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Natural Math

Maria Droujkova
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Maria Droujkova
2801 Calhoun St.
New Orleans, LA 70118
(504) 866-2508
e-mail: maria@math.tulane.edu
http://www.naturalmath.com

ABSTRACT

The Natural Math project's main goal is to create mathematical curriculum around concepts of higher math (algebra, calculus and "post-calculus" subjects), presented in a way that makes them available with minimal prerequisites. In particular, the results of the project make it possible and desirable to teach higher math to very young children and math-anxious adults.

1. INTRODUCTION

Young people (ages 4 to 10) can learn higher mathematics. They can discover concepts for themselves, develop original algorithms, and take many elements of teaching into their own hands. Learning can be arranged in such a way that mathematics comes to students naturally, painlessly, and very fast.

I attempt to prove the above statements by presenting examples of lessons that are part of the Natural Math project. The main goal of the project is to create mathematics curriculum that is rich in concepts yet readily available to people of almost all ages and all levels of mathematical education. One of the methods to achieve that goal is to arrive at higher mathematical ideas through intuition, common knowledge and common language, adding rigor later.

Several rules are strictly followed in all experiments:
• Only volunteers can participate. Adults cannot volunteer children against their will. Every child can stop the program or start it again at any time.
• Students have to discover the key concepts for themselves. Teacher provides the environment that makes it possible.
• Students are never given any algorithms, but are led to generalize their concrete experiences into algorithms.
• Students have "veto rights" in choice of activities; mentors can ask them to lift the veto as a favor, but can't insist.

2. CHILDREN CAN DO IT.

Most lesson descriptions go much faster than the real lessons, because some examples and explanations are excluded.

1. Very young children can understand almost all arithmetic concepts.

By "arithmetic" I mean concepts that can be presented without algebraic generalizations, using only numbers. For example, the idea of multiplication is arithmetic, and the idea of derivative is not; the idea of negative numbers is arithmetic, and the idea of inverse function is not. This definition is intentionally fuzzy: I do not want to create the impression that children *cannot* do certain math before a certain age. We cannot prove general negative statements of that sort because any experiment necessarily uses particular teaching methods, and negative results only give information about the teaching methods used. If you have doubts, think about this: what if people used as much mathematics in everyday life as they use language? Wouldn't everybody learn a lot of math by the age of about three, together with his mother tongue(s)? (Now, that would be some experiment!)

Example 1.1. Genevieve (age 4) learns coordinates.

Genevieve could count to 10 when we started. We had some fun with zero (Montessori, p.329), when I asked: "Give me one marble, please... Now give me zero marbles! Jump two times... Now jump zero times," etc. I introduced negative numbers as something that comes before zero as we count. I cut out small pieces of paper, wrote numbers (-10 to 10) on them, and we hung them on a long cardboard in the appropriate order. I drew and cut out an animal (a dragon) that "lived on the number line," walking back and forth. Genevieve enjoyed questions of the sort: "If the dragon is at -2 and goes 3 steps to the left, where will it land?" I always followed by: "See, negative two minus three is negative five."

Then I pointed out that the dragon has wings and can fly, and demonstrated by moving the dragon next to a wall (e.g., two steps to the right, three steps down). We drew the movements on graphing paper with a coordinate system, and Genevieve was able to find coordinates of given points, and to find points when given coordinates, translating that into "steps" at first.

Example 1.2. Cecali (age 9) studies percents.
I wrote down: "3%" and explained that this phrase in mathematical language means: "Three for every hundred." Then we discussed questions of the sort: "What is 2% of 300?" At first, I had to translate it: "If we have 2 for every hundred, how many will we have for three hundreds?" After about 5 minutes, I did not have to translate anymore. Cecali did not know decimals yet, so we could not move much farther than, for example: "Find 7% of 250." It never ceases to surprise me that problems that are "too simple" often mean trouble: Cecali could not find 15% of 100, and it took a while for her to figure out that "degenerate" proportion: 15 for every hundred, then how many for a hundred?. *It is a general tendency: if the example is so simple that it does not represent the concept anymore, young people almost always have difficulty with it.* See also Example 2.3, where Kirk could not understand the function f(x) = x.

