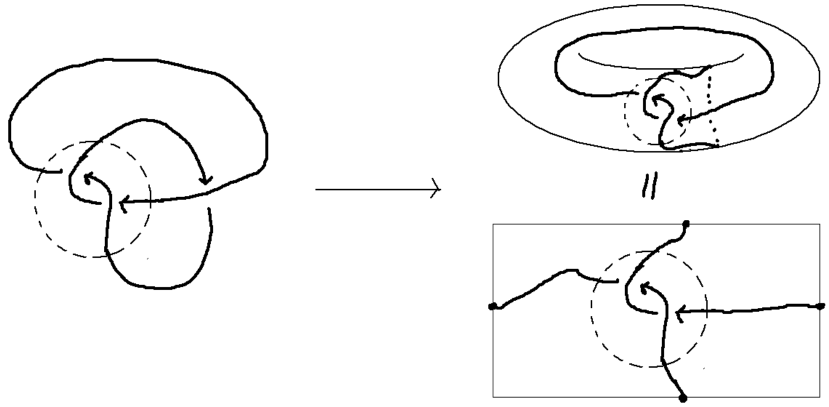


Figure 4.8 A canonical projection of the Trefoil

Example 4.3.4. Consider the above example but with the isolated crossing mirrored as in figure 4.9.

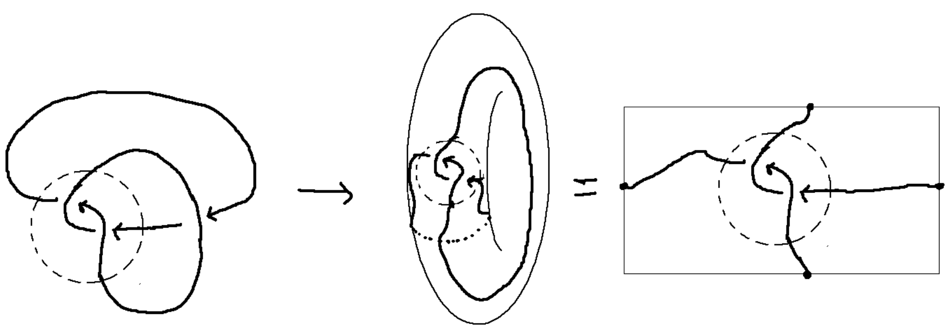


Figure 4.9 A canonical projection of the unknot

Notice that this knot is related by Reidemeister moves to the unknot in the plane. But its canonical projection is *identical* in the abstract torus to the canonical projection in 4.3.3!

In fact this was the example (figure 4.3) we used to demonstrate that realizations are not unique (the different embeddings of the torus give different realizations!)

This points to a certain indistinguishability that we will point to in section 4.4.

Before that though we consider the fundamental quandles of canonical projections.

Proposition 4.3.5. *Let k be a knot diagram in the sphere of some knot. Fix some crossings of k to take the canonical projection of. Let $X = \{a_i\}$ be the set of arcs of k, $R_1 = \{a_i = a_j \triangleright a_k\}$ be the set of relations generated by the crossings of k we will not remove, and let $R_2 = \{a_i = a_j \triangleright a_k\}$ be the set of relations generated by the crossings of k which we will remove. With this partition of the relations, the fundamental quandle of k is*

$$Q = \langle X \mid R_1, R_2 \rangle .$$

Then the fundamental quandle of the canonical projection of k with respect to the crossings R_2 is

$$Q' = \left\langle X \mid R_1, \{a_i = a_j \ : \ (a_i = a_j \triangleright a_k) \in R_2\} \right\rangle$$

.

Proof. We prove this for a single crossing removal since removing many crossings amounts to removing one crossing many times. Consider the region near the single crossing we will remove in the projection and label the arcs near the crossing as in figure 4.10.

