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SOME PROPERTIES OF COMPACTIFICATIONS

By Melvin Henriksen and J. R. Isbell

Introduction. A *compactification* of a topological space X is a compact (Hausdorff) space containing a dense subspace homeomorphic with X. Since only completely regular spaces have compactifications, all spaces mentioned here will be completely regular unless the contrary is assumed explicitly. This paper is a study of properties of the sets of points which may be added to a space in compactifying it. We find several properties $\mathcal{P}$ such that for all spaces X

(*) *if the complement of X in* one *of its compactifications has property $\mathcal{P}$, then the complement of X in* any *of its compactifications has property $\mathcal{P}$.*

A list of such properties is provided in Theorem 2.2. It includes compactness, local compactness, σ-compactness, the Lindelöf property, and paracompactness.

Recall Čech's result [3] that any compactification AX of X is a continuous image of the Stone-Čech compactification βX of X under a mapping which takes $\beta X - X$ onto $AX - X$. The essence of the reason that (*) holds for the listed properties is that the restriction of this mapping to $\beta X - X$ preserves these properties in the strong sense that the domain has the property if and only if the range does. Indeed, this latter mapping is an example of what we call a *meshing map*; namely a mapping f of a space X onto a space Y which has an extension $\bar{f}$ over some compactification AX of X onto some compactification BY of Y, which maps $AX - X$ homeomorphically onto $BY - Y$. We call a property $\mathcal{P}$ a *meshing property* if whenever $f\colon X \to Y$ is a meshing map, then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$. Then a necessary and sufficient condition for (*) to hold for a property $\mathcal{P}$ is that $\mathcal{P}$ be a meshing property (Theorem 2.6).

All of the properties listed in our first paragraph are actually preserved by a wider class of mappings, namely those mappings $f\colon X \to Y$ such that f maps X continuously onto Y, f is closed, and for each $y \in Y$, the set $f^{-1}(y)$ is compact. We call these *fitting maps*, and the properties they preserve (in the strong sense given above), *fitting properties*. Every meshing map is fitting (and hence every fitting property is a meshing property); the converse is true for locally compact spaces, where these mappings coincide with those *proper* mappings in the sense of Leray [12] that are onto. Many fitting maps that are not meshing are provided by the fact that the projection map of the product of a non-locally compact space Y and a compact space containing at least two points onto Y fails to be meshing (Corollary 1.7). However, meshing properties that are not fitting are harder to find. We have identified one such (Example 2.1), but we have not found any among the more familiar topological properties.

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In general, we have a necessary and sufficient condition for a fitting map f: $X \to Y$ to be meshing (Theorem 1.8), which reduces in case Y is normal to the following: any two distinct points x_1, x_2 of X have neighborhoods U_1, U_2 such that $f[U_1] \cap f[U_2]$ is compact (Corollary 1.9). Examples show that the class of meshing maps is not closed under functional composition, though fitting maps obviously are (1.11).

For any property $\mathcal{P}$, we say that X has property $\mathcal{P}$ *at infinity* if $\beta X - X$ has property $\mathcal{P}$. If $\mathcal{P}$ is a meshing property, then so is $\mathcal{P}$ at infinity (Theorem 2.8). Moreover, $\mathcal{P}$ is a fitting property if and only if $\mathcal{P}$ at infinity is (Theorem 2.7).

Note that compactness at infinity is just local compactness. X is locally compact at infinity if and only if the set $R(X)$ of all points at which X fails to be locally compact is compact (Theorem 3.1); and in general, for any meshing property $\mathcal{P}$, X has property $\mathcal{P}$ at infinity at infinity if and only if $R(X)$ has property $\mathcal{P}$ (Theorem 2.9). X is Lindelöf at infinity if and only if every compact subset of X is contained in a compact set of countable character (Theorem 3.6). In particular, every metrizable space is Lindelöf at infinity.

In §1, fitting maps and meshing maps are studied, while §2 is devoted to fitting properties and meshing properties. Section 3 is concerned with some special properties at infinity. Finally, in an appendix (§4), we show that among all Hausdorff spaces, regularity is a fitting property, while complete regularity is not.

1. **Fitting and meshing mappings.** We are concerned almost exclusively with subspaces of compact (Hausdorff) spaces, that is, with completely regular (Hausdorff) spaces. *Indeed, throughout the sequel, except in the appendix* (§4), *the word "space" will be used to abbreviate "completely regular space".* If a space X is (homeomorphic with) a dense subspace of a compact space BX, then BX is called a *compactification* of X.

For any space X, let $C(X)$ denote the set of continuous real-valued functions on X, and let $C^*(X)$ denote the subset of all bounded $\phi \in C(X)$. Among the compactifications of X, we will be concerned particularly with the *Stone-Čech* compactification βX. It is characterized among the compactifications of X (to within a homeomorphism keeping X pointwise fixed) by the fact that every $\phi \in C^*(X)$ has a (unique) continuous extension over βX. In addition, we shall depend heavily on the following result of Čech [3; 831].

1.1 Lemma (Čech). *Any compactification BX of a space X is the image of βX under a (unique) continuous mapping f that keeps X pointwise fixed; furthermore $f[\beta X - X] = BX - X$.*

Note that both f and its restriction to $\beta X - X$ are closed continuous mappings such that the inverse image of a point is compact. More generally we define:

Definition. *A closed continuous mapping f of a space X onto a space Y such that for each $y \in Y$, the set $f^{-1}(y)$ is compact, is called a fitting map.*

Remark. It seems best to emphasize at this point that (except in the appendix) the domain space X, and the range space Y, of a fitting map, are by fiat completely regular spaces (as per the convention introduced above). We observe that although this does involve a loss of generality, it will not affect our main applications of the concept of a fitting map, since we are concerned mainly with spaces that have compactifications. In an appendix (§4), we will discuss the precise extent of the loss of generality; but otherwise we will restrict our discussion to completely regular spaces.

The following simple lemma will be needed frequently below.

1.2 LEMMA. *A continuous mapping f of a space X onto a space Y is a fitting map if and only if both*

(i) *for each $y \in Y$, and each neighborhood U of $f^{-1}(y)$ in X, there is a neighborhood V of y in Y such that $f^{-1}[V] \subset U$, and*

(ii) *for each compact subset K of Y, the set $f^{-1}[K]$ is compact.*

Proof. It is well known (and not difficult to verify directly) that (i) holds if and only if f is closed [11; 97]. Hence (i) and (ii) together imply that f is a fitting map.

Suppose conversely that f is a fitting map, let K be a compact subset of Y, and let $\mathfrak{U}$ be any open covering of $f^{-1}[K]$. By hypothesis, for each $k \in K$, there is a finite subset $\mathfrak{U}_k$ of $\mathfrak{U}$ that covers $f^{-1}(k)$. Let U_k denote the union of the elements of $\mathfrak{U}_k$. By (i), for each $k \in K$, there is an open neighborhood V_k of k in Y such that $f^{-1}[V_k] \subset U_k$. Since K is compact, there is a finite subset $\{k_1, \cdots, k_n\}$ of K such that $\bigcup_{i=1}^{n} V_{k_i} \supset K$. Clearly $\bigcup_{i=1}^{n} \mathfrak{U}_{k_i}$ is a finite subfamily of $\mathfrak{U}$ that covers $f^{-1}[K]$, so (ii) holds, and the lemma is proved.

A continuous mapping of a locally compact space into a locally compact space satisfying condition (ii) of Lemma 1.2 is called a *proper mapping* by Leray and Bourbaki, who showed that every such mapping is closed [12, no. 22] [2; 103]. More generally, *any mapping f of a space X onto a k-space Y satisfying* (ii) *is closed.* (Recall that Y is a k-space provided every subset of Y intersecting every compact subset of Y in a closed set is itself closed. Every locally compact space, and every space satisfying the first axiom of countability is a k-space [11; 231].)

For, let S be any closed subset of X, and let K be any compact subset of Y. Note that by (ii), $S \cap f^{-1}[K]$ is compact, so $f[S] \cap K = f[S \cap f^{-1}[K]]$ is closed. Since X is a k-space, $f[S]$ is closed.

It will be shown next that if Y is not a k-space, then f need not be a closed mapping.

We first introduce some notation that will be used throughout the sequel. If α is any ordinal, let $W(\alpha)$ denote the space of ordinal numbers less than α in the interval topology. As usual ω_0 and ω_1 denote respectively the first infinite and the first uncountable ordinal numbers.

1.3 EXAMPLE. Let H denote the subspace of $W(\omega_1 + 1)$ obtained by deleting all countable limit ordinals. Observe that every compact subset of H is finite, so H is not a k-space. Let X denote the set H with the discrete topology. Then the identity map of X onto H satisfies (ii) but is not closed.

In particular, every proper map of a locally compact space onto a locally compact space is a fitting map. It is also true that a continuous mapping f of locally compact space X onto a locally compact space Y is proper if and only if f has a continuous extension over the one point compactification αX of X onto the one point compactification αY of Y [12], [2]. Observe that this extension sends the point at infinity of αX onto the point at infinity of αY.

We next introduce another generalization of the concept of proper mapping based on this latter property. For any mapping f on a set X, the restriction of f to a subset S of X will be designated by $f \mid S$.