Cecali promised to help me with taxes next year.

2. Children can easily understand "lower-level abstractions"
that require few steps of reasoning, such as unknowns, more-less, etc. They are not always able to express their knowledge in mathematical (or any other) language, so it is quite a challenge to communicate with them.

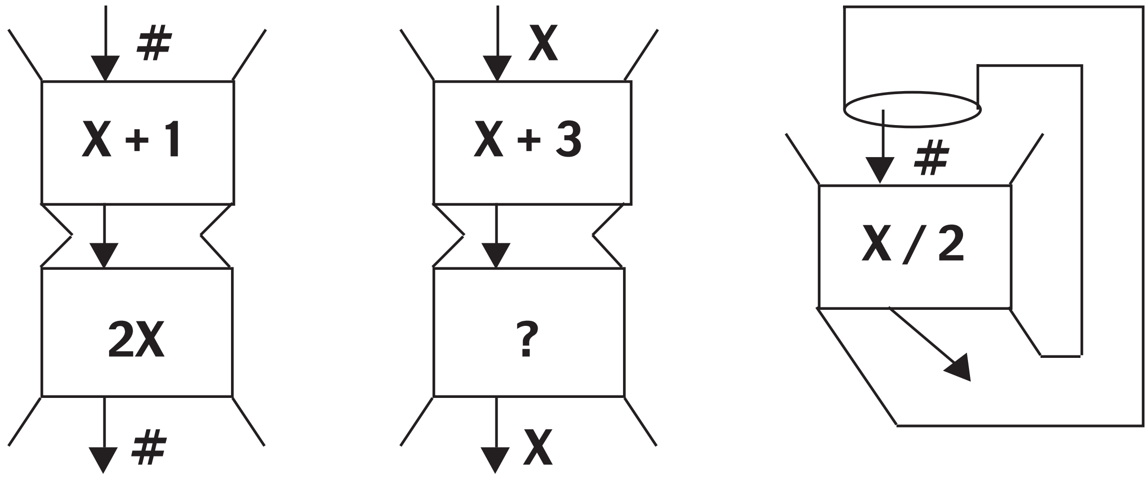
Example 2.1. Genevieve (age 4) solves equations.
We started by playing with beads and a little basket. I put two beads in the basket without showing to Genevieve how many were there. I demonstratively added one more bead, and then shown the total of three and asked: "How many were there before?" The little girl thought for about half a minute (it's a long time); she was very concentrated on the task, and finally figured it out.

One of the major difficulties in my project is that *most adults seem to be unable to tolerate the sight of a child quietly thinking for a long time.* Most people who observe my lessons have to be restrained from "helping" the thinking child: they want to explain, to give hints, to reformulate the problem. I warn people that they should not interfere; many parents tell me that it is very hard for them. It is interesting that little children often are willing to think about a question much longer than adults. Often children say, "Please, don't tell me the answer!" Many elementary math curricula concentrate on memorization, which means that most teacher's questions have to be answered immediately. It follows that children do not have a chance to think. Sometimes I even have to convince those of my older students who go to school that it is OK to think before answering.

We played the same "guessing game" many times, and I started to move it onto paper by drawing and writing. I first called the unknown "something," writing: _ + 1 = 2 and saying: "Something plus one is two, what is that something?" I mentioned the letter notation (x + 1 = 2), but it did not "sink in." Unfortunately, Genevieve did not know how to read yet, but she was amused by the fact that she could "read mathematics:" she learned to recognize numbers up to 10, "+", "-" and "=". She proudly told her parents about her success.

Example 2.2. Jasmine (age 6) plays the "More-Less Game."
The rules of the game are simple: the "host" takes a number, and the others have to guess it. If the guess is wrong, the "host" says if it is more or less than the number he has. "Host" put some marbles in a bag without showing how many (it gave the winner satisfaction of confirming the right guess by counting marbles). Jasmine had a lot of fun with the game. She learned the idea of "less and more" and figured out the beginning of the bisection method (the most efficient way to play is to find an interval containing the number and then bisect it repeatedly).