Separate the arc labels X into two sets $X = Y \cup \{a, b, c\}$. Since we only remove one crossing, we have that $R_2 = \{c = a \triangleright b\}$ and

$$Q = \langle X \mid R_1, R_2 \rangle = \langle Y, a, b, c \mid R_1, c = a \triangleright b \rangle .$$

Now consider the fundamental quandle of the canonical projection. By construction, the arc and crossing information is identical except at this one crossing. In particular, there is one fewer arc since $c = a$ and we no longer have the relation $c = a \triangleright b$. Since $c = a$, this amounts to replacing every c in

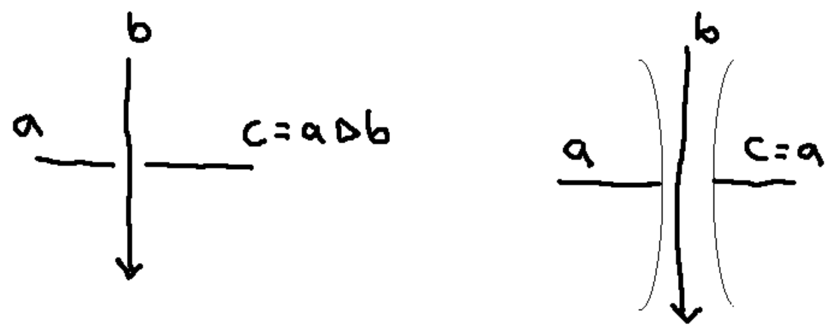


Figure 4.10 Quandle Relations near a removed crossing

R_1 by a. However, it is equivalent to keep c as an arc and introduce a new relation $a = c$. This is exactly what we need:

$$Q' = \langle Y, a, b, c \mid R_1, c = a \rangle .$$

□

Remark. This relationship is not very strong in general as we will see in a moment.

Example 4.3.6. Consider the canonical projection of the Trefoil onto the torus in example 4.3.3. We saw in example 2.4.8 that the fundamental quandle of the Trefoil is

$$\langle a, b, c \mid a = c \triangleright a, b = a \triangleright b, c = b \triangleright a \rangle .$$

Notice that the canonical projection in exmaple 4.3.3 is exactly the knot diagram in the proof of 4.2.1 whose fundamental quandle turned out to be the singleton quandle

$$\langle x \mid \rangle .$$

Finally, we saw in example 2.4.14 that the fundamental quandle of the Trefoil is distinct from the singleton quandle.

In particular, the fundamental quandle of a canonical projection need not be the same as the original fundamental quandle.

4.4 Canonical Projections of Unknotted Diagrams

The canonical projection of single crossings onto the torus gives a partial answer on the torus to our motivating question for this chapter: which knot diagrams on the single holed torus have the singleton fundamental quandle?

Theorem 4.4.1. *Any single crossing canonical projection of the unknot has singleton fundamental quandle.*

Before we get to the proof let's discuss this. First, recall that a canonical projection of a knot K first projects K (uniquely) onto the sphere giving us a diagram which is related by Reidemeister moves to the unknot on the sphere. However, our diagram may have crossings. Choosing one of them to remove gives us a single crossing canonical projection.

Even though we started with the unknot, we may not have an unknot as our canonical projection as in example 4.3.4. What the theorem is stating is that even if we don't necessarily end up with an unknot, we do necessarily end up with a diagram which has singleton fundamental quandle (in other words there's a relationship to the unknot - who's fundamental quandle in the plane is the singleton quandle).

Proof. We (surprisingly) can use the theory of knotoids (see Turaev (2010)) to prove this. Basically, a knotoid is like a knot diagram in a surface, except that instead of being a loop, the curve is a projection of a line segment. In order to prevent ourselves from being able to simply untie knotoids by pushing the endpoints, we restrict ourselves to just the usual Reidemeister moves for equivalence. We should think of knotoids as knots but with a cut in them. Ok, now we can get to the proof!

Let k be a knot diagram on the sphere which is related by Reidemeister moves to the unknot. Fix the crossing C of k to be the crossing we will remove in the canonical projection.

First, we construct the fundamental quandle for k by labelling each arc of k. Since the symbols don't matter let us specially label the crossing C as in figure 4.11.

