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# The Use of Realistic Imagery to Represent the Relationships in a Four-Dimensional Coordinate System

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**SUMMARY**

First, this paper will describe the fundamentals of dimensionality and the relationships of the elements in a four-dimensional coordinate system. Second, it will outline a number of analogies between the elements and techniques of descriptive geometry with the elements and techniques for creating images of one-point perspective. Then, it will describe, at least, two of the ways realistic imagery can be used to illustrate all of the relationships in a four-dimensional coordinate system. The use of realistic imagery will allow one to decipher the illusions inherent in diagrams of higher dimensions. This will also allow for the illustration of a four-dimensional coordinate system with the images of any chosen space.

## 1. INTRODUCTION

A four-dimensional coordinate system, in the terms of descriptive geometry, is composed of one, two and three dimensional elements, as will be explained and illustrated. It is now common for us to consider the space we live in and the way our eyes see objects in space to be "three-dimensional" or "3-D." These "realistic" pictures are actually illusionistic images of perspective also termed one-point perspective, three-point perspective, linear perspective, centric perspective, artificial perspective, scientific perspective and geometric perspective. The easiest way to explain and capture an image of one-point perspective is with a photograph. One-point perspective can also be recreated with any aesthetic medium: drawing, oil painting, sculpture, hologram, virtual reality or any other technique for creating a motion picture. A detailed explanation of the composition of an image in one-point perspective and an introduction to the history of perspective in art is in Section 3.

## 2. DESCRIPTIONS OF DIMENSIONALITY

*Four-Dimensional Descriptive Geometry* by C. Ernesto S. Lindgren and Steve M. Slaby is a step-by-step explanation with accompanying diagrams showing the ways the lower dimensional elements of descriptive geometry, commonly called lines, planes, and boxes, form a four-dimensional coordinate system. These are the same geometric elements in a three-dimensional Cartesian coordinate system: three mutually perpendicular one-dimensional lines forming three mutually perpendicular two-dimensional planes, all intersecting at a zero-dimensional point, to form a three-dimensional coordinate system.

Similarly, a four-dimensional coordinate system has four mutually perpendicular one-dimensional lines, six two-dimensional planes and four three-dimensional spaces intersecting at a zero-dimensional point (Figure 1, hypertetrahedron). There are other relationships and requirements for the elements of a four-dimensional coordinate system. Still, these descriptions suffice as an extremely abbreviated and simplified explanation of dimensionality and the terms higher-dimensional, multi-dimensional and n-dimensional.

These examples and descriptions do not encompass n-dimensional topological models.

## 3. SIMILAR CONCEPTS OF SPACE

It is also common for us to associate one, two and three dimensions with an image of one-point perspective: a one-dimensional line with the line of sight perpen-

**Figure 1** (hypertetrahedron) corresponds to the diagram on page 17 in Lindgren and Slaby's book.

dicularly intersecting a projected plane in the middle ground of an image, two-dimensional surfaces with that projected plane and the surface the image is on and a three-dimensional space with the whole image. Leonardo da Vinci achieved these results relying on his studies of perspective. From those studies, he developed a compositional structure for recreating accurately scaled images of one-point perspective.

His fresco, *Last Supper*,1495-1497, is the most noted example of linear perspective because its compositional lines are blatantly apparent, and it still exists. The compositional lines appear as the edges of the picture frames, walls and ceiling beams. When visually continued, these lines converge with the line of sight at the point of infinity (marked by the figure of Jesus) to create an accurate image of one-point perspective. This line of sight is also perpendicular to the projected plane, in the middle ground, where the Apostles are arranged. It is acceptable to consider this plane, where the Apostles are arranged, separate from the general compositional structure denoted by the architecture and the figure of Jesus. This plane divides the space in front of the figures from the cavernous space of the room behind the figures. The relevant aspects of this composition are the following. There is a line of sight (a one-dimensional line) perpendicular to a two-dimensional plane that can be interpreted as dividing the three-dimensional space into two three-dimensional spaces. For more details about the composition of this work, see John Canaday's book, *What Is Art*?

This type of compositional structure naturally presents the appearance of a foreground, middle ground and background. The usage of these terms in this paper will be limited to these very simple and traditional definitions. The foreground of an image, generally the lower third of the image, shows the space the artist was standing in. Therefore, the foreground shows the space represented by the photograph. The middle ground shows where the line of sight perpendicularly intersects the projected plane at the horizon line. The background is the space behind the foreground and the projected plane in the middle ground.