Example 2.3. Kirk (age7) constructs "Function Machines"
"Function machine" is a machine that does something to numbers you put in it. I drew some mechanism for Kirk, and asked her to give me numbers to put in the machine until she could guess what the machine did. We had: in 1 out 2, in 5 out 6, in 2 out 3, and Kirk said: "The second number is one more." I concluded: "The machine adds one." Then we took turns creating "function machines," and the drawings and the formulas got fancier and more interesting. This game is a big hit with every student to whom I show it. Kirk



(a) (b) (c)

Figure 1: The Function Machines
Small cards are used to construct the "Function Machines." (a) Composition of functions (b) Inverse: what should be on the empty card is the machine always returns the same number we put in? (c) Iterations

was very puzzled when I constructed the machine that did: in 2 out 2, in 10 out 10 (f(x)=x). Compare to Example 1.2.

"Function Machines" can effectively lead even very young students to concepts such as composition of functions, inverse, iterations, and so on (Figure 1).

3. Children can develop "higher-level abstractions."
Sometimes the depth of their understanding is surprising, given very little knowledge.

Example 3.1. Emily (age7) investigates matrices.
This lesson was inspired by a book by D. Cohen (Cohen, pp. 5-8). Emily and I pretended to "go shopping." We chose what to buy and how much, and I wrote prices as follows:

books dolls $

$$(3 \quad 2) \bullet \begin{pmatrix} 2 \\ 5 \end{pmatrix} = ?$$

Emily computed the total and wrote: "16". 1 told her that usually people call the thing we just did "multiplication of vectors," and spelled the word "vector." Then we had two days of shopping:

	roses	skirts	dolls		$		
day 1	1	3	2		1		20
day 2	2	1	1	•	5	=	9
					2		

I said that the traditional name for such an arrangement of numbers is "matrix," and spelled the word, explaining that we multiplied a matrix by a vector. Usually students quickly pick up appropriate "math language."

To motivate multiplication of two matrices, I suggested "comparison shopping" using names of local groceries:

	item 1	item 2	item 3		S	WD			
day 1	1	2	0	•	3	5		5	9
day 2	2	1	3		1	2	=	22	15
					5	1			

Linear algebra students often have problems with the following: as a result, we are getting rows and columns that correspond to days and shops, but which is which? What does each entry in the answer matrix mean? To explore that, I asked: "Which shop is cheaper?" After some confusion, Emily figured out that the columns were for shops and the rows were for days. Emily liked questions such as: "If you are a manager of Winn Dixie, will you use day 1 or day 2 in your advertising campaign? What if you are Shwegmann's manager? What to do if you are a customer?"

At first, Emily was using addition to find totals, e.g., 5 + 5 + 5 instead of 3 * 5. After a while, she seemed to remember some of these "facts." *She was doing multiplication faster by the minute, without going through a single drill on "times tables."* She and other children with whom we studied "shopping matrices" were able to *figure out the rules for matrix multiplication without any explanations*.

Next time, we coded every letter in the alphabet by its number, so "a" was "1", "b" was "2"... Emily chose words and coded them, for example, (4, 15, 7) for "dog". After Emily coded several words, I told her that it is very easy to guess what the code is. What about making a secret code? She asked me why people would want secret codes, so we talked about spies for a while. I suggested to divide every number by two (now "dog" was (2, 7 1/2, 3 1/2)). We sent each other secret messages and gave "keys" to them.

Message: (1 1/2, 1/2, 10). Key: multiply by 2

She decoded it quickly, and gave me a coded message of this sort with no difficulty. But it did not prepare me for her reaction to the next one I gave:

Message: $\begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 2 & 5 \\ 3 & 6 \\ 12 & 1 \end{pmatrix}$ Key: multiply by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Emily: ???

Me: Just like our shopping! (pointing to the page with "shopping matrices")
Emily (immediately): *But we do not write like that*.
Me: Like what? What do you mean?
Emily: I said, we do not write like that!