Now if we let the set of all labels away from a, b, c be the set X and the relations generated by the crossings of the diagram k that are not C be the set R then we may write the fundamental quandle of k as in proposition 4.3.5:

$$Q_k = \langle a, b, c, X \mid R,\ c = a \triangleright b \rangle$$

Note that since k is related by Reidemeister moves in the plane to the unknot and since the fundamental quandle of a knot is invariant under Reidemeister moves we have

$$Q_k = \langle x | \ \rangle = \{x\}$$

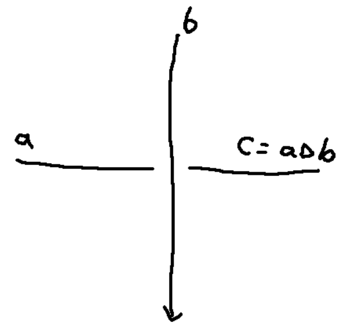


Figure 4.11 Assigning a labeling to the crossing C.

the singleton quandle. In particular, any two formal strings of generators and ▷'s in Q_k are "equal" as quandle elements.

Now, let p be the canonical projection (on the torus) we get from removing the crossing C from the diagram k. By proposition 4.3.5 we can write the fundamental quandle of p as

$$Q_p = \langle a, b, c, X \mid R, a = c \rangle .$$

Our goal is to show that this quandle is also the singleton quandle. We do this by considering the following knotoid (defined in Turaev (2010)) in the plane. Take the diagram k and construct the knotoid t by leaving k the same except near the crossing C make a cut on the c arc as in figure 4.12.

Examining figure 4.12, we see that there is a new arc which we have labeled d. Before we utilize the fundamental quandle of a knotoid let us discuss why this concept is an invariant for knotoids. Put simply: the Reidemeister moves act locally, and never interact with the cut, so the fundamental quandle will still respect the equivalence classes induced by the Reidemeister moves! In particular, the argument we made back in section 2.4 still holds: the map from knotoids to their fundamental quandles is still a functor!

Ok! Let the fundamental quandle of this knotoid be

$$Q_t = \langle a, b, c, d, X \mid R, d = a \triangleright b \rangle .$$

Note that d does not appear in any relations in R since it only exists near the crossing C. In particular, the only relation involving d is $d = a \triangleright b$. We

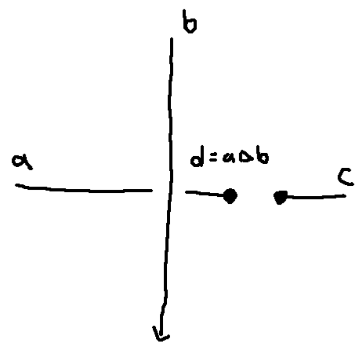


Figure 4.12 The crossing C in the knotoid t cut near the crossing.

may remove the symbol d by replacing all of its appearences with $a \triangleright b$ and have the same quandle. Since d appears no where in R, making this substitution does not change any relation in R! In particular we have

$$Q_t = \langle a, b, c, X \mid R \rangle .$$

Now, Turaev (2010) proves that for knot diagram A, B related by Reidemeister moves and knotoids a, b created from A and B by small cuts (as we have done above), that a and b are related by Reidemeister moves if we consider them to be embedded on the sphere $\mathbb{S}^2$. Since our knot diagram k is given to be related to the unknot on the sphere, we have that the knotoid t is related by Reidemeister moves to the unknotted knotoid in figure 4.13.

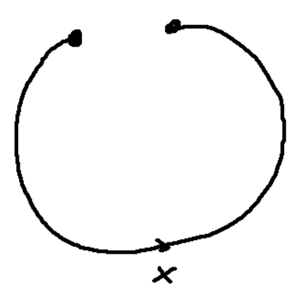


Figure 4.13 The unknotted knotoid.