DEFINITION. *A continuous mapping f of a space X onto a space Y such that there exist compactifications AX of X and BY of Y, and a continuous extension $\bar{f}$ of f over AX onto BY such that $\bar{f} \mid (AX - X)$ is a homeomorphism onto $BY - Y$, is called a meshing map of X onto Y.*

It is obvious that every continuous mapping of a compact space onto a compact space is fitting, so *every meshing map is a fitting map*. Moreover, if we choose the compactifications in the definition above to be αX and αY, it becomes clear that any proper map, and hence any fitting map, of a locally compact space X onto a locally compact space Y is a meshing map. In general, however, the assumption that a map is meshing is much more restrictive than the assumption that it is fitting, as will be seen below (Corollary 1.7).

If S is a subset of a space X, then we use S^- to denote the closure of S in X.

1.4 LEMMA. (a) *If f is a fitting map of a space X onto a space Y, and if S is a subset of X, then $f \mid S$ is a fitting map (onto $f[S]$) if and only if $S = f^{-1}[f[S]] \cap S^-$.*

(b) *If f is a meshing map of a space X onto a space Y, and if S is a closed subset of X, then $f \mid S$ is a meshing map (onto $f[S]$).*

Proof. (a) Assume first that S is a dense subset of X. Then $f^{-1}[f[S]] \cap S^- = f^{-1}[f[S]]$. If $S = f^{-1}[f[S]]$, then for each $y \in f[S]$, the compact set $f^{-1}(y)$ is contained in S, and for any relatively closed subset T in S, the image of T under $f \mid S$ is the relatively closed set $f[T] \cap f[S]$. Conversely, suppose that $f \mid S$ is fitting, and consider any $x \in X - S$, and $y \in f[S]$. By hypothesis, the set $f^{-1}(y) \cap S$ is compact and hence has an open neighborhood U in S whose closure in X does not contain x. Then $f[S - U]$ is a closed subset of $f[S]$ not containing y. But $f(x)$ is a limit point in Y of $f[S - U]$, so $y \neq f(x)$. Hence $S = f^{-1}[f[S]]$, and we have (a) in case S is dense.

If S is a closed subset of X, then obviously $f \mid S$ is fitting, and $S = f^{-1}[f[S]] \cap S^-$.

Finally let S be any subset of X, and assume first that $S = f^{-1}[f[S]] \cap S^-$. As noted above, $f \mid S^- = f_1$ is fitting. But S is dense in S^-, and $f_1^{-1}[f_1[S]] = f^{-}{}_1[f[S]] \cap S^- = S$, so by the above, $f_1 \mid S = f \mid S$ is a fitting map. Conversely if $f \mid S$ is a fitting map, then regarding it as the restriction to S of the fitting map $f \mid S^-$, we obtain easily that $S = f^{-1}[f[S]] \cap S^-$. Hence we have proved (a).

(b) By hypothesis, there exist compactifications AX of X and BY of Y and an extension $\bar{f}$ of f over AX onto BY such that $\bar{f} \mid (AX - X) = f_1$ is a homeomorphism onto $BY - Y$. Let S be a closed subset of X, and let T denote its closure in AX. Then $\bar{f} \mid T$ is an extension of f_1 over T whose restriction to the subset $T - S$ of $AX - X$ is a homeomorphism onto $f[T] - f[S]$, so $f|S$ is a meshing map.

Our next lemma is concerned with the Stone-Čech compactification. For any space X, we abbreviate $\beta X - X$ by X^*. If A is a subset of βX, we designate its closure in βX by A^β.

If f is any continuous mapping of a space X onto a space Y, and if BY is any compactification of Y, then by a theorem of Stone [15, Theorem 88], f has a (unique) continuous extension f_B over βX onto BY. Clearly $f_B[X^*]$ contains $BY - Y$, but these two sets will not coincide in general. However, since any continuous mapping of a compact space onto a compact space is fitting, and immediate consequence of Lemma 1.4 is

1.5 LEMMA. *If f is a continuous map of a space X onto a space Y, then the following statements are equivalent.*

(a) *There exists a compactification BY of Y such that $f_B[X^*] = BY - Y$.*

(b) *If BY is any compactification of Y, then $f_B[X^*] = BY - Y$.*

(c) $f_\beta[X^*] = Y^*$.

(d) *f is a fitting map.*

In particular, the restriction of f_β to X^* is a fitting map onto Y^* if and only if f is a fitting map.

For any space X, let $R(X)$ denote the set of points of X at which X is not locally compact. Since its complement is open, $R(X)$ is closed. Furthermore, in any compactification BX of X, the closure of $BX - X$ consists precisely of $BX - X$ and $R(X)$.

Observe also that *if f is a fitting map of a space X onto a space Y, then $f[R(X)] = R(Y)$.* For, by Lemma 1.2 (ii), $f[R(X)] \subset R(Y)$. If $y \in R(Y) - f[R(X)]$, then the compact set $f^{-1}(y)$ has a compact neighborhood U disjoint from $R(X)$, so by Lemma 1.2 (i), there is a neighborhood V of y in Y such that $f^{-1}[V] \subset U$. Hence V^- is a compact neighborhood of y. This contradiction yields $R(Y) \subset f[R(X)]$, so we are done.

Recall that if f is a continuous mapping of a space X onto a space Y, then a *cross section* of f is a continuous mapping g of Y into X such that (the composite function) $f \circ g$ is the identity map of Y onto itself.

1.6 THEOREM. *If a meshing map f of a space X onto a space Y has a cross section, then $f \mid R(X)$ is a one-one mapping (and hence a homeomorphism) onto $R(Y)$.*

Proof. Since f is a meshing map, there exist compactifications AX of X and BY of Y, and an extension $\bar{f}$ over AX onto BY such that $\bar{f} \mid (AX - X)$ is a homeomorphism onto $BY - Y$. By hypothesis, f has a cross section g mapping Y into X, and by the theorem of Stone cited above, g has a continuous extension g' over βY *into* AX. Now $f \circ g$ is the identity map of Y onto itself, so $\bar{f} \circ g'$ maps βY *onto* BY. But $\bar{f} \mid (AX - X)$ is a one-one mapping onto $BY - Y$, and $\bar{f}[g'[\beta Y]] \supset BY - Y$, so $g'[\beta Y] \supset AX - X$. Since $g'[\beta Y]$ is compact, $g'[\beta Y]$ contains the closure of $AX - X$ in AX, which in turn contains $R(X)$. We conclude that $R(X) \subset g'[\beta Y] \cap X$.

We will show next that $S = g'[\beta Y] \cap X \subset g[Y]$. To do so we note first that since $\bar{f} \circ g'$ coincides with the identity on Y, and maps βY onto BY, by Lemma 1.1, $\bar{f} \circ g'$ sends $\beta Y - Y$ onto $BY - Y$. Hence $g'[\beta Y - Y] = AX - X$, so $S \subset g'[Y] = g[Y]$.

Finally, since g is a cross section of f, f is a one-one mapping on $g[Y]$, and hence on $R(X)$.

If K is a compact space, Y is any space, and $X = K \times Y$, it is apparent that $R(X) = K \times R(Y)$. Moreover, the projection map f of X onto Y obviously has a cross section; choose any $k \in K$ and map Y onto $\{k\} \times Y$ in the natural way. Hence if we observe that the projection map f is obviously a fitting map, we have immediately from Theorem 1.6:

1.7 COROLLARY. *If K is any compact space containing at least two points, and Y is any space that is not locally compact, then the projection map of $K \times Y$ onto Y is a fitting map that is not a meshing map.*

1.8 THEOREM. *A fitting map f of a space X onto a space Y is meshing if and only if for any pair x_1, x_2 of distinct points of $R(X)$ such that $f(x_1) = f(x_2)$, there exist neighborhoods U_1 of x_1 and U_2 of x_2 in βX such that $f_\beta[U_1] \cap f_\beta[U_2]$ is compact and is contained in Y. In particular, if $f \mid R(X)$ is a one-one mapping, then f is a meshing map.*

Proof. Note first that for any continuous map f of X onto Y, the condition given above for x_1 and x_2 holds automatically unless both x_1 and x_2 are in $R(X)$. For, if x_1 has a compact neighborhood U_1 in X, then $f_\beta[U_1]$ is already a compact subset of Y. If $f(x_1) \neq f(x_2)$, then $f(x_1)$ and $f(x_2)$ have disjoint compact neighborhoods V_1, V_2 in βY. So, if $U_i = f_\beta^{-1}[V_i]$ $(i = 1, 2,)$ then $f_\beta[U_1]$ and $f_\beta[U_2]$ are disjoint.

Now suppose that f is a meshing map, so that there exist compactifications AX of X and BY of Y, and an extension $\bar{f}$ over AX onto BY mapping $AX - X$ homeomorphically onto $BY - Y$. By Lemma 1.1, there exist mappings i_A of βX onto AX and i_B of βY onto BY keeping X (respectively Y) pointwise fixed and sending X^* (respectively Y^*) onto $AX - X$ (respectively $BY - Y$). Observe that $\bar{f} \circ i_A = i_B \circ f_\beta$, since they both send βX onto BY, and coincide with f on X. Let V_1, V_2 be disjoint closed neighborhoods of the distinct points x_1, x_2 of $R(X)$ in AX, and let $U_i = i_A^{-1}[V_i]$ $(i = 1, 2)$. The set $(\bar{f} \circ i_A[U_1]) \cap (\bar{f} \circ i_A[U_2]) = \bar{f}[V_1] \cap \bar{f}[V_2]$ is compact, and since $\bar{f}$ is a one-one mapping on $AX - X$, $\bar{f}[V_1 - X]$ and $\bar{f}[V_2 - X]$ are disjoint. Therefore $(\bar{f} \circ i_A[U_1]) \cap (\bar{f} \circ i_A[U_2])$ is a compact subset of Y. But this latter set coincides with $(i_B \circ f_\beta[U_1]) \cap (i_B \circ f_\beta[U_2])$. So, since i_B maps Y^* onto $BY - Y$, it follows that $f_\beta[U_1] \cap f_\beta[U_2]$ is a compact subset of Y, as required.