Only then I noticed that she was waving her hand *up and down*. She instantly understood that the answer will be a *column* vector (something most college students would notice much later) and was trying to explain that it is improper in English to write from top to bottom, not from left to right. English is not my native language; she was trying to teach me as she does occasionally (I always appreciate it and thank her properly). It is amazing that she understood such an abstract fact in a few seconds. Consider steps she had to take:

1) recall the process of matrix multiplication
2) understand, *without performing any operations*, that she will get a column vector
3) connect it with orientation of writing, something most people don't even notice
4) find a way to explain it (having very limited math vocabulary).

After I wrote transposes, Emily agreed to decode the message (her name), which proves that her column vs. row consideration was not a guess.

Emily's Matrix Poem

A little fun a little bit
And then we coded the alphabet
And on the coach was a pet
Matrix matrix matrix

A lot of fun a lot of it
We coded words and wrote it
and on the table was a pet
Matrix matrix matrix

A little fun a lot of it
We played with coordinate
And on the ceiling was a pet
Matrix matrix matrix

A lot of fun a little bit
We played a game and stitched it
And on the chair was a pet
Matrix matrix matrix

And on the bed was a pet
Matrix matrix matrix
Matrix!

Emily was so excited about matrices that she wrote a poem, reproduced above, which I used to talk about combinatorics.

4. Children can construct and use algorithms for solving problems.

That also requires some *classification* of the problems (to figure out what algorithm to use), which is an abstraction in itself. Sometimes it is desirable to lead children to some particular algorithm; in this case, restrictions can be used to make other methods "illegal." Restrictions can be presented as "the rules of the game" (see Example 4.1 below), which saves a lot of explanations by referring to the "game culture" that is a common frame of reference (cf., Davis, pp. 107-140) for many children. The "game frame," with its notions of fairness, fixed rules, sharing, competing, and taking turns, is a very powerful teaching tool especially suited for mathematics.

Example 4.1. Aidar (age 7) solves equations by "reversing."

After a "hands-on" introduction to equations (see Example 2.1) I explained that it is a tradition in mathematics to write letters, especially "X", instead of - or "something," and demonstrated: _+1=4 is the same as X+1=4
X=3

Some education psychologists argue that young children can not "reverse," i.e., cannot see that subtraction is the inverse of addition. Sure enough, the idea that: X = 4 - 1 was not something that naturally occurred to my young students. They had no need for "reversing," being perfectly able to find the solution by guessing. My task was to create that need. Since my students were not fluent in large numbers and fractions, it was impossible to present examples that are usually used to motivate high school students who only want to guess, such as 5X=7 or X+1997=2870. I tried to ask (about X + 1 = 4): "What are you doing with 4 and 1 to get 3?" which produced a lot of puzzled looks and answers of the sort: "You find what you should add to 1 to get 4." After many more futile attempts (allow me to spare the description) I finally found a game that led students to "reversing."

The game uses a calculator (TI-82, for example) where all numbers and operations show and stay on the screen. Here is my talk with Aidar (translated from Russian):

Me: Do you remember how to solve equations? Try this one: X+2=5
Aidar: X is 3
Me: Here is a new game with this calculator (some explanations were required about the "enter" key). We want to get the answer of the equation on the screen, but we can't just type it in. The rule of the game is to use any buttons except the number that is the answer, so you can't use "3" here. Try it!

Aidar typed the following:
5+2 "clear" (he has noticed it does not work before he pressed "enter" to find the answer)
2 "clear"
5-2 "enter"

We took turns solving equations with the calculator. Aidar did not have to use "clear" button anymore. It worked in a similar way with all my young students: after understanding the rules of the game, they figured out the "reversing" method of solving equations, without any hints from me. I used marbles to demonstrate the method again: "There are some marbles hidden in my hand. I add two more, and now we have five. To find out how many were there before, we can just take away the two marbles I added, in order to undo what I did."

This example makes one wonder what does the phrase: "Children are not able to understand concept X until age Y" mean? Very often, it means that before age Y children do not have any *experiences* that require understanding of concept X. Hence, if a teacher provides such experiences, students might understand concepts early.