We see immediately that the fundamental quandle of this knotoid is

$$\langle x \mid \rangle .$$

But t and this "unknotoid" are related by Reidemeister moves, which the fundamental quandle is an invariant of (knotoids related by Reidemeister moves must have isomorphic fundamental quandles). Thus we have that in fact

$$Q_t = \langle a, b, c, X \mid R \rangle \cong \langle x \mid \rangle .$$

Now, it follows that Q_t has one element. In particular, the generators a and c are equal by the relations of R. So we may freely add the relation $a = c$ (since it is already implied by the others) so that

$$Q_t = \langle a, b, c, X \mid R, a = c \rangle = \langle x \mid \rangle .$$

But wait! This is exactly the presentation of the fundamental quandle of the canonical projection of k, p! It follows that

$$Q_p = \langle a, b, c, X \mid R, a = c \rangle = Q_t = \{x\}.$$

And so the fundamental quandle of p is the singleton as desired!

□

Remark (Indistinguishability). This theorem gives us a reason why the the left knot diagram k in figure 4.5 has singleton fundamental quandle. In example 4.3.4 we saw that this diagram was the canonical projection of an unknotted diagram! What is really important here is that the diagram k in the abstract torus is both the canonical projection of an unknotted diagram *and* the trefoil! So even though the trefoil is not the unknot, since the diagram k is also a canonical projection of an unknot it will have singleton fundamental quandle!

We formalize this notion in the following corollary.

Corollary 4.4.2. *Let k be a knot diagram on the sphere with a crossing C such that mirroring just C results in an unknotted diagram* (*that is related by Reidemeister moves to the unknot*)*.*

Then the canonical projection of k removing the crossing C will have singleton fundamental quandle.

Proof. Let k be as above and let m be the knot diagram identical to k except with the crossing C mirrored. Since m is unknotted by supposition, the canonical projection of m removing C will have singleton fundamental quandle by theorem 4.4.1.

Now, consider the canonical projection of k removing the crossing C. The knot diagram will have identical crossing and arc information as k except with C removed and opposite arcs of C set equal. But this description also perfectly describes m! Intuitively, by removing the crossing C the knot diagram "forgets" which direction C was in! Thus the canonical projections of k and m are identical. Since the fundamental quandle is an invariant of knot diagrams, the fundamental quandle of k is also trival. □

Remark. Knot diagrams with the property that switching a single crossing unknots them are called "knots with unknotting number 1." It turns out that the unknotting number is an invariant and that all unknotting number 1 knots are prime (that is, cannot be realized as two nontrivial knots glued together) Scharlemann (1985).

4.5 Bridge Projections

We can extend the notion of the canonical projection removing one crossing to removing an entire bridge.

Definition 4.5.1 (Bridge)**.** An *over bridge* in a knot diagram is a length of curve of the diagram such that the curve is "over" in every crossing it is involved in.

Conversely an *under bridge* in a knot diagram is a length of curve of the diagram such that the curve is "under" in every crossing it is involved in.

We can extend canonical projections by removing entire bridges instead of simply crossings.

Definition 4.5.2 (Bridge Projections)**.** Let K be a knot in space. First project K onto the sphere (trivially) and fix a particular knot diagram k which is equivalent to this projection which has at least n bridges. A *bridge projection* of K onto the n-holed torus is given by "removing" the bridges of k by extruding toruses out of the sphere. This process looks

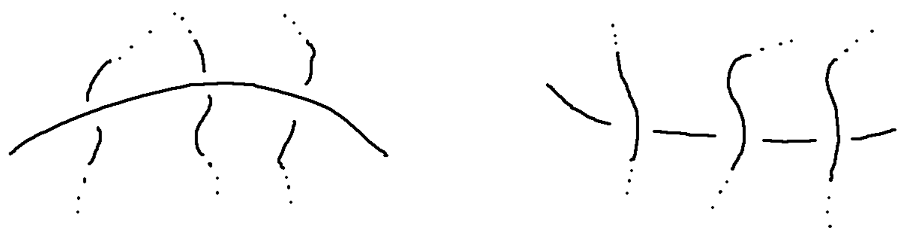


Figure 4.14 An over bridge (left) and under bridge (right).

like figure 4.15.