Suppose conversely that f is fitting and that our condition on pairs of distinct points of $R(X)$ holds. We define AX as the quotient space of βX by the decomposition whose elements are the single points of X and the sets $f_\beta^{-1}(p)$, $p \in Y^*$. We will show next that the quotient mapping h of βX onto AX is closed by applying Lemma 1.2 (i). Observe first that $X - R(X)$ is an open subset of βX, every open subset of which is the inverse image of an open set in AX. Next observe from the definition of AX that for each $q \in AX - X$, there is a $p \in Y^*$ such that $h^{-1}(q) = f_\beta^{-1}(p)$, so any neighborhood of $h^{-1}(q)$ in βX contains the inverse image under f_β of a neighborhood in AX of q. Finally, we consider any point $x \in R(X)$, any open subset U of βX containing x, and the compact set $\beta X - U$. For each point p of $\beta X - U$ in X, there exists by hypothesis a compact neighborhood V_p of p in βX and a compact neighborhood V_p' of x such that $f_\beta[V_p] \cap f_\beta[V_p']$ is a compact subset of Y. (Recall the remarks made in the first paragraph of the proof.) If $p \in (\beta X - U) \cap X^*$, then by Lemma 1.5, $f_\beta(p) \neq f_\beta(x)$, so we may repeat the same construction, and even conclude that $f_\beta[V_p]$ and $f_\beta[V_p']$ are disjoint. Then $\beta X - U$ is covered by finitely many of the sets V_p ; let V denote the union of the elements of such a finite family and let V' denote the intersection of the corresponding sets V_p' . Then $f_\beta[V] \cap f_\beta[V']$ is a compact subset of Y; so since V and V' are disjoint, and h sends X^* onto $AX - X$, the sets $h[V]$ and $h[V']$ are disjoint. But then $AX - h[V]$ is a neighborhood of $h(x)$ whose inverse image is contained in U. Thus, by Lemma 1.2 (i), h is a closed mapping, so AX is a compactification of X [11; 148].

From the construction of AX, it is apparent that the mapping $\bar{f}$ of AX onto βY defined to coincide with f on X, and to map each point $q = h[f_\beta^{-1}(p)]$ in $AX - X$ to p in Y^*, is a continuous extension of f such that $\bar{f} \mid (AX - X)$ is a one-one mapping onto Y^*. Since $\bar{f}$ is a fitting map, we may conclude from Lemma 1.4 (a), that $\bar{f} \mid (AX - X)$ is a homeomorphism onto Y^*.

1.9 Corollary. *A fitting map f of a space X onto a normal space Y is a meshing map if and only if for each pair of distinct points x_1 , x_2 of $R(X)$ there exist neighborhoods V_1 of x_1 and V_2 of x_2 in X such that $f[V_1] \cap f[V_2]$ is compact.*

Proof. We need only show that if Y is normal, then for any pair of distinct points x_1 , x_2 of X and closed neighborhoods (in X) V_1 , V_2 of x_1 , x_2 , the set $S = f[V_1] \cap f[V_2]$ is compact if and only if $T = f_\beta[V_1^\beta] \cap f_\beta[V_2^\beta]$ is a compact subset of Y. Since f_β is continuous, $f_\beta[V_i^\beta] \subset (f_\beta[V_i])^\beta$ $(i = 1, 2)$. Thus $f_\beta[V_1^\beta] \cap f_\beta[V_2^\beta]$ is a compact subset of $(f[V_1])^\beta \cap (f[V_2])^\beta = (f[V_1] \cap f[V_2])^\beta$, since $f[V_1]$ and $f[V_2]$ are closed subsets of the normal space Y [10, Lemma 7], [17]. Now if S is compact, then $S^\beta = S$, so T is a compact subset of Y. If T is a compact subset of Y, then since S is closed in Y, the set S is compact.

A simple generalization of the fact that every fitting map of a locally compact space onto a locally compact space is a meshing map follows immediately from

Theorem 1.8; *if X is any space such that $R(X)$ has at most one point, then a fitting map of X onto a space Y is a meshing map.*

Note also that the simpler criterion given in Corollary 1.9 can be applied if X is known to be normal; for in this case it follows that Y is normal [18]. We obtain also from the construction in the proof of Theorem 1.8:

1.10 COROLLARY. *If f is a meshing map of a space X onto a space Y, then $f_\beta \mid X^*$ is a meshing map onto Y^*.*

Proof. Consider the compactification AX of X constructed in the proof of Theorem 1.8, recall that the extension $\bar{f}$ of f over AX maps AX onto βY, and that $g = \bar{f} \mid (AX - X)$ is a homeomorphism onto Y^*. Let T denote the closure of $AX - X$ in AX. Then g^{-1} is a homeomorphism of Y^* onto the dense subset $AX - X$ of T, so T is a compactification of Y^*. Then, if i_A denotes the mapping of βX onto AX given by Lemma 1.1, then $i_A \mid (X^*)^\beta$ is a continuous extension of $f_\beta \mid X^*$ that sends $(X^*)^\beta - X^*$ homeomorphically onto $T - (AX - X)$. Hence $f_\beta \mid X^*$ is a meshing map.

It is obvious that fitting maps are closed under composition. That is, if f is a fitting map of X onto Y, and g is a fitting map of Y onto a space Z, then $g \circ f$ is a fitting map of X onto Z. We conclude this section with an example to show that this need not be the case for meshing maps.

1.11 EXAMPLE. Let Z denote any normal non-locally compact space such that $R(Z)$ is compact. (The space H of Example 1.3 has this property.) Let X denote the sum of two disjoint copies Z_1, Z_2 of Z, let Y denote the quotient space obtained by identifying corresponding points of $R(Z_1)$ and $R(Z_2)$. The quotient mapping f of X onto Y is obviously a fitting map, so by Corollary 1.9, it is a meshing map. Finally, Y admits a quotient mapping g onto Z obtained by identifying those pairs of corresponding points of Z_1 and Z_2 not already identified in passing from X to Y. Again, g is obviously a fitting map, and is a one-one mapping on $R(Y)$, so by Theorem 1.8, it is a meshing map. But $g \circ f$ is the projection map of the product of Z and a compact space consisting of two points onto Z, so by Corollary 1.7, $g \circ f$ is not a meshing map.

2. Fitting and meshing properties.

DEFINITION. *A property $\mathcal{P}$ of topological spaces is called a fitting (respectively meshing) property if whenever f is a fitting (respectively meshing) map of a space X onto a space Y, then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.*

Since every meshing map is a fitting map, every fitting property is a meshing property. After introducing some notation, we will exhibit a meshing property that is not fitting.

For any cardinal number $\mathfrak{m}$, let $\exp \mathfrak{m} = 2^{\mathfrak{m}}$, and for $n = 1, 2, \cdots$, let $\exp^{n+1} \mathfrak{m} = \exp(\exp^n \mathfrak{m})$. We also let $\mathfrak{m}^* = \sum_{n=1}^{\infty} \exp^n \aleph_0$. For any set A, we designate the cardinal number of A by $\mid A \mid$.

For any space X, let $R^1(X) = R(X)$, and for $n = 1, 2, \cdots$, let $R^{n+1}(X) = R(R^n(X))$.

2.1 EXAMPLE. For any space X, let X have property $\mathcal{P}_1$ if there exists a positive integer n such that $| R^n(X) | < \mathfrak{m}^*$. We observe first that $\mathcal{P}_1$ is not a fitting property, for if X is the product of a compact space of power at least $\mathfrak{m}^*$ and the space Q of rational numbers, then the projection mapping of X onto Q is fitting, but for all n, $R^n(X) = X$; thus $| R^n(X) | \geq \mathfrak{m}^*$, while $| R(Q) | = \aleph_0$. We show next that $\mathcal{P}_1$ is a meshing property.

Let f be a meshing map of space X onto a space Y. Recall that if f is any fitting map, then $f[R(X)] = R(Y)$. Since by Lemma 1.4, $f \mid R(X)$ is a fitting map, we obtain that for $n = 1, 2, \cdots$, $f[R^n(X)] = R^n(Y)$, so Y has property $\mathcal{P}_1$ if X has $\mathcal{P}_1$. Assume next that Y has property $\mathcal{P}_1$. Since f is a meshing map, there exist compactifications AX of X, and BY of Y such that $AX - X$ and $BY - Y$ are homeomorphic. Recall also that $R(X)$ is contained in the closure T of $AX - X$ in AX, and the well known fact that if a space U is dense in a space V, then $| V | \leq \exp^2 | U |$. Then $| R(X) | \leq | T | \leq \exp^2 | AX - X | = \exp^2 | BY - Y | \leq \exp^2 | BY | \leq \exp^4 | Y |$. By Lemma 1.4, $f \mid R(X)$ is a meshing map, so applying this latter argument successively to the spaces $R^n(X)$, we obtain $| R^{n+1}(X) | \leq \exp^4 | R^n(Y) |$, $n = 1, 2, \cdots$. Hence X has property $\mathcal{P}_1$, so $\mathcal{P}_1$ is a meshing property.