Example 4.2. Emily (age 7) adds large numbers.

Emily wanted to learn how to operate with large (3-4 digit) numbers. She understood the idea of place value and the fact that it is convenient to add tens to tens, hundreds to hundreds, etc. However, she wanted to do operations from right to left and did not know what to do after the following step: 32+81=?

$$\begin{array}{r} 3\ 2 \\ +\ 8\ 1 \\ \hline 11\ 3 \end{array}$$

The problem was that I could not understand what exactly she was doing (I only saw that she was writing from left to right). I had to iterate my suggestions, by trial and error getting closer and closer to her yet unknown to me technique (numerical method of pedagogy). She refused to use approaches that were too far from her own. And I am happy she did, because she invented something original. One wonders how many inventions are lost because inventors do not defend them strongly enough. Finally, we figured out how to make her algorithm work. She added easily, if slowly, and had no problems with 3- and 4-digit examples:

  1 7 3
+ 2 6 9
  3 13 12
  4 ~~3~~ 4 2
  4 4 2

Emily added from left to right (first 1+2, then 7+6, then 3+9), then "carried," looking at the next number while considering numbers from left to right. If I had to teach "long addition" by "telling them the rule," I would use Emily's algorithm rather than conventional algorithm, because it is more straightforward. Emily and I started to do "long multiplication" on the same day the lesson above happened.

5. Children can understand some topics from almost all branches of mathematics, including "postcalculus."
Presentation has to be adjusted, of course. See examples from other sections, plus:

Example 5.1 Kirk (age 7) plays with cyclic groups.
Young people love cyclic groups. There is something fascinating in being able to do arithmetic with finite amount of numbers.

With Kirk, we started by looking at the ordinary clock. Numbers never grow past 12, so if it is one o'clock now, 15 hours later it will be 4, not 16. 1 pointed to the wall clock and wrote: 15 + 1 = 4. With much laughter, Kirk solved several problems of this sort. Then we talked about other planets, where the day can be longer or shorter than 12 hours. Kirk chose the 3-hour-long day and drew the "space alien clock" with numbers 1, 2, and 3. Then I asked: "If it's two o'clock on that planet, what time will it be 14 hours later?" I expected Kirk to count hours around the clock (2,3,1,2,3,1 ...); however, she immediately said: "One o'clock." She solved it faster than I did (I used the fastest way, i.e., remainders). She did other problems as fast, so it was not a lucky guess (however, she was much slower next week when we returned to the topic). Judging by the time of her response, she was using a very efficient method to solve these problems (cf., Woods, Resnick and Groen), without being told about any methods whatsoever!

I explained that mathematicians invented a way to write about alien clocks without confusion: they write 2+14= 1 (mod 3) (I read it aloud). Kirk had no problems with notation. We explored alien clocks of different modulo, using examples with positive and negative numbers, such as:

2-4=3(mod5)

Kirk developed a vague notion of remainders when she noticed patterns in numbers:

4=1(mod 3)
5=2(mod 3)
6=0(mod 3)
7=1(mod 3) again, and so on.

Cyclic groups is now one of Kirk's favorite topics in mathematics. The topic can be used to talk about division and multiplication ("What time will it be on the planet 13 hours after midnight?" or "How many hours are there in 7 days?") and is an effective way to introduce remainders.

Example 5.2 Emily (age 8) starts to understand linear independence of vectors.
We played a game: we take a piece of graphing paper and draw some "obstacle course" on it, made of any objects, e.g., trees, lakes, castles... The objective of the game is for one of us to guide the other through that maze from the start to some treasure at the finish. The difficulty is that we can only use commands in numbers (I told a science fiction story to go with it, about commands received through a very primitive radio that could translate only numbers), so one has to give coordinates of vectors to guide, for example (1,0) for one step to the right. We played the game for a while to make sure that all relevant terminology and concepts were exposed (such as "vector addition," "vector coordinates," etc.).