Leonardo da Vinci was not the first person to study perspective during the Renaissance. Leon Battista Alberti is credited for writing the first definitive treatise on centric perspective entitled *de Pictura,* 1435. The most authoritative and cited English translation to date is by Cecil Grayson entitled *On Painting and On Sculpture.* The origin of geometric perspective in art would not be complete without mentioning Alberti's dedication in 1436 of his treatise to Filippo Brunelleschi. Filippo Brunelleschi is alleged, by his biographer Antonio Manetti, to have created the first two, now lost, paintings of scenes in perspective of Florence, Italy. The exact dates of the panels are not known; however, in Samuel Y. Edgerton Jr.'s book, *The Renaissance Rediscovery of Linear Perspective,* Edgerton dates at least one of the panels to 1425 because of other paintings by Masaccio and Masolino dating from circa 1425-1427. Edgerton's assertions are thoroughly researched and are enlightenedly reified with a photographic recreation of Brunelleschi's first painting in perspective.

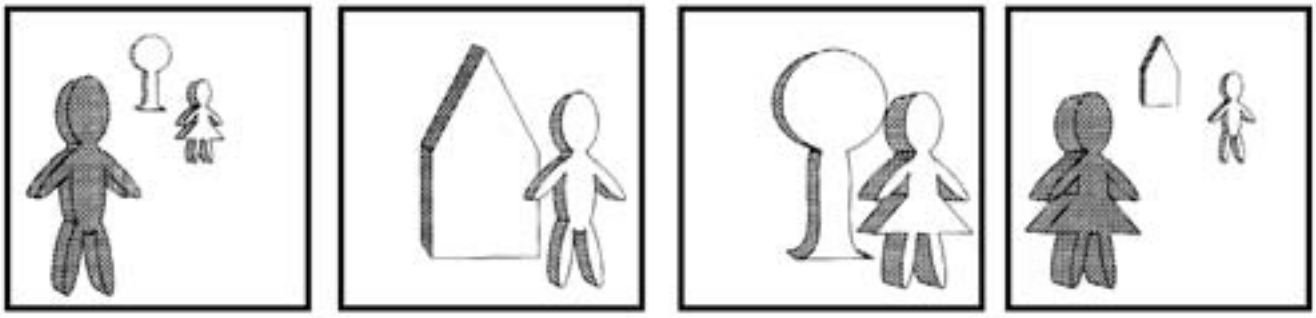
The writings of William M. Ivins, *On the Rationalization of Sight* and particularly *Art and Geometry,* link the understandings various cultures throughout the ages had of optics and geometries to the developments of linear perspective and modern geometries. Irwin Panofsky's book, *Renaissance and Renascences in Western Art,* thoroughly documents all of the aesthetic developments leading to accurately scaled, realistic images in perspective. The connectivity of art and geometric techniques through the phenomenon of perspective has been documented by many writers during the past six centuries.

In summation, if any aesthetic medium can be used to simultaneously represent the lower dimensions, then it can be used to illustrate the higher dimensions. In other words, given the assimilation of the elements of descriptive geometry with the elements of pictorial composition, any aesthetic medium can use realistic imagery to visually present the geometric relationships of the lower dimensional elements in higher four-dimensional coordinate systems.

#### 4. GEOMETRIC RELATIONSHIPS

Lindgren and Slaby's four-dimensional hypertetrahedron, as mentioned before, is composed of four mutually perpendicular one-dimensional lines, six two-dimensional planes and four three-dimensional spaces. The mutually perpendicular requirements of the four reference lines invariably leads to the mutually perpendicular relationships of the two and three-dimensional elements. The following four conditions of perpendicularity, on page 22 in Lindgren and Slaby's book, must also be met:

**Figure 2**



| Line 3 | Line 1 | Line 4 | Line 2 |
|---|---|---|---|
| Plane μ | Plane η | Plane α | Plane β |
| Space 1 | Space 2 | Space 3 | Space 4 |

1. The *four spaces* of the system, taken in three's, determine the four lines that are perpendicular to each other and belong to the same point.

2. *Four lines* taken in three's, determine the four spaces that are perpendicular to each other.