We get an analogous theorem for single bridge projections as for single canonical projections also using knotoids.

Theorem 4.5.3. *Any single bridge projection of the unknot has singlton fundamental quandle.*

Proof. The proof follows in much the same way as in the canonical projection. First it will be easier to work with under bridges. So if we have an over bridge, look at the knot from the other side before projecting it onto the sphere. Since the fundamental quandle is an invariant of knot diagrams on the sphere (which correspond exactly to knots in space) projecting the knot onto the sphere from the other side does not change anything except that we get to work with an under bridge.

Now we follow a similar argument. Let k be a knot diagram related to the unknot in the sphere with a chosen under bridge. First, we make a labelling of the arcs of the under bridge in figure 4.16.

Notice that since we do not know the orientation of the b_i arcs we cannot definitively write down the exact relations at the crossings. We know that the first n are each either

$$a_i = a_{i-1} \triangleright b_{i-1} \qquad \text{or} \qquad a_i = a_{i+1} \triangleright b_i$$

and the last is similarly either

$$c = a_n \triangleright b_n \qquad \text{or} \qquad a_n = c \triangleright b_n$$

We use the fact that the quandle operation must be a bijection (that is for each $\cdot \triangleright b$ we have an inverse $\cdot \triangleright^{-1} b$) to write the dependencies going one

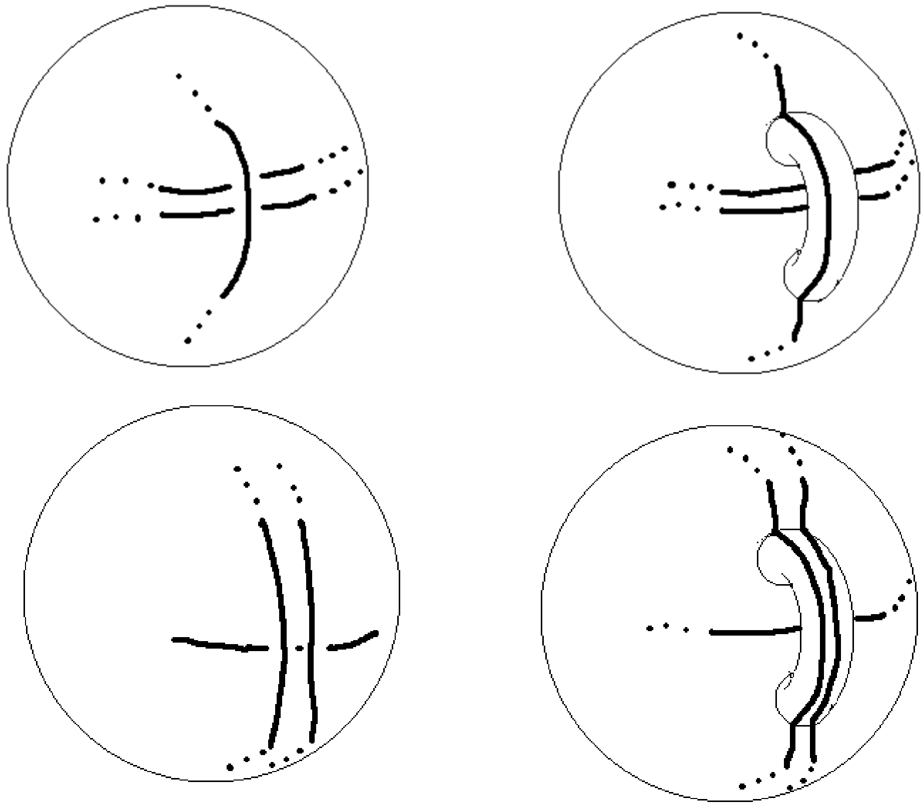


Figure 4.15 Removing an over bridge (top) and under bridge (bottom) by extruding a torus.