When the game started to get too easy for Emily, I suggested the new rules: now we guide the broken robot that can make only two kinds of steps, but as many of them as we want, in positive and negative directions (Picture 3). Emily was guiding the robot at first, but when time came to make steps in negative direction, she did not know what to do. I asked her if I could guide the robot, and she said, relieved and reluctant at the same time: "We guided each other before, but now we guide the robot, so it is OK for you to do that."

During any lessons students are very vulnerable: just by being there they admit to the whole world that they do not know something, can't do something, and need a special person, the teacher, to help them. Too easily it can be turned into a threatening situation. The teacher has to exercise extreme caution and sensitivity not to hurt students' feelings. It is especially true if students are unable to do some task or if they make a mistake. *Students sometimes feel hurt by the gentlest offer of help*. So, I try to provide excuses for them to ask for help, or play along when they invent excuses as Emily did. Sometimes I would just recommend students to think some more, of course. Let me mention here that if the teacher has to administer any sort of formal tests that affect students' life in any way, the situation almost always gets ugly, sensitivity or not.

Next, I drew two vectors, (- 1, 1) and (-2,2) and asked: "If the robot can only go in the steps like that, will we be able to get him to the treasure?" Emily saw immediately that the robot would only go along a straight line. I told her that mathematicians say that vectors like that are "not independent." After practicing with couples of vectors (in the last few examples Emily could see independence by coordinates, without having to draw vectors), we were ready to play the next "maze game." I drew two independent vectors and added the third, not parallel to each of the two. I asked: "Do we need the third vector, or is two of them enough?" And after Emily said that two is enough, I explained that three are not independent again, because all "steps" of the third kind can be achieved with the first two vectors.