Any of the properties more usually encountered in general topology that we have examined turn out to be fitting whenever they are meshing. Our next theorem gives a list of such properties. First we recall some definitions for the sake of completeness.

A space is *σ-compact* if it is the union of countably many compact subspaces; *countably compact* if every countable open covering of it has a finite subcover. A space has the *Lindelöf property,* provided every open covering of it has a countable subcover. A collection of subsets of a space is *locally finite* if every point has a neighborhood meeting only finitely many elements of the collection. *A refinement* of a covering $\mathfrak{U}$ of a space is a covering $\mathfrak{V}$ such that every element of $\mathfrak{V}$ is a subset of some element of $\mathfrak{U}$. A space is *paracompact* (respectively *countably paracompact*) if every open (respectively countable open) covering has a locally finite open refinement.

2.2. THEOREM. *The following properties are fitting properties*: (a) *compactness,* (b) *σ-compactness,* (c) *the Lindelöf property,* (d) *countable compactness,* (e) *local compactness,* (f) *paracompactness, and* (g) *countable paracompactness.*

Proof. Let f denote a fitting map of a space X onto a space Y. Let (A_1), $\cdots$, (G_1) denote respectively the assertion that if X has property (a), $\cdots$, (g), then Y has, and let (A_2), $\cdots$, (G_2) denote the assertion obtained from these by interchanging X and Y.

It is clear from the definitions involved (using only the continuity of f) that (A_1), (B_1), (C_1), and (D_1) hold. We next prove

(E_1). For any $y \in Y$, it is clear, since $f^{-1}(y)$ is compact, and X is locally compact, that $f^{-1}(y)$ has a compact neighborhood U. By Lemma 1.2, there is a neighborhood V of y in Y such that $f^{-1}[V] \subset U$. Then V^- is a closed subset of the compact set $f[U]$, and hence is the desired compact neighborhood of y.

(F_1). E. Michael has shown in [13] that any closed continuous image of a paracompact space is paracompact.

(G_1). We use the following characterization of countable paracompactness due to Ishikawa [9]. A space is countably paracompact if and only if for every countable descending chain of closed subsets $\{F_i\}$ with empty intersection, there is a countable descending chain $\{U_i\}$ of open sets whose closures have empty intersection such that $F_i \subset U_i$ for $i = 1, 2, \cdots$.

Let $\{F_i\}$ be any countable descending chain of closed subsets of Y with empty intersection. Then, since f is continuous, $\{f^{-1}[F_i]\}$ is a countable descending chain of closed subsets of X with empty intersection. Since X is countably paracompact, there is a countable descending chain $\{U_i\}$ of open sets such that $U_i \supset f^{-1}[F_i]$ for $i = 1, 2, \cdots$, and such that $\bigcap_{i=1}^{\infty} U_i^-$ is empty. For $i = 1, 2, \cdots$, let $G_i = Y - f[X - U_i]$. Since f is closed, each G_i is an open set containing F_i and G_{i+1}. It suffices to prove that $\bigcap_{i=1}^{\infty} G_i^-$ is empty. Suppose, on the contrary, that there is a $y \in \bigcap_{i=1}^{\infty} G_i^-$. Now for each i, $f^{-1}(y) \subset U_i$, so $\{f^{-1}(y) \cap U_i^-\}$ is a descending chain of compact sets with empty intersection. Hence there is an i_0 such that $U_{i_0}^-$ and $f^{-1}(y)$ are disjoint. That is, $f^{-1}(y)$ is contained in the interior of $X - U_{i_0}$. So, by Lemma 1.2, there is a neighborhood V of y such that $f^{-1}[V] \subset X - U_{i_0}$. But then $V \subset f[X - U_{i_0}]$, so y is not in $G_{i_0}^-$. Hence $\bigcap_{i=1}^{\infty} G_i^-$ is empty, whence Y is countably paracompact.

(A_2) follows immediately from Lemma 1.4, and (B_2) follows immediately from (A_2). We next prove

(D_2). Suppose that X is not countably compact, and let D denote an infinite closed discrete subset of X. Now f is a closed mapping, so since every subset of D is closed, every subset of $f[D]$ is closed. Thus $f[D]$ is closed and discrete. Moreover, since for each $y \in Y$, the set $f^{-1}(y)$ is compact, and hence has only finitely many elements in common with D, the set $f[D]$ must be infinite. Hence Y is not countably compact.

(E_2). If $x \in X$, and V is a compact neighborhood of $f(x)$, then by the continuity of f and Lemma 1.2, $f^{-1}[V]$ is a compact neighborhood of x. Hence if Y is locally compact, then X is locally compact.

It remains to prove (C_2), (F_2), and (G_2). Let $\mathfrak{U}$ denote an arbitrary open covering of X. For each $y \in Y$, there is a finite subfamily $\mathfrak{U}_y$ of $\mathfrak{U}$ that covers the compact set $f^{-1}(y)$, and by Lemma 1.2 there is a neighborhood V_y of y in Y such that $f^{-1}[V_y]$ is contained in the union U_y of the elements of $\mathfrak{U}_y$. If Y is a Lindelöf space, there is a countable subcover $\{V_{y_1}, \cdots, V_{y_n}, \cdots\}$ of the open covering $\{V_y\}$ of Y, so $\bigcup_{i=1}^{\infty} \mathfrak{U}_{y_i}$ is clearly a countable subcover of $\mathfrak{U}$. Hence we have (C_2). If Y is paracompact, the open covering $\{V_y\}$ has a locally finite open refinement $\{W_\alpha\}$. Since $\{W_\alpha\}$ is a refinement of $\{V_y\}$, for each α, there is a $y(\alpha) \in Y$ such that $f^{-1}[W_\alpha] \subset U_{y(\alpha)}$. It is easy to verify that the open covering $\{f^{-1}[W_\alpha] \cap U: U \in \mathfrak{U}_{y(\alpha)}\}$ is a locally finite refinement of $\mathfrak{U}$, so

(F_2) holds. Finally, if Y is countably paracompact, we may assume that $\mathfrak{U}$ is countable. For each $y \varepsilon Y$, let V_y' denote the union of all the open subsets of Y, whose inverse images are contained in U_y. Since a countable set has only countably many finite subsets, there are only countably many distinct elements in the open covering $\{V_y'\}$. The remainder of the proof that X is countably paracompact proceeds as in the paracompact case, so we have (G_2). This completes the proof of the theorem.

The list of properties given in Theorem 2.2 does not exhaust the class of fitting properties. For example, it is clear that *the negation of a fitting* (*or a meshing*) *property is fitting* (*respectively meshing*). Some additional fitting properties may be deduced from work of Hanai [7], who, in addition, has proved some parts of our Theorem 2.2 (with essentially the same argument) under the additional assumption that both X and Y are normal.

It seems appropriate to mention some properties that are not fitting, although they are related to some of those given in Theorem 2.2.

As noted in [18], a closed continuous image of a normal space is normal. It has also been noted in [13] that the joint property of being normal and countably paracompact is carried forward by closed continuous mappings. It is not true, however, that either of these two properties is even a meshing property.

2.3 EXAMPLE. Let $X = W(\omega_1) \times W(\omega_1 + 1)$, and let $Y = W(\omega_1)$, and let f be the projection map of X onto Y. As we know, f is a fitting map, and since X and Y are locally compact, it is also a meshing map. But, as is well known, Y is normal and countably compact (hence countably paracompact), while X is nonnormal [11; 163–164].

This example depends on the fact that if K is compact, and Y is any space, the projection map of $K \times Y$ onto Y is a fitting map. Hence, *if $\mathcal{P}$ is a fitting property, and if a space X has property $\mathcal{P}$, then the product of X with any compact space has property $\mathcal{P}$.*

We cannot, however, replace "fitting" by "meshing" in this last assertion, as can be seen by examining the property $\mathcal{P}_1$ of Example 2.1.

Recall that a space X is called pseudo-compact if $C(X) = C^*(X)$, i.e., if every $\phi \varepsilon C(X)$ is bounded, and that a normal pseudo-compact space is countably compact [8, Theorem 30]. If f is a continuous map (in particular, if f is a fitting map) of a pseudo-compact space X onto a space Y, then Y is pseudo-compact. For if $\phi \varepsilon C(Y)$ is unbounded, then the composite function $\phi \circ f$ is an unbounded element of $C(X)$. On the other hand, pseudo-compactness is not even a meshing property.

2.4 EXAMPLE. Let $Y = W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}$, and let $X = W(\omega_1 + 2) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\} - \{(\omega_1 + 1, \omega_0)\}$. It is well known that Y is pseudo-compact but not countably compact [8; 69]. But X is the topological sum of Y and the closed countable discrete subspace $\{(\omega_1 + 1, n) : n < \omega_0\}$ of X, so X is not pseudo-compact. Let f denote the mapping of X onto Y defined by letting $f(y) = y$ for all $y \varepsilon Y$, and by letting $f(\omega_1 + 1, n) = (\omega_1, n)$ for all $n < \omega_0$. Clearly f is a fitting map. (Moreover, for each $y \in Y$, the set $f^{-1}(y)$ has at most two points). Since X and Y are locally compact, f is a meshing map, so pseudo-compactness is not a meshing property.