way:

$$a_{i+1} = a_i \rhd b_i \qquad \text{or} \qquad a_{i+1} = a_i \rhd^{-1} b_i$$

$$c = a_n \rhd b_n \qquad \text{or} \qquad c = a_n \rhd^{-1} b_n$$

Regardless, let us name the equality of a_{i+1} (for $1 \leq i \leq n-1$) crossing r_i and note that r_i is of the form $a_i = \ldots$, whatever it is. Let C be the relation of the last crossing of the form $c = \ldots$ Just as last time we group all the other arcs X and the other relations R so that the fundamental quandle is

$$Q_k = \langle X, \{a_i\}, \{b_i\}, c \mid R, \{r_i\}, C\rangle \cong \{x\}.$$

Now, we make a cut to get a knotoid which separates c from the relations r_i in figure 4.17.

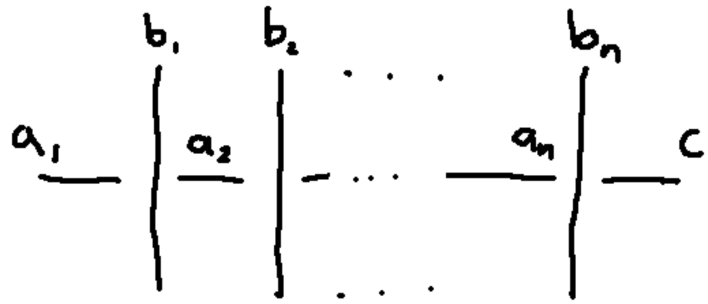


Figure 4.16 Labelling the arcs of an under bridge.

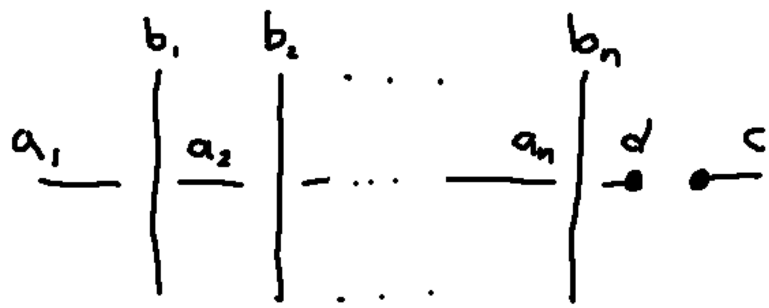


Figure 4.17 The knotoid generated by cutting near an underbridge

Let D be the new relation between d and a_n, either $d = a_n \triangleright b_n$ or $d = a_n \triangleright^{-1} b_n$ Just as before, the fundamental quandle for this knotoid is

$$Q_t = \langle X, \{a_i\}, \{b_i\}, c, d \mid R, \{r_i\}, D\rangle \cong \{x\}.$$

Because of how we set up our relations (using the fact that $\cdot \triangleright b_i$ has an inverse) we can remove d by using its relation $d = a_n \triangleright^{\pm 1} b_n$. But we know that d appears nowhere so this change does not affect the fundamental quandle. Similarly each a_{i+1} can be removed this way until we are left with only a_1. Then our fundamental quandle becomes

$$Q_t = \langle X, a_1, \{b_i\}, c \mid R\rangle \cong \{x\}.$$

The intuition here is that all of the a_i are determined exactly by b_i and a_1, so we don't need to know about them to determine the fundamental quandle.

Finally, since there is only one element in Q_t we may add in the relation $a_1 = c$ to get

$$Q_p = Q_t = \langle X, a_1, \{b_i\}, c \mid R, a_1 = c\rangle = \{x\}$$

which is the fundamental quandle of the bridge projection. $\square$

Chapter 5

Future Work

5.1 Mesh Projection

In chapter 4 we defined canonical projections and extended them to bridge projections both of which had this property that projections of unknotted diagrams have singlton fundamental quandle. There is a further generalization which I'll call a mesh projection since the crossings we remove look like a mesh. The picture looks like figure 5.1.