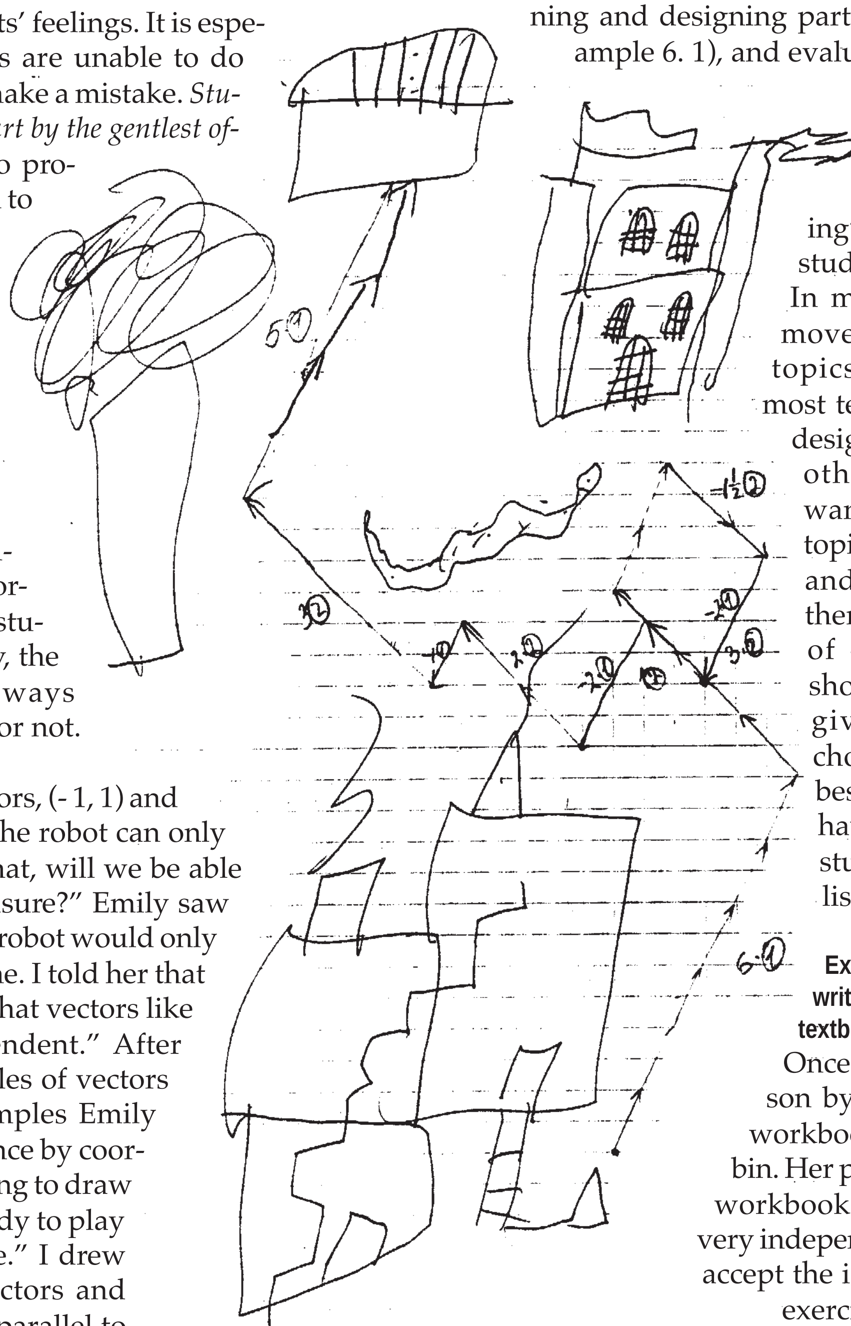


Figure 2: The Maze and the Broken Robot
Our robot could only make two kinds of steps, (1,2) and (-2,2). Yet he reached the treasure, a bookshelf (Emily loves to read). The game can be used to discuss linear combinations and different coordinate systems.

6. Many elements of teaching can be successfully managed by even the youngest students.

These elements include, but are not limited to, planning and designing parts of curriculum (Example 6. 1), and evaluation (Example 6.2). One of the simplest and most efficient tools I use to arrange "self-teaching" is taking turns with students in all activities. In most cases, students move to more advanced topics much faster than most teachers would move designing curriculum; in other cases, students want to stay longer on topics that interest them and/or are difficult to them. One is reminded of nutrition studies showing that toddlers, given full freedom, choose the diet that is best for them (I do not have the reference; the study results were published in periodicals).

Example 6.1 Kirk (age7) writes her own math textbook.

Once Kirk started our lesson by throwing her math workbook into the recycling bin. Her parents and I liked her workbooks, but Kirk, being a very independent person, did not accept the idea of doing regular exercises. She asked me if I would write a better book for her, and I suggested that she can write her own math text, and promised to help. We meet once a week and discuss the book's progress, and we exchange e-mail. Kirk invents math games, creates exercises, and writes stories to accompany them. I show her some math she does not know yet, and she transforms and internalizes it. Here is her story to accompany a series of simple arithmetic exercises she put in the book; she also draws pictures for every page:

"Kirk's mom told her she may have an Oreo, but she is too small to reach the shelf. Every time Kirk solves something, she grows. Help her to get the cookies."

An example of interesting problems Kirk raises:

"One human year is about seven dog years. How old is my dog in his years, if he is eight in our years?" It led to an interesting discussion of proportions (and biology: we talked about other creatures' lives).

Most of Kirk's math education is planned by her parents (she is homeschooled). However, the little part of her learning that her book constitutes seems to be important to her, and, of course, much fun.

Example 6.2 Aidar (age 7) creates assessment.
Aidar initiated a "game of school": one of us would give the other exercises and then evaluate them. I was playing the student first, and Aidar graded me (A+). I noticed that he was nervous about the grades. Then he was playing the student:

Me (returning his paper with solved exercises): Very good!
Aidar (looking for the grade and not finding it): Where is my grade?
Me: I don't like grades.
Aidar: Oh... (thinking for a while, sadly looking at the paper with huge A+ he wrote as a "gift" for me, then getting an idea) But you can make tiny little marks next to every problem that is solved right!

Aidar has found a way to make softer (and more informative) evaluations. We followed his suggestion, of course.

In summary, children can do a lot of quality mathematics before the age of 10. It does not even take much time (none of my students spend more than two hours a week on the lessons). The next section tries to answer the question: "What is it for?"