3. *Four lines,* taken in pairs, determine six planes, which, taken in three's, form four groups of planes belonging to the same line and mutually perpendicular.

4. Any one *line* of the reference system is perpendicular to the space determined by the other three lines in the system.

All of the resulting relationships of these conditions will be stated and illustrated by the end of the paper. However, this is not the place for a reprint of all of the geometric theorems in Lindgren and Slaby's book, some of which were originally appropriated from Henry P. Manning's book, *Geometry of Four Dimensions.* This paper will show at least one of the ways images of one-point perspective can be used to illustrate all of the relationships in a four-dimensional coordinate system.

The next step is to state a pattern for establishing four images of spaces that can be used to visually present all of the before stated elements and conditions of belonging and perpendicularity. This pattern was wholly derived from the requirements of the four reference lines. This is a logical approach for illustrating a linear coordinate system because the existence and relationships of the elements in the coordinate system are dependent upon the reference lines of the system. Then, in an analogous manner, once the four reference lines are represented by four images, the images themselves create the necessary one, two and three dimensional elements. This allows the inherent compositional structure of the images to present the one, two and three-dimensional elements in the higher four-dimensional coordinate system. The easiest way to establish four images to visually present the requirements of the four reference lines is with four photographs derived from three points in opposition. The two photographs taken from the two outer points are directed toward the central point, while the two photographs taken from the central point are directed toward the outer points.

Then each of these photographs or images must simultaneously represent a line, plane and three-dimensional space. This is achieved when the spaces are numbered, from left to right, 1, 2, 3, 4 and respectively designated as planes μ, η, α and β and lines 3, 1, 4 and 2. Planes γ and δ, the other two planes in the system, must be mutually perpendicular to planes μ, η, α and β. These relationships occur when plane γ shows the ground or horizontal plane of spaces 2 and 3, and plane δ shows the ground or horizontal plane of spaces 1 and 4. The side of viewing the spaces between the three points in opposition does not matter or if the photographs are numbered from right to left, as long as space 1 is associated with plane μ and line 3, space 2 with plane η and line 1, space 3 with plane α and line 4 and space 4 with plane β and line 2; the geometry is composed of reciprocal statements, therefore, the result will be the same four arrangements. The reciprocal nature of this geometry allows for, at least, these two aesthetic interpretations of any space chosen to illustrate a four-dimensional coordinate system.

When choosing a space, remember coordinate systems

were developed for measuring space, so each photograph should obviously show the same natural space. In the natural space used to illustrate the system described in this paper, there is a house to the far left, inside of space one, a man at the plane dividing spaces 1 and 2, a woman at the plane dividing spaces 3 and 4 and a tree to the far right, inside of space 4. The front surfaces of the man, woman, house and tree are not shaded, while their sides and backs are shaded (Figure 2).

#### 5. DESCRIPTIONS OF DIAGRAMS

Now, all of the conditions of belonging and perpendicularity of a fluctuating hypertetrahedron will be indicated with words and diagrams in lieu of real time images.

Plane $\eta$, belonging to spaces 1 and 2, is shown in clear view as a reference plane of space 2 because the foreground of the image shows the space of space 2 (Figures 3a-3e, page 10). This image also shows line 1 perpendicularly intersecting space 1 at plane $\eta$. Reference line 1 and spaces 2, 3 and 4 and planes $\alpha$, $\beta$ and $\gamma$ become perpendicular to space 1 when reference lines 2, 3 and 4 are made perpendicular to line 1 and each other in three groups two lines, while the third line remains back to add depth to space 1. If the third line is not used to add depth to space 1, then space 1 collapses or unfolds.

Plane $\beta$, belonging to spaces 2 and 4, is shown in clear view as a reference plane of space 4 because the foreground of the image shows the space of space 4 (Figures 4a-4e, page 10). This image also shows line 2 perpendicularly intersecting space 2 at plane $\beta$. Reference line 2, spaces 1, 3 and 4 and planes $\alpha$, $\beta$ and $\mu$ become perpendicular to space 2 when reference lines 1, 3 and 4 are made perpendicular to line 2 and to each other in three groups of two lines, while the third line remains back to add depth to space 2. If the third line is not used to add depth to space 2, then space 2 collapses or unfolds.