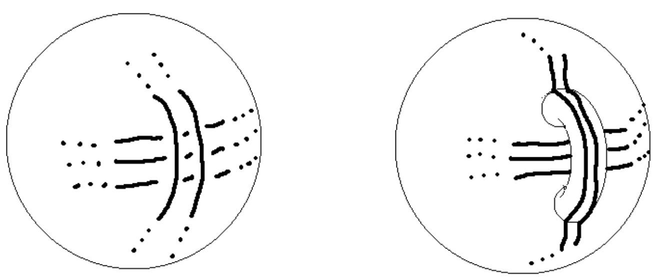


Figure 5.1 Removing a "mesh" by extruding a torus.

Interestingly every knot diagram on the torus is the mesh projection of a knot. The way we can see this is by considering an arbitrary diagram on the abstract torus, and move all of its crossings into an open disk (which can be done since there are finitely many crossings). Now, the remaining strands look like figure 5.2. Either standard embedding of the torus in space gives the knot whose mesh projection is this diagram.

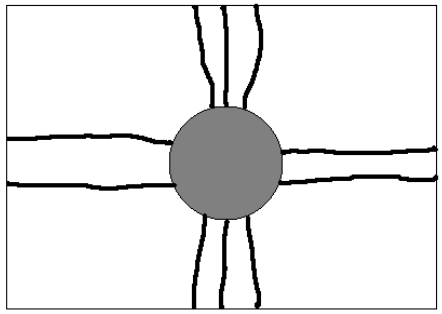


Figure 5.2 Removing a "mesh" by extruding a torus.

I was unable to prove an analog for the triviality of projections of the unknot in this case but it seems possible. Since every projection of a knot onto the torus is a mesh projection, we have the following conjecture:

Conjecture 5.1.1. *Every projection of the unknot onto the torus has singlton fundamental quandle.*

5.2 Complete the Trivial Quandle Classification for the Torus

In chapter 4 we found a class of knots on the torus which had singlton fundamental quandle. In particular, these were canonical projections (and more generally bridge projections) of unknotted diagrams on the sphere (which need not be unknotted on the torus, as we saw).

However, these are not all the diagrams in the torus which have singlton fundamental quandle. In particular, there is a whole class of knots called torus knots which are knots that can be projected onto the torus with no crossings! These have singlton fundamental quandle because they have one arc and no crossings which gives them fundamental quandle

$$Q = \langle x \mid \rangle = \{x\}.$$

An example of a torus knot is the trefoil shown in figure 5.3 Are these the only knots with singlton fundamental on the torus? Is there some way

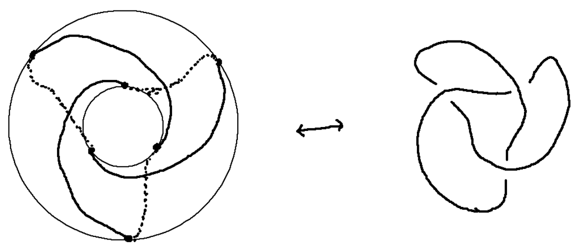


Figure 5.3 A projection of the trefoil onto the torus with no crossings.

in which both torus knots and these projections are in some larger class together?

We might expect that mesh projections of the unknot and these torus knots to be the only ways in which the fundamental quandle of a diagram can be the singleton quandle, although this is very far from being proven. Since every knot diagram on the torus which is the mesh projection of the unknot has an unknot for a realization, we can rearrange this claim into the following powerful conjecture:

Conjecture 5.2.1. *If the fundamental quandle of a knot diagram on the torus is the singleton quandle then one of its realizations* (*for some embedding of the torus into space*) *is ambient isotopic to a torus knot.*

The converse is not true because the trefoil is a torus knot and its trivial projection onto the torus has quandle distinct from the singleton quandle. Despite the lack of a converse, this conjecture would give us the following intuition: every knot diagram on the torus with singleton fundamental quandle is a projection of a knot realizable with a diagram with no crossings on the torus. In this way, every knot diagram with singleton fundamental quandle is "basically" unknotted.