2.5 THEOREM. *If $\mathcal{P}$ is a fitting (respectively meshing) property that is inherited by subspaces of spaces having $\mathcal{P}$ that are both open and closed, then $\mathcal{P}$ is inherited by closed (respectively closed locally compact) subspaces.*

Proof. Suppose that Y is a space satisfying $\mathcal{P}$, and let A be a closed (respectively closed locally compact) subspace of Y. Let X denote the sum of disjoint copies Y_1 of Y and A_1 of A. Let f denote the mapping that sends each point of X onto the point of X from which it was obtained. Then f is clearly a fitting map. Moreover, if A is locally compact, then $R(X) \subset Y_1$, so $f \mid R(X)$ is a one-one mapping, and we may conclude from Theorem 1.8 that f is meshing. In any case A_1 is an open and closed subset of X, so from the hypothesis, A has property $\mathcal{P}$.

Theorem 2.5 serves to clarify Example 2.4. Obviously, any open and closed subspace of a pseudo-compact space is pseudo-compact, but a closed locally compact subspace need not be, so by Theorem 2.5, pseudo-compactness is not a meshing property.

We observe also that *if $\mathcal{P}$ is a meshing property inherited by subspaces that are open and closed, and is possessed by some nonempty space, then $\mathcal{P}$ is possessed by all compact spaces.* For, if Y is any nonempty space satisfying $\mathcal{P}$, and K is any compact space, then Y is the image of the topological sum X of K and Y under the fitting map f that coincides with the identity on Y and sends K to some fixed point of Y. Since f is a one-one mapping on $R(X) \subset Y$, by Theorem 1.8, f is a meshing map, so X and hence K has property $\mathcal{P}$.

DEFINITION. *A space X is said to have property $\mathcal{P}$ at infinity if X^* has property $\mathcal{P}$.*

2.6 THEOREM. *The following conditions on a space X are equivalent if and only if $\mathcal{P}$ is a meshing property.*

(a) *X has property $\mathcal{P}$ at infinity.*

(b) *For some compactification AX of X, the space $AX - X$ has property $\mathcal{P}$.*

(c) *For any compactification AX of X, the space $AX - X$ has property $\mathcal{P}$.*

Proof. Suppose first that $\mathcal{P}$ is a meshing property. For any compactification AX of X, the restriction to $\beta X - X$ of the mapping i_A of βX onto AX given by Lemma 1.1 is a meshing map by Corollary 1.10. Hence the equivalence of (a), (b) and (c) follows immediately from the cited lemma.

Suppose next that $\mathcal{P}$ is not a meshing property. Then there exists a meshing map f of a space X onto a space Y such that one of X, Y has property $\mathcal{P}$, while the other does not. Also there exist compactifications AX of X and BY of Y, and a continuous extension $\bar{f}$ of f over AX onto BY such that $\bar{f} \mid (AX - X)$ is a homeomorphism onto $BY - Y$. Let $Z = AX \times [0, 1] - (X \times \{1\})$. Then $AZ = AX \times [0, 1]$ is a compactification of Z such that $AZ - Z$ and X are homeomorphic. Let Z_1 denote the quotient space of AZ by the decomposition whose elements are the single points of Z, and the sets $f^{-1}(p) \times \{1\}$, $p \in Y$. It is easily verified that the quotient mapping is closed, so the quotient space BZ is a compactification of Z such that $BZ - Z$ and Y are homeomorphic. Hence (b) and (c) (applied to the space Z) fail to be equivalent. This completes the proof of the theorem.

All of the fitting (hence meshing) properties of Theorem 2.2 are inherited by closed subspaces, but for their negations, this is not true. Under this stronger hypothesis, the following improvement of Theorem 2.6 is valid. *If $\mathcal{P}$ is a meshing property inherited by closed subspaces, and if X is a subspace (not necessarily dense) of a compact space Z such that $Z - X$ has property $\mathcal{P}$, then X has $\mathcal{P}$ at infinity.*

2.7 THEOREM. (a) *A property $\mathcal{P}$ is fitting if and only if $\mathcal{P}$ at infinity is a fitting property.*

(b) *A property $\mathcal{P}$ is inherited by open-closed subsets if and only if $\mathcal{P}$ at infinity is inherited by open-closed subsets.*

Proof. Note first that if $\mathcal{P}$ is fitting, then $\mathcal{P}$ at infinity is fitting by Lemma 1.5. Next, suppose that $\mathcal{P}$ is inherited by open-closed subsets, and let A be an open-closed subset of a space X having property $\mathcal{P}$ at infinity. Since every $\phi \in C^*(A)$ has a continuous extension over X, it follows that the closure A^β of A in βX is βA. Thus $A^* = A^\beta - A$ is an open and closed subset of X^*, and hence has property $\mathcal{P}$, i.e., A has $\mathcal{P}$ at infinity.

Before proceeding further, we give a preliminary construction. For any space X, let ω_α be an uncountable regular initial ordinal such that $| \beta X | < | W(\omega_\alpha) |$, let $X_1 = \beta X \times W(\omega_\alpha + 1) - (X \times \{\omega_\alpha\})$, and let $X_2 = \beta X_1 \times W(\omega_{\alpha+1} + 1) - (X_1 \times \{\omega_{\alpha+1}\})$. Now for any uncountable regular initial ordinal ω_β, every $\phi \in C(W(\omega_\beta))$ is known to be eventually constant, (cf. e.g. [8]). Hence since $| \beta X | < | W(\omega_\alpha) |$, every $\phi \in C^*(X_1)$ has a continuous extension over $\beta X \times W(\omega_\alpha + 1)$, so $\beta X_1 = \beta X \times W(\omega_\alpha + 1)$. Similarly $\beta X_2 = \beta X_1 \times W(\omega_{\alpha+1} + 1)$. Thus X_1^* is homeomorphic with X, and X_2^* is homeomorphic with X_1. It follows that $X_2^{**} = \beta X_2^* - \beta X_2^*$ and X are homeomorphic.

Now, suppose that $\mathcal{P}$ at infinity is a fitting property, let f denote a fitting map of a space X onto a space Y, and let X_2 and Y_2 denote the spaces obtained by the construction above. We define a mapping f_β^* over βX_2 onto βY_2 by letting $f_\beta^*(p, \alpha_1, \alpha_2) = (f_\beta(p), \alpha_1, \alpha_2)$ for all $p \in \beta X$, $\alpha_1 \leq \omega_\alpha$ and $\alpha_2 \leq \omega_{\alpha+1}$. (As usual, f_β denotes the continuous extension of f over βX onto βY.) Let $f' = f_\beta^* \mid X_2$. Since f is a fitting map, f' is a fitting map.

Now since $\mathcal{P}$ at infinity is a fitting property, $\mathcal{P}$ at infinity at infinity is a fitting property by the first part of the proof. Hence X_2^{**} has $\mathcal{P}$ if and only if Y_2^{**} has $\mathcal{P}$. But by our construction X_2^{**} and X, respectively Y_2^{**} and Y, are homeomorphic. Hence $\mathcal{P}$ is a fitting property.

Finally, suppose that $\mathcal{P}$ at infinity is inherited by open-closed subspaces. Using the first part of our proof, we conclude that $\mathcal{P}$ at infinity at infinity is inherited by open-closed subspaces. Now let A be an open-closed subspace of a space having property $\mathcal{P}$. It is easily seen that A_2 is homeomorphic to an open-closed subspace of X_2, and hence that A_2^{**} is homeomorphic to an open-closed subspace of X_2^{**}. Noting again that X_2^{**} and X, respectively A_2^{**} and A, are homomorphic, we conclude that A has property $\mathcal{P}$. This completes the proof of the theorem.

We conclude this section with two theorems whose proofs we find convenient to give together.

2.8 THEOREM. *If $\mathcal{P}$ is a meshing property, then*

(a) *$\mathcal{P}$ at infinity is a meshing property, and*

(b) *$\mathcal{P}$ at infinity is inherited by closed subspaces if and only if $\mathcal{P}$ is inherited by closed subspaces.*

2.9 THEOREM. *Let $\mathcal{P}$ be a meshing property. Then a space X has property $\mathcal{P}$ at infinity at infinity if and only if $R(X)$ has property $\mathcal{P}$.*

Proof. Part (a) of 2.8 follows immediately from Corollary 1.10. Next, we prove 2.9.

By definition, X has property $\mathcal{P}$ at infinity at infinity if and only if X^* has property $\mathcal{P}$ at infinity. By 2.8 (a), this latter property is meshing, so by Theorem 2.6, X has property $\mathcal{P}$ at infinity at infinity if and only if $(X^*)^\beta - X^* = R(X)$ has property $\mathcal{P}$.

It remains to prove 2.8 (b). Suppose first that $\mathcal{P}$ is inherited by closed subspaces, and let A be a closed suspace of a space X having $\mathcal{P}$ at infinity. Then X^* has property $\mathcal{P}$, and $A^\beta - A$ is a closed subspace of X^*. So by hypothesis $A^\beta - A$ has property $\mathcal{P}$. Thus, by Theorem 2.6, A has $\mathcal{P}$ at infinity.