Plane $\mu$, belonging to spaces 1 and 3, is shown in clear view as a reference plane of space 1 because the foreground of the image shows the space of space 1 (Figures 5a-5e, page 11). This image also shows line 3 perpendicularly intersecting space 3 at plane $\mu$. Reference line 3, spaces 1, 2 and 4 and planes $\beta$, $\delta$ and $\eta$ become perpendicular to space 3 when reference lines 1, 2 and 4 are made perpendicular to line 3 and to each other in three groups of two lines, while the third line remains back to add depth to space 3. If the third line is not used to add depth to space 3, then space 3 collapses or unfolds.

Plane $\alpha$, belonging to spaces 3 and 4, is shown in clear view as a reference plane of space 3 because the foreground of the image shows the space of space 3 (Figures 6a-6e, page 11). This image also shows line 4 perpendicularly intersecting space 4 at plane $\alpha$. Reference line 4, spaces 1, 2 and 3 and planes $\gamma$, $\mu$ and $\eta$ become perpendicular to space 4 when reference lines 1, 2 and 3 are made perpendicular to line 4 and to each other in three groups of two lines, while the third line remains back to add depth to space 4. If the third line is not used to add depth to space 4, then space 4 collapses or unfolds.

#### 6. CONCLUSIONS

Realistic images of one point perspective can help us decipher the illusions in four-dimensional coordinate systems. The realistic images allow us to visually determine the "ground" planes, "side" planes and "top" planes of each of the three-dimensional spaces in a four-dimensional coordinate system. People can now visualize themselves, others, architecture or natural environments in four-dimensional coordinate systems. This art should nurture an understanding of optical illusions and Euclidean, Cartesian and non-Euclidean concepts of space. In turn, these contributions will hopefully emphasize the importance of art in the development of other disciplines.

#### 7. ACKNOWLEDGEMENTS

The extraordinary lifetime achievements of Steve M. Slaby and C. Ernesto S. Lindgren served as a catalyst for this breakthrough in higher dimensional illustration. The author extends his deepest thanks for their unreserved, enthusiastic support. Additional thanks to Arthur L. Loeb, at Harvard University, for his thoughtful contributions. Finally, the superb graphic illustrations have evolved to their current calculated clarity with the advice and tenacious computer graphic imaging, typesetting and editing expertise of George D. Beelen Jr. of Youngstown, Ohio.

