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# Review: J.M. Aarts and T. Nishiura, Dimension and Extensions (Amsterdam, London, New York, and Tokyo, 1993)

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# BOOK REVIEW

*Dimension and extensions*, by J. M. Aarts and T. Nishiura. North-Holland Math.
Library, Amsterdam, London, New York, and Tokyo, 1993, xii + 331 pp.,
$106.50. ISBN 0 444 89740 2

Johann de Groot would have loved this book! In his 1942 doctoral dissertation
he considered the problem of compactifying a (separable metrizable) space $X$ by
adding a metrizable remainder of minimal dimension. This latter dimension is called
the *compactness deficiency*, def $X$, of the space $X$. He formulated a conjecture on
how to characterize this number internally that captured the imagination of scores
of mathematicians. In 1980 Roman Pol disproved this conjecture but only after a
new branch of dimension theory had been developed and had taken on a life of its
own. This excellently written, exciting book is a portrait of a living and dynamic
area as well as a monument to de Groot built by his former student, Jan Aarts,
and his able coauthor, Togo Nishiura. It should be required reading for anyone
interested in dimension theory. Regrettably, its high price will prevent it from
being on as many bookshelves as it should.

The first chapter sets the stage for the rest of the drama of the book. It concen-
trates on spaces that are separable and metrizable where the small inductive, the
large inductive, and the classical Lebesgue covering dimensions coincide. For such
a space $X$, define cmp $X$ inductively as follows: (i) cmp $X = -1$ if $X$ is compact;
(ii) for any nonnegative integer $n$, cmp $\leq n$ if for each point $p$ and each closed set
$G$ not containing $p$ there is a partition $S$ between $p$ and $G$ with cmp $S \leq n-1$;
(iii) cmp $X = n$ if cmp $X \leq n$ and cmp $X \nleq n-1$; and (iv) cmp $X = \infty$ if the
inequality cmp $X \leq n$ fails for each $n \geq 0$. (This would be the definition of ind $X$
if "$X$ is compact" in (i) were replaced by "$X = \varnothing$".) In his thesis de Groot showed
that def $X = $ cmp $X$ (for $X$ separable and metrizable) if $n = -1$ or 0, and he
conjectured that this equality held for every positive integer. The corresponding
analog of the large inductive dimension is denoted by Cmp. Many other dimension
functions were introduced in an attempt to split the de Groot problem into more
tractable pieces, and it was less than obvious that one was more pertinent than the
other. In 1960 Sklyarenko introduced an integer-valued function that associates
with separable metrizable $X$ an integer the authors denote by Skl $X$ as follows:
Skl $X \leq n$ if $X$ has a base $\mathcal{B} = \{B_i : i = 1, 2, \dots\}$ such that for any $n+1$ different
indices $i_0, \dots, i_n$, the intersection of the boundaries of the $U_{i_j}$ for $0 \leq j \leq n$ is
compact.

Inductive definitions are not easy to work with, and it took until 1982 for Pol
to produce an example of a separable metrizable space $X$ with cmp $X = 1$ and

Cmp $X = \det X = 2$. Later, Kimura and then Hart showed that the gap between $\det X$ and $\text{cmp } X$ could be made arbitrarily large. In 1988 Kimura was able to show that $\det X = \text{Skl } X$. While this may appear to put the finishing touch to the de Groot problem, it is far from the end of the story—even for the separable metrizable case. Compactifying is not the only way to extend a space. For example, the corresponding questions for completions are investigated. Covering dimension is also generalized. The many unsolved problems in this chapter demonstrate how alive this subject remains even in the separable metrizable case.

While the material described above occupies less than a quarter of the book, it sets the tone for the rest of it. Once one lifts the restriction to separable metrizable spaces, the classical dimension functions part company, as do their generalizations in the spirit of de Groot. The authors consider generalizations of dimensions modulo classes of topological spaces. A unified approach to the various compactifications that arise in this kind of dimension theory is developed. A lot of new material is presented, and unsolved problems abound. To give the rest of the book the attention it deserves would require an encyclopedic review and would embarrass the reviewer if he failed to convey the excitement one gets by reading this volume.

Many topics considered are motivated and placed in proper historical perspective. Generality appears naturally, not just for its own sake. While a prior knowledge of classical dimension theory is helpful, the book is largely self-contained except for references  to older books for some proofs of theorems. The exposition is masterful, and it is clear that the authors are more interested in communication than in displaying their (considerable) erudition.

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