Suppose conversely that $\mathcal{P}$ at infinity is inherited by closed subspaces, and let A be a closed subspace of a space satisfying $\mathcal{P}$. Let H denote the space defined in Example 1.3, and let $M = \beta X \times H - \{(p, \omega_1): p \in X^*\}$. If $D = H - \{\omega_1\}$, then $M = (\beta X \times D) \cup (X \times \{\omega_1\})$. It is clear that $R(M) = X \times \{\omega_1\}$, so by 2.9, M has property $\mathcal{P}$ at infinity at infinity. Let $N = (A^\beta \times D) \cup (A \times \{\omega_1\})$. Then N is a closed subset of M, so by 2.8 (a), and the part of 2.8 (b) just proved, N has property $\mathcal{P}$ at infinity at infinity. But $R(N) = A \times \{\omega_1\}$, so applying 2.9 again, we conclude that A has property $\mathcal{P}$. Hence we have proved Theorems 2.8 and 2.9.

3. **Some properties at infinity.** In this section, we discuss in varying detail the fitting properties given in Theorem 2.2, and give some pertinent examples.

Compactness at infinity is equivalent, of course, to local compactness.

A space X is locally compact at infinity if and only if X is compact at infinity at infinity, so by Theorem 2.9, we have

3.1 THEOREM. *A space X is locally compact at infinity if and only if $R(X)$ is compact.*

Čech has discussed the concept of σ-compactness at infinity (using the term topologically complete) and obtained the equivalences of our Theorem 2.6 for this special case [3; 837 ff.]. We have the following simple result.

3.2 THEOREM. *A space X is σ-compact at infinity if and only if whenever X is a dense subspace of a space Y, X is a G_δ in Y.*

Proof. If X is a G_δ in βX, then X^* is an F_σ in βX, i.e., X is σ-compact at infinity. Conversely suppose X is σ-compact at infinity, and is dense in a space Y. By Theorem 2.6, $\beta Y - X$ is σ-compact, and hence is an F_σ-subset of βY, so X is a G_δ in βY, and thus is a G_δ in Y.

We recall the well-known result that a metrizable space is a G_δ in every (compact) space containing it if and only if it is completely metrizable (cf. e.g., [3; 838] or [11; 207]). In other words, *a metrizable space is σ-compact at infinity if and only if it is completely metrizable.* As we will see later (Corollary 3.7), every metrizable space is Lindelöf at infinity.

3.3 THEOREM. *If a space X is both σ-compact, and σ-compact at infinity, then $X - R(X)$ is an (open) dense subset of X.*

Proof. Suppose on the contrary that $R(X)$ contains a nonempty open subset U of X. Since X is completely regular, U contains an open F_σ-subset V of X. Since X is σ-compact at infinity, X, and hence the open subset V of X, is a G_δ in βX. Recall that every compact space is of the second category in itself [11; 200]. Now V is also a G_δ in V^β, and $V^\beta - V$ is of the first category in V^β, so V must be of the second category in V^β. But, since X is σ-compact, V is the union of countably many compact sets, each of which is nowhere dense since $V \subset U$. This contradiction yields the theorem.

Examination of the space of rational, respectively irrational numbers (regarded as subspaces of the one-point compactification of the real line) shows that the conclusion of Theorem 3.3 need not follow if one assumes merely that X is σ-compact, respectively σ-compact at infinity. Also, the converse of Theorem 3.3 is false in the following strong sense; if $X = H$, the space defined in Example 1.3; then $H - R(H) = H - \{\omega_1\}$ is dense, but H is neither σ-compact, nor σ-compact at infinity. We also note that one cannot conclude from the hypothesis of Theorem 3.3 that X must contain a dense locally compact and σ-compact subspace. For example, let $A = W(\omega_1 + 1)$, let B denote the subspace of $W(\omega_0^2 + 1)$ obtained by deleting all the limit ordinals except ω_0^2, and let X denote the space obtained from the topological sum of A and B (regarded as disjoint point-sets) by identifying $\omega_1 \in A$ with $\omega_0^2 \in B$. Then X is locally compact at all but one point of X, so $X - R(X)$ is not σ-compact. However, for metrizable spaces, we have:

3.4 COROLLARY. *If X is a σ-compact completely metrizable space, then $X - R(X)$ is an (open) dense σ-compact subset of X.*

Proof. As remarked above, a completely metrizable space is σ-compact at infinity. So, by Theorem 3.3, $X - R(X)$ is dense, and being an open subset of a σ-compact metric space, is σ-compact.

Finally, it follows immediately from Theorem 3.3 that *a homogeneous σ-compact space is σ-compact at infinity if and only if it is locally compact.*

We recall a few definitions. For a closed subset F of a space X, a *basis at* F is a collection $\{U_\alpha\}$ of neighborhoods of F such that every open set containing F contains some U_α. The *character* of F is the least cardinal number of a basis at F. If X is a subspace of a space Y, we denote the character of F as a subset of X, respectively Y, by $K_X(F)$, $K_Y(F)$.

3.5 LEMMA. *If F is a compact subset of a dense subspace X of a space Y, then $K_X(F) = K_Y(F)$.*

Proof. If $\{U_\alpha\}$ is a basis at F as a subset of Y, then $\{U_\alpha \cap X\}$ is a basis at F as a subset of X, so $K_X(F) \leq K_Y(F)$. Conversely, if $\{V_\alpha\}$ is a basis at F as a subset of X, and if for each α, U_α denotes the closure of V_α in Y, then $\{U_\alpha\}$ is a basis at F as a subset of Y. For, if U is an open subset of Y containing F, then since F is compact, U contains a closed neighborhood U' of F in Y. By hypothesis, $U' \cap X$ contains some V_α, so $U_\alpha \subset U' \subset U$. Thus $\{U_\alpha\}$ is a basis at F in Y. Hence $K_X(F) \geq K_Y(F)$, and the lemma is proved.

3.6 THEOREM. *A space X is Lindelöf at infinity if and only if every compact subset of X is contained in a compact set of countable character.*

Proof. Suppose that X^* is Lindelöf, and let F be a compact subset of X. For each $p \in X^*$, there is an open F_σ-subset U_p of βX containing p and disjoint from F. (For, there is a $\phi \in C(\beta X)$ such that $\phi[F] = 0$, and $\phi(p) > 0$.) The open covering $\{U_p : p \in X^*\}$ of X^* has a countable subcover. If U is the union of its elements, then U is an open F_σ-subset of βX disjoint from F. Then $\beta X - U$ is a closed G_δ-subset of βX containing F. It is easily seen that any closed G_δ in a compact space has countable character. So, by Lemma 3.5, $\beta X - U$ has countable character as a subset of X.

Conversely, suppose that every compact subset of X is contained in a compact set of countable character, and let $\{V_\alpha\}$ be any open covering of X^*. For each α, let U_α be an open subset of βX such that $V_\alpha = U_\alpha \cap X^*$. The complement F of the union of all the elements of $\{U_\alpha\}$ is a compact subset of X. Let Z be a compact set of countable character (as a subset of X) containing F. By Lemma 3.5, Z has countable character as a subset of βX, and hence $\beta X - Z$ is a σ-compact subset of βX containing X^*. So $\{U_\alpha\}$, and hence $\{V_\alpha\}$, has a countable subcover.

3.7 COROLLARY. *Every metrizable space is Lindelöf at infinity.*

The class of metrizable spaces, and the class of spaces that are Lindelöf at infinity share the following property.

3.8 THEOREM. *A countable product of spaces each Lindelöf (σ-compact) at infinity is Lindelöf (σ-compact) at infinity.*

Proof. Denote the spaces by X_i, $i = 1, 2, \cdots$, and denote the product of their Stone-Čech compactifications βX_i by BX. For each i, let R_i denote the product of X_i^* with all the βX_j with $j \neq i$. Then each R_i, being the product of a Lindelöf (σ-compact) space and a compact space, is Lindelöf (σ-compact). But $BX - X = \bigcup_{i=1}^{\infty} R_i$, and hence is a Lindelöf (σ-compact) space. Thus, by Theorem 2.6, the product of the X_i's is Lindelöf (σ-compact) at infinity.

An arbitrary product of spaces Lindelöf at infinity need not be Lindelöf at infinity; for example consider an uncountable product of countable discrete spaces.

3.9 THEOREM. *If X is Lindelöf at infinity, and every subspace of X is Lindelöf, then every subspace of X is Lindelöf at infinity.*

Proof. If A is any subspace of X, then $A^{\beta} - A$ is the union of $A^{\beta} \cap X^*$ and $A^{-} - A$. The first is a Lindelöf space since it is a closed subspace of the Lindelöf space X^*, while the second is a Lindelöf space by hypothesis. Hence $A^{\beta} - A$ is a Lindelöf space, so by Theorem 2.6, A is Lindelöf at infinity.