**8. BIBLIOGRAPHY**


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FIGURE 3a

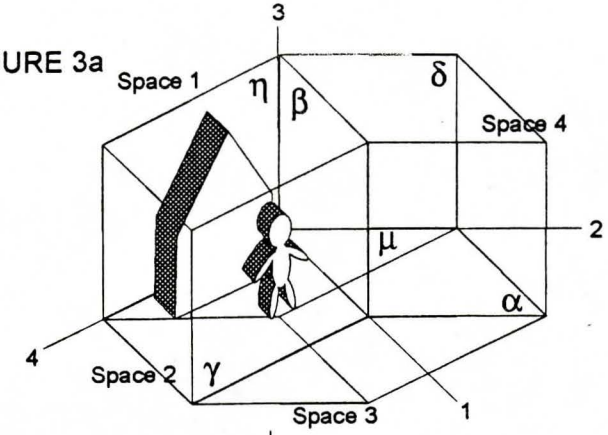



FIGURE 3b

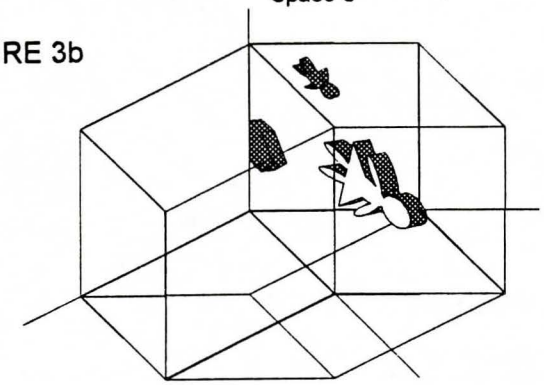


FIGURE 3c

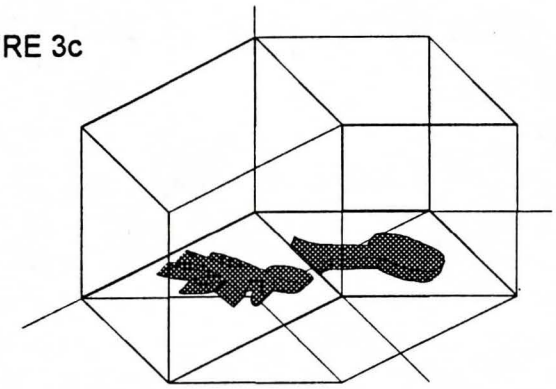


FIGURE 3d

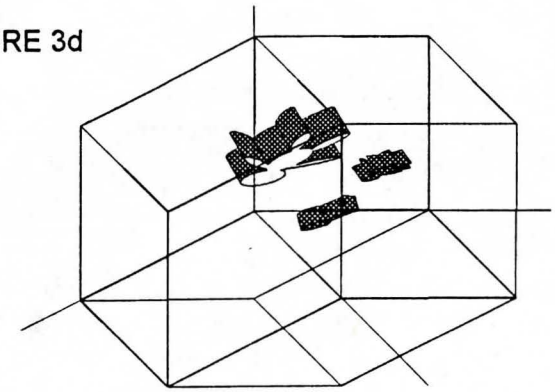


FIGURE 3e
COMPOSITE

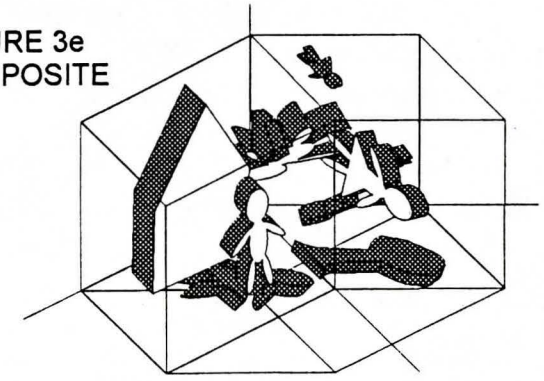


FIGURE 4a

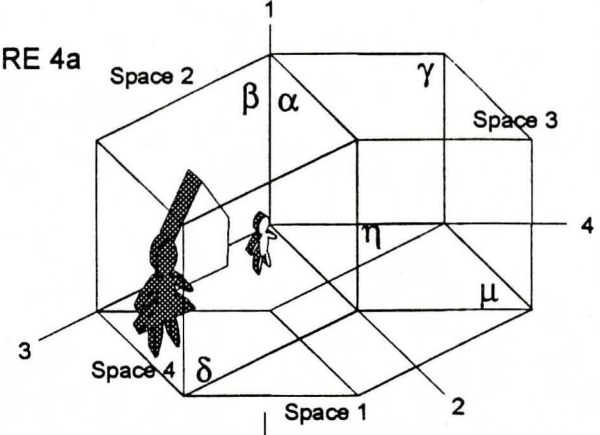



FIGURE 4b