5.3 Tribrackets on Surfaces

We discussed tribrackets in 2.6 and looked at extending cocycle invariants of tribrackets, but I could not come up with non-trivial examples. Maybe a place to extend tribrackets is to surfaces as we did for quandles. This could be really fruitful because in general surfaces are not cut into two pieces by any curve (a usual curve cuts the torus into one open disk). Thus, since tribrackets color these regions, they might have the power to say something about exactly how diagrams are projection on their surface (whereas the quandle is more of an intrinsic structure). For instance, the most free tribracket that colors the unknot may be wildly different depending on projection as in figure 5.4.

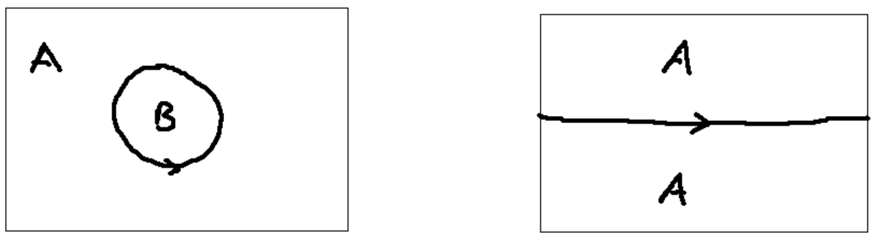


Figure 5.4 The unknot projected onto the abstract torus two ways with different tribracket colorings

5.4 Virtual Knots

There is a connection between virtual knots and work of this thesis.

Definition 5.4.1 (Kauffman (1998)). A virtual knot diagram is the same as in 4.1.1, except that we allow an addition crossing called a "virtual crossing," which we label as in figure 5.5. We say two virtual knots are "the same" or isomorphic if they are related by a finite set of normal Reidemeister moves and/or the virtual Reidemeister moves in figure 5.6

In particular, virtual knots can be thought of as knot diagrams on surfaces along with the ability to add and remove handles to the surface (called stabilization), which is what the "virtual crossings" represent Chrisman and Todd (2017). Thus, the stabilization is the difference between the knot and virtual knot situations, and the work of this thesis is a kind of middle ground between the two. In particular, quandles turn into so called virtual quandles or virtual bi-quandles (bi-quandles are a generalization of quandles in the normal case) Kauffman and Manturov (2004). These objects have an extra

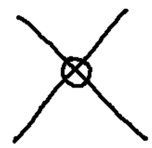


Figure 5.5 The symbol for a virtual crossing

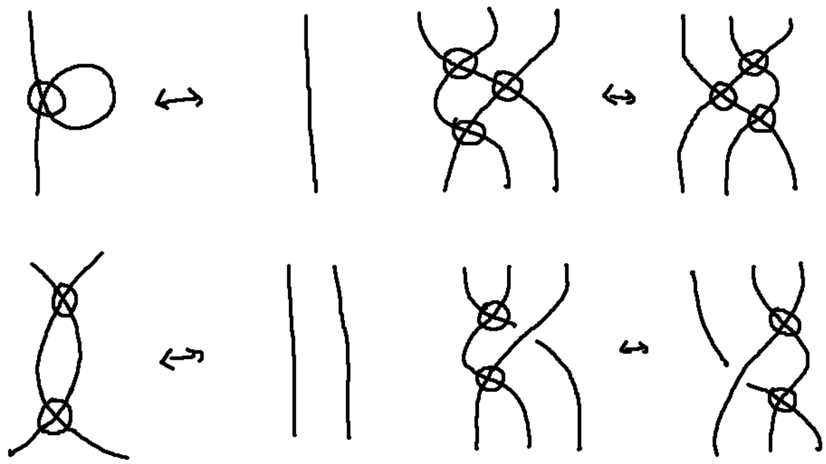


Figure 5.6 Virtual Reidemeister moves (Kauffman (1998))

one-input operation to keep track of "virtual crossings." Since this thesis does not use stabilization, it doesn't make sense to talk about the virtual crossings since they don't move around on a fixed surface. What is the relationship here? Can virtual quandles give us some intuition about what kinds of quandles we should expect on diagrams, or do quandles already capture the case of virtual quandles without stabilization?

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