3.10 THEOREM. *Let $\mathcal{P}$ be the property of being compact, σ-compact, Lindelöf, countably compact, paracompact, countably paracompact, or normal. Then an arbitrary topological sum of spaces having property $\mathcal{P}$ at infinity has property $\mathcal{P}$ at infinity.*

Proof. Denote the spaces in question by $\{X_\alpha\}$, their topological sum by X, the sum of their Stone-Čech compactifications by Y, and let $\alpha Y = Y \cup \{p\}$ denote the one-point compactification of Y. Note that $\alpha Y - X$ consists of the union of all the X_α^*'s and $\{p\}$. For all the properties $\mathcal{P}$ above except σ-compactness and normality, the theorem follows easily from the fact that any neighborhood of p in $\alpha Y - X$ contains all by finitely many of the X_α^*'s, the definition of $\mathcal{P}$, and Theorem 2.6.

Suppose next that each X_α^* is σ-compact, and write $X_\alpha^* = \bigcup_{i=1}^{\infty} K_{i\alpha}$, where each $K_{i\alpha}$ is compact. For $i = 1, 2, \cdots$, let $K_i = \bigcup_\alpha (K_{i\alpha} \cup \{p\})$. Clearly $\alpha Y - X = \bigcup_{i=1}^{\infty} K_i$, and each K_i is compact, so by Theorem 2.6, X is σ-compact at infinity.

It was noted in Example 2.3, that normality is not a meshing property, so more care must be taken in this case. Note first that $\beta X = \beta Y$, since βY is a compactification of X, and since every $\phi \in C^*(X)$ has a continuous extension over βY. Then X^* is the union of all the spaces X_α^* and the compact space $Y^* = \beta Y - Y$. If F, G are disjoint closed subsets of X^*, let $F_1 = F \cap Y^*$, and $G_1 = G \cap Y^*$. Let U_1 be an open neighborhood of F_1 in X^* whose closure is disjoint from G, and let V_1 be an open neighborhood of G in X^* whose closure is disjoint from U_1^-. Since $(F \cup G) - (U_1^- \cup V_1^-)$ is contained in the union of finitely many of the X_α^*'s, the remainder of the proof that F and G have disjoint neighborhoods is routine.

The only fitting property of Theorem 2.2 not mentioned in Theorem 3.10 is local compactness. This is because a countable sum of spaces locally compact at infinity need not be locally compact at infinity. For example if S is the subspace of the Euclidean plane consisting of $\{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\} - \{(1, y): y \neq 0\}$, then S is obviously locally compact at infinity. But, by Theorem 3.1, the topological sum of countably many distinct copies of S is not locally compact at infinity.

3.11 THEOREM. *Let $\mathcal{P}$ be a meshing property inherited by closed subspaces, let X be a space having property $\mathcal{P}$, let Y be a compactification of X, and let Z be a compactification of $Y - X$. Then Z has property $\mathcal{P}$. In particular, $X^{**} = \beta X^* - X^*$ has property $\mathcal{P}$.*

Proof. In view of Theorem 2.6, it suffices to show that X^{**} has property $\mathcal{P}$. Now $(X^*)^\beta - X^* = R(X)$ is a closed subset of X. Since $\mathcal{P}$ is inherited by closed subspaces, $(X^*)^\beta - X^*$ has property $\mathcal{P}$, so by Theorem 2.6, X^* has property $\mathcal{P}$ at infinity. That is, X^{**} has property $\mathcal{P}$.

For any space X, let $X^{(1)} = X^*$, and let $X^{(n+1)} = (X^{(n)})^*$, for $n = 1, 2, \cdots$.

3.12 COROLLARY. *If $\mathcal{P}$ is a meshing property inherited by closed subspaces, and if X has both property $\mathcal{P}$ and property $\mathcal{P}$ at infinity, then $X^{(n)}$ has property $\mathcal{P}$ for $n = 1, 2, \cdots$.*

An interesting hierarchy of Lindelöf space is created if one applies Corollary 3.12 to the space of rational numbers.

We have little to say about paracompactness, or countable paracompactness at infinity; all positive results in this direction are concerned with linearly ordered spaces.

It is known that every linearly ordered space is countably paracompact [1] [5, Theorem 9.5]. Unfortunately, not every subspace of a linearly ordered space is a linearly ordered space; for example the subspace of the real line consisting of -1 and the strictly positive real numbers is not a linearly ordered space. However, it is easy to conclude that *every subspace of a linearly ordered space is countably paracompact.* For as is easily verified, every subspace of a space is countably paracompact (respectively paracompact) if and only if every open subspace is. (In the paracompact case, this is shown in [4].) Now every open subspace U of a linearly ordered space is a sum of maximal disjoint open intervals, and every open interval is a linearly ordered space in the induced topology, and hence, by the above, is countably paracompact. Thus U, being a topological sum of countably paracompact spaces, is countably paracompact.

By way of application of the above, we have:

3.13 THEOREM. (a) *Every linearly ordered space is countably paracompact at infinity.*

(b) *Every linearly ordered space satisfying the first axiom of countability is paracompact at infinity.*

Proof. (a) Every linearly ordered space X is a dense subspace of the compact space X^+ obtained by adding endpoints (if necessary) to its Dedekind completion. By the above $X^+ - X$ is countably paracompact, so by Theorem 2.6, X is countably paracompact at infinity.

(b) Let $\{V_\alpha\}$ denote any open covering of $X^+ - X$. For each α, choose an open subset U_α of X^+ such that $V_\alpha = U_\alpha \cap (X^+ - X)$. The family $\{U_\alpha\}$ covers an open subset U of X^+, and U is the sum of a family $\{I_\beta\}$ of maximal disjoint open intervals. Since X satisfies the first axiom of countability, each I_β is σ-compact and hence paracompact, so U is paracompact. Thus $\{U_\alpha\}$ has a locally finite open refinement $\{W_\gamma\}$, and $\{W_\gamma \cap (X^+ - X)\}$ is clearly a locally finite open refinement of $\{V_\alpha\}$.

The remainder of this section will be devoted to giving pertinent examples.

From the above one may also easily infer that *every subspace of a compact linearly ordered space satisfying the first axiom of countability is paracompact.* (For, every open subset of such a space is a sum of σ-compact spaces.)

We cannot conclude that every subspace of a compact linearly ordered space satisfying the first axiom of countability is Lindelöf. For, let L denote the set of all real numbers (x, y) with $0 \leq x \leq 1, 0 \leq y \leq 1$, lexicographically ordered, with the interval topology. As is well known [11; 164], L is compact and satisfies the first axiom of countability. But $\{(x, \frac{1}{2}): 0 \leq x \leq 1\}$ is an uncountable discrete subset of L.

As we will see below, there exist compact spaces satisfying the first axiom of countability that contain nonnormal subspaces, and spaces satisfying the first axiom of countability that are not even normal at infinity (Examples 3.16 and 3.17).

3.14 Example. *A countable (hence σ-compact) space that is not paracompact at infinity.* Let N denote the countable discrete space, let p be a point of N^*, and let X denote the subspace $N \cup \{p\}$ of βN. Then X is a countable space, but $X^* = N^* - \{p\}$ is not paracompact. For, if X^* were paracompact, then being locally compact, it would be a sum of locally compact σ-compact subspaces A_α [2; 107]. Then p must be a limit point in N^* of at least two A_α's, or else p would be a G_δ in N^*, and hence a G_δ in βN. But every closed G_δ of βN contained in N^* has power $\exp^2 \aleph_0$ [8, Theorem 49], so this is impossible. Choose two such A_α's and call them A_1, A_2. By [6, Theorems 2.6 and 2.7], every bounded continuous real-valued function on an open F_σ-subset of N^* has a continuous extension over N^*. But the function $\phi \in C^*(A_1 \cup A_2)$ such that $\phi[A_1] = 0$ and $\phi[A_2] = 1$ has no continuous extension over p. We conclude from this contradiction that X^* is not paracompact.

3.15 Example. *A paracompact space that is paracompact at infinity, but not Lindelöf at infinity.* It suffices to take X^*, where X is the product of an uncountable discrete space and the space of rational numbers.

3.16 Example. *A nonnormal space satisfying the first axiom of countability that is Lindelöf at infinity.* Let S denote a linearly ordered space whose order type is $2 \cdot \lambda$, where λ is the order type of the real line R. That is, S is obtained by R by replacing each $r \varepsilon R$ with two copies r_1, r_2 of itself, such that r_2 is an immediate successor of r_1. Let $S_2 = \{r_2 \varepsilon S: r \varepsilon R\}$ in the relative topology induced by S. A basic neighborhood of r_2 in S_2 is a half-open interval, closed at r_2, and with r_2 as left end point. In [14], Sorgenfrey constructed an hereditarily Lindelöf space whose product with itself is not normal. It is easily seen that S_2 and this latter space are homeomorphic. Let S^+ denote the compactification of S obtained by adding a left and a right end point to S. Then S_2 is dense in S, and $S^+ - S_2$ and S_2 are homeomorphic, so by Theorem 2.6, S_2 is Lindelöf at infinity.

Let $X = S_2 \times S_2$. By Theorem 3.8, X is Lindelöf at infinity, but as noted above, X is not normal.

We remark also that the space S_2 is homeomorphic to the subspace $\{(x, 1): 0 < x < 1\}$ of the space L constructed just after Theorem 3.13.