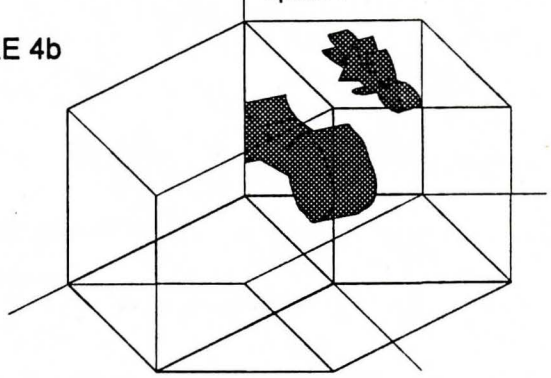


FIGURE 4c

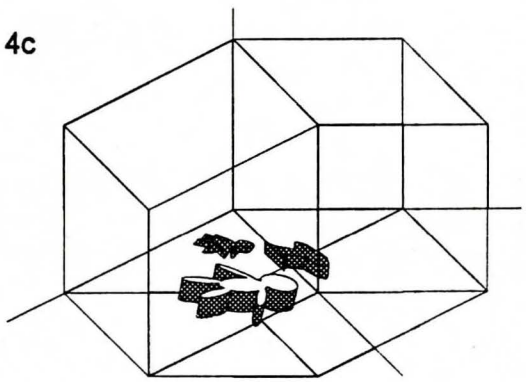


FIGURE 4d

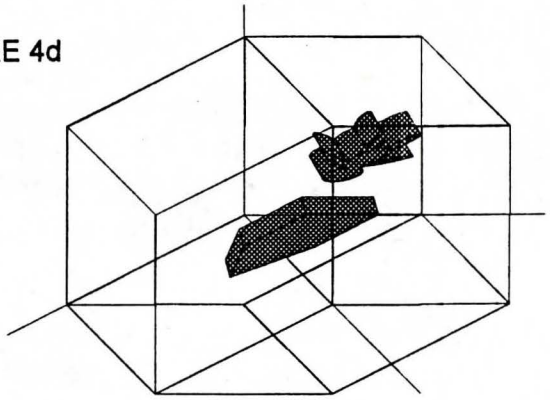


FIGURE 4e
COMPOSITE

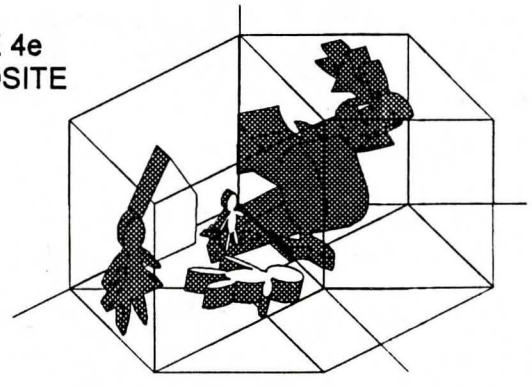


FIGURE 5a

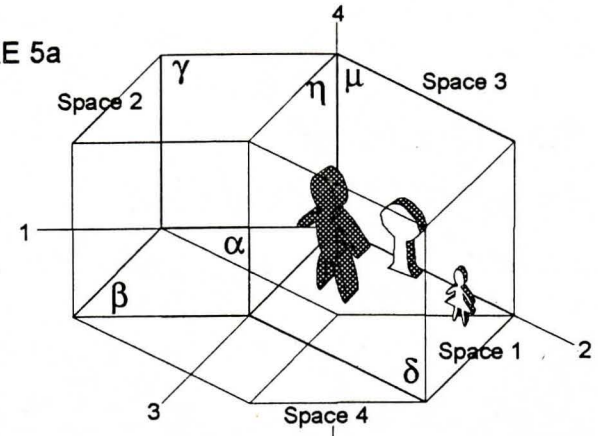
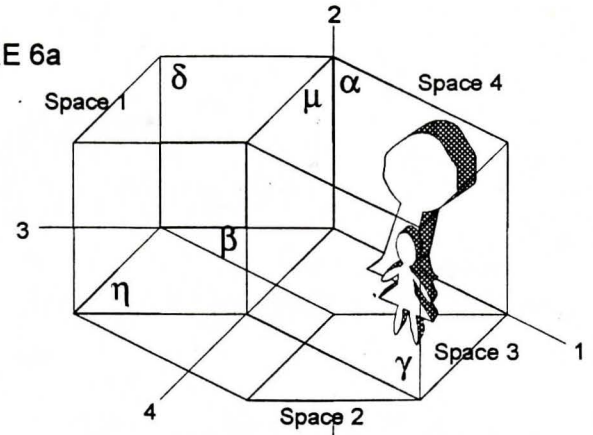



FIGURE 6a

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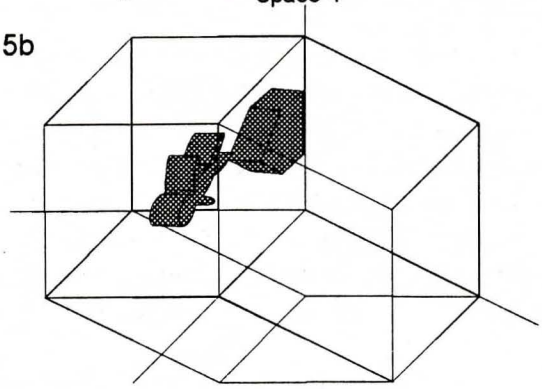