3. WHY TEACH HIGHER MATH TO YOUNG PEOPLE?
1) "Bird's eye view." There is more understanding of "what mathematics is about." My young students would not think, as many people of their age (and, unfortunately, too many adults): "Math is addition, subtraction, and some multiplication."

2) Fun. Young students like higher math more than primitive arithmetic (if rigor is not pushed too far). It leaves more place for creativity. The best time for learning is when you like what you learn. None of my linear algebra students at Tulane University were inspired enough by the subject to write a poem (as Emily in Example 3.1). Children often enjoy (or don't mind) learning material that is boring and/or difficult for adults (consider languages, for example).

3) New frames of reference. Giving students even a vague notion of new concepts creates frames of reference in their minds. It builds a base for further teaching. It helps to break the vicious circle of "you can't start to learn it unless you understand it, but you can't understand it until you learn it." Students are initiated into richer mathematical culture, and it is done gently, without the undue strain people experience in college when they have to learn too much at once. They have more time to get used to that culture and to absorb it.

4) Advanced problem solving methods. Experiences in higher mathematics lead students from purely arithmetical methods to algebraic methods involving various symbolic operations, abstractions, etc. This differs from many programs for "gifted" where children are given the most difficult problems they can possibly solve using elementary methods they already know. As a child, I often felt "cheated" having to solve problems with great difficulties using primitive methods, only to learn later that those tricky problems turn into elementary exercises in more advanced methods. I am sure everybody can supply his own examples. My big personal sore was in elementary physics of movement (velocity, acceleration, etc.) where problems were hard to solve without using derivatives, but were just standard exercises with derivatives. About half a year I wasted because of that seems to be more than enough time to learn necessary calculus.

5) Meaningful exercise. Higher mathematics is "arith-

metic-intense," presenting excellent review opportunities. Many people do not seem to learn subject #n until they use it for subject #(n+l) (compare to the usual lament of students taking, for example, a differential equations course: "If I took calculus now, I would get an "A" so easily!")

6) Future science education. Many people who are successful in mathematics say that they are very comfortable with the parts of mathematics they learned early (Danger! People who are *forced* to learn things they are not ready for may not benefit). For example, Professor Vladimir Arnol'd says (Lui): "Many Russian families have the tradition of giving hundreds of such problems [very nonstandard old merchant problems] to their children, and mine was no exception. Very young children start thinking about such problems even before they have any knowledge of numbers. Children five to six years old like them very much and are able to solve them ... The feeling of discovery that I had then [as a child] was exactly the same as in all the subsequent much more serious problems ..." Anecdotal evidence shows that *gentle* exposure to higher mathematics at an early age is very helpful for future understanding. In practice, "gentle" means "less rigorous, very intuitive, with no formal tests."

7) Time saving. Learning arithmetic through higher math saves time, for obvious reasons (returning to concepts several times makes the periods "in between" work for understanding; review is imbedded in learning of new things; students see arithmetic as a mere tool for higher math, which reduces fear considerably). In practice it means that people who are interested in mathematics and science can start doing graduate-level research several years earlier than usual. Those who are not interested in math receive *an opportunity to spend considerably less time with the subject, reaching the same or better results*, and doing work that can even change their attitude towards math.

8) Application to remedial education. Methods of teaching that work well for young people can be successfully used with older people having learning problems. After all, young students and "lower track" students face the same challenge of *limited prerequisites*. Of course, the situation is usually worse for people who learned math in a wrong way: they require *rehabilitation* more than anything. My experience shows that the methods I use with children do a good job rehabilitating math-anxious people.

Clearly, more research is needed in the area of teaching higher mathematics to people with limited prerequisites. I hope that more mathematicians and education specialists will address the exciting pedagogical problem of making mathematics more available. It can and should result, in particular, in very different curricula for young people, whose learning potential is too often grossly underestimated. I also hope that more people will realize that students themselves hold in their hands solutions to many pedagogical problems, and that students need some freedom in learning, as well as a lot of unobtrusive help, in order to solve these problems.

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