3.17 Example. *A Lindelöf space satisfying the first axiom of countability that is not normal at infinity.* Using the notation of Example 3.16, let $Y = S^+ \times S^+ - S_2 \times S_2$, then as noted above, Y is a Lindelöf space satisfying the first axiom of countability, but since $S^+ \times S^+ - Y = S_2 \times S_2$ is not normal, and since Y is dense in $S^+ \times S^+$, the space Y^* cannot be normal. (For, recall that a closed continuous image of a normal space is normal [18], and that $S_2 \times S_2$ is the image of Y^* under a fitting map by Lemma 1.1.)

Note also that Y is a nonnormal subspace of $S^+ \times S^+$.

4. Appendix. In this section, we examine the significance of our hitherto standing hypothesis that all spaces are completely regular. It turns out that

4.1. *Neither the property of being Hausdorff, nor the property of being completely regular is preserved by fitting maps.*

4.2. *Among Hausdorff spaces, regularity is a fitting property, while complete regularity is not; in fact, if f is a fitting map of a Hausdorff space X onto a Hausdorff space Y, either X or Y may be completely regular without the other being completely regular.*

A single example will establish 4.1. Consider a space X whose points are the real numbers in $[0, 1)$, and two additional points, called $1'$ and 1^*. The subset $[0, 1)$ is open, and carries its usual topology. A basic neighborhood of $1'$ consists of $1'$ itself together with an open interval $(1 - \epsilon, 1)$ for some $\epsilon > 0$, and a basic neighborhood of 1^* may be obtained by replacing $1'$ by 1^* in the above. Let Y denote the space $[0, 1]$ in its usual topology. Then Y is a compact (Hausdorff) space, and hence is completely regular. But the mapping defined by sending $1'$ and 1^* onto 1, and leaving all other points fixed is clearly a fitting map.

It is shown in [11; 148] that if $f: X \to Y$ is fitting, and X is regular or Hausdorff, then so is Y. Conversely, suppose that $f: X \to Y$ is fitting, Y is regular, and X is Hausdorff. For any $x \varepsilon X$, and any closed set F not containing x, consider first the compact set $F \cap f^{-1}(f(x))$. Since X is a Hausdorff space, there is an open set U containing this set whose closure does not contain x. Then $F - U$

is a closed set disjoint from $f^{-1}(f(x))$. Hence $f[F - U]$ is a closed set not containing $f(x)$. Since Y is regular, there is an open V^- containing $f[F - U]$ whose closure does not contain $f(x)$. Then the set $U \cup f^{-1}[V]$ is an open subset of X containing F whose closure does not contain x. Thus X is regular, and we have the first part of 4.2.

Essentially Tychonoff's original construction [16] of a regular space Y that is not completely regular exhibits Y as the image under a fitting map of a completely regular space, as we will see shortly.

Let T denote the compact space $W(\omega_1 + 1) \times W(\omega_1 + 1)$, and let $B = T - \{(\omega_1, \omega_1)\}$. It is well known, and easily verified by a cofinality argument, that $\beta B = T$. In particular, if $E_- = \{(\alpha, \omega_1): \alpha < \omega_1\}$, and $E_+ = \{(\omega_1, \alpha) : \alpha < \omega_1\}$, then E_- and E_+ are not completely separated, i.e., there is no $\phi \in C(B)$ such that $\phi[E_-] = 0$, and $\phi[E_+] = 1$, for no such ϕ could have a continuous extension over T. More generally, every $\psi \in C(T)$ is constant on a neighborhood of (ω_1, ω_1). We will refer to the sets E_- and E_+ as the *edges* of B.

For each $n \geq 1$, let T^n and B^n denote respectively distinct copies of the spaces T and B, and denote the edges of B^n by E_-^n and E_+^n. Let W denote the topological sum of all the spaces T^n, let $\alpha W = W \cup \{p\}$ denote the one point compactification of W, and finally let $X = \alpha W - \cdot\{(\omega_1, \omega_1)_n : n = 1, 2, \cdots\}$. It is obvious that X is completely regular.

Let Y denote the quotient space obtained from the decomposition of X whose elements are as follows: for each n and α, the pair of points $(\omega_1, \alpha)_n$ of E_+^n and $(\alpha, \omega_1)_{n+1}$ of E_-^{n+1}; for every other point x of X, the single point x. It is easily checked that the quotient mapping $f : X \to Y$ is a fitting map. As noted above, Y must be regular and Hausdorff. Now, each $\psi \in C(Y)$ must be constant on the image under f of a cofinal subset of each of the edges of each B^n. Since $W(\omega_1)$ has no countable cofinal subset, and by dint of the identifications made, there is an $\alpha_0 < \omega_1$; and a real number r such that $\psi(y) = r$ for all y in $f[\{(\alpha, \omega_1)_n \in E_-^n : \alpha > \alpha_0\}]$ or $f[\{(\omega_1, \alpha)_n \in E_+^n : \alpha > \alpha_0\}]$, for $n = 1, 2, \cdots$. Hence $f(p) = r$ as well. Thus p and the closed set E_-^1 not containing it fail to be completely separated, so Y is not completely regular.

We next exhibit a fitting map of a noncompletely regular space upon a completely regular quotient space. First, for $n = 1, 2, \cdots$, let S^n denote the image under f of the sum P^n of the spaces $B^1, \cdots, B^n$. Note that each S^n is a completely regular subspace of the regular space Y constructed above. Let E_0^n denote the image of E_-^1 under f (in S^n), and for $i = 1, 2, \cdots, n$, let E_i^n denote the image of E_+^i under f, in S^n. Note that if $i \neq j$, then E_i^n and E_j^n are not completely separated. Now for each $n \geq 1$, and $i \leq n$, and each $k \geq 1$, let $V_k(E_i^n)$ be the open subset of S^n which is the image under f of the subset of P^n consisting of all B^j satisfying $-k < j - i < k + 1$ (and $1 \leq j \leq n$), less the edges E_{i-k}^n and E_{i+k+1}^n (if they exist). Observe that the closure of $V_k(E_i^n)$ is contained in $V_{k+1}(E_i^n)$ for all $k = 1, 2, \cdots, n - 1$.

Let X_1 denote the topological sum of all the spaces S^n. The desired space X' will consist of X_1 and a compact set X_2 which we will construct next.

Let F denote the set of all functions θ defined on the set N of positive integers and taking values in the set of nonnegative integers such that $\theta(n) \leq n$ for all $n \in N$. The set F becomes a partially ordered set if we let $\theta \leq \theta'$ mean $\theta(n) \leq \theta'(n)$ for all $n \in N$. Let G be a maximal subset of F satisfying (a) G is totally ordered, and (b) for any θ, θ' in G with $\theta' > \theta$, the function $\theta' - \theta$ assumes no value infinitely often. Let X_2 denote the linearly ordered space obtained by adding end points to the Dedekind completion of G. As is well known, (cf., e.g. [5]) X_2 is compact and connected.

Let $X' = X_1 \cup X_2$, and let X_1 be open in X'. It remains to define neighborhoods in X of points of X_2. For each $x \in X_2$, we give a neighborhood basis consisting of sets $U(a, b, n_0, \phi)$, where $a, b \in X_2$ are such that $[a, b]$ is a closed neighborhood of x in X_2, n_0 is a nonnegative integer, and ϕ is a function on N to the set of nonnegative integers such that $\phi(n) \to \infty$ as $n \to \infty$. The set $U(a, b, n_0, \phi)$ consists of (1) all the points of X_2 in $[a, b]$, and (2) for all $\theta \in G \cap [a, b]$, the union of all the sets $V_{\phi(n)}(E^n_{\theta(n)})$ for which $n > n_0$.

It is not difficult to prove that every continuous real-valued function on X' is constant on X_2, so X' is not completely regular. However, X' is a Hausdorff space. For, obviously any pair of points of X' not both in X_2 have disjoint neighborhoods. Consider two points a, b of X_2 and suppose that $a < b$. Then there are $\theta, \theta' \in G$ which that $a < \theta < \theta' < b$ in X_2. By definition of G, there is an $n_0 \in N$ such that $n > n_0$ implies $\theta'(n) > \theta(n) + 4$. If $\phi(n)$ is defined as 0 for $n \leq n_0$, and $\theta'(n) - \theta(n) - 2$ for $n > n_0$, and we let l and r denote respectively the left and right-hand end points of X_2, then $U(l, \theta, n_0, \phi)$ and $U(\theta', r, n_0, \phi)$ are disjoint neighborhoods of a and b respectively. So X' is a Hausdorff space.

Let Y' be the quotient of X by the decomposition of X whose elements are the compact set X_2, and all the single points of X_1. Evidently the quotient mapping is a fitting map, and it remains only to show that Y' is completely regular. This will follow once we show that every neighborhood V of X_2 in X contains all but finitely many of the completely regular subspaces S^n.

Since X_2 is compact, V contains some $U = U(l, r, m, \phi)$. Suppose U fails to contain S^n for infinitely many $n \in N$. Then there is a sequence σ of edges E^n_j disjoint from U for arbitrarily large n. Let L be the set of all θ in G such that for infinitely many E^n_j in σ, we have $j > \theta(n)$, and let α denote the least upper bound in X_2 of L. Unless α is l or r, there are θ_1 in L and $\theta_2 > \alpha$ in G such that $\theta_2 - \theta_1 < \phi$, so U must contain infinitely many of the E^n_j's. Similar arguments show that $\alpha = l$ or $\alpha = r$ are likewise impossible. Hence no such sequence exists, and U contains infinitely many of the spaces S^n. This completes the proof of 4.2.

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