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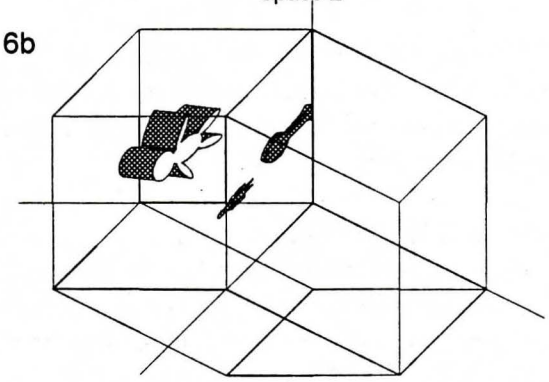

FIGURE 5c

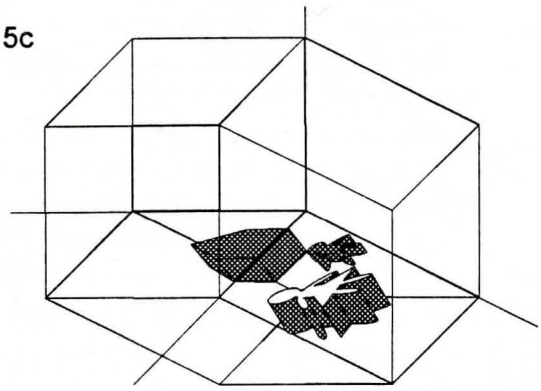

FIGURE 6c

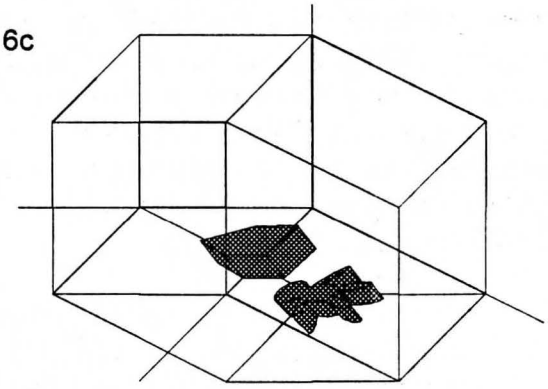

FIGURE 5d

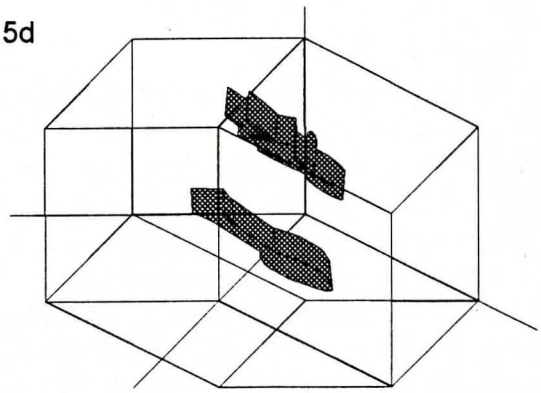

FIGURE 6d

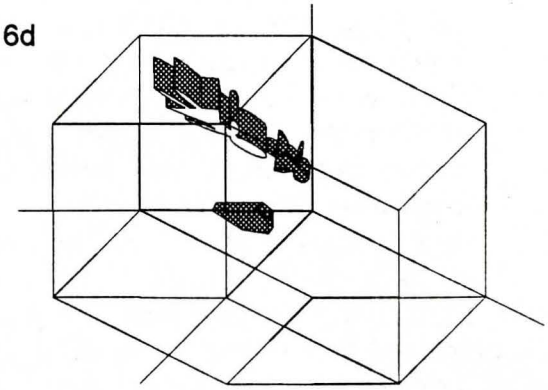

FIGURE 5e
COMPOSITE

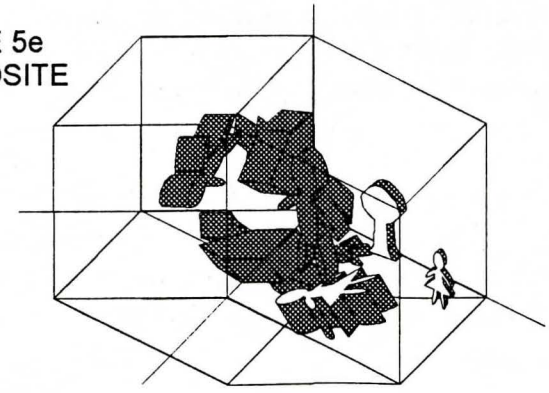

FIGURE 6e
COMPOSITE