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Solving Equations: A Make-Work Project for Math Teachers and Students

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Synopsis

The purpose of this article is to share a particular view I have towards solving equations in the school mathematics classroom. Specifically, I contend that solving equations in the math classroom is a make-work project for math teachers and students. For example, math teachers take a predetermined value that makes a statement true, and then proceed to make it harder and harder and harder for their students to determine the value that makes the statement true. However, math teachers do so with the explicit purpose of teaching their students how to reveal the solution that they themselves have concealed. Stated in make-work project parlance, the math teacher digs a hole with the explicit purpose of teaching, then having the students fill the hole that they dug.

In *school* mathematics, an important distinction for what will follow, I contend that solving equations is a make-work project for math teachers and students. Let me explain.

1. The Revealing

In school mathematics, to solve an equation means, in essence, to determine the value that makes a particular statement true. Let us consider, as an example, the equation

$$x + 1 = 3.$$

Clearly, there are a number of values that make the statement false. For instance, for $x = 1$, the statement is false. For $x = 32$, the statement is false. Of course, I could go on listing values that make the statement false, but I will not. As we all know, for $x = 2$, the statement is true, that is, the value that makes the statement $x + 1 = 3$ true is 2. In other words, we have solved the equation. I contend, from a temporal perspective, that a majority of school mathematics is spent determining the value that makes a given statement true, that is, solving equations.

As students move on in their school mathematics career, they quickly move beyond being able to determine the value that makes a statement true through guess-and-test methods (as demonstrated in the previous paragraph). All too soon, it becomes quite difficult for students to solve the equations they are given by their teacher. Consider, by way of another example, solving

$$3x + 3 = 9$$

for x. Of course, one could take a guess-and-test approach to determine the value that makes this statement true; however, this technique is often discouraged in favour of a more procedural approach to solving equations. After a lesson entitled (something along the lines of) *Factoring to Solve Equations*, students will (hopefully) rewrite

$$3x + 3 = 9$$

as

$$3(x + 1) = 3 \cdot 3.$$

In doing so, they may now apply what they had learned in a previous lesson entitled (something along the lines of) *Solving Equations with Division* and will end up with

$$x + 1 = 3.$$

The value that makes the statements $3x + 3 = 9$ and $x + 1 = 3$ true is one and the same, that is, 2. Thus, while a student may not be able to determine the value that makes the statement $3x + 3 = 9$ true, they may be able to determine the value that makes the statement $x + 1 = 3$ true. If not, recalling an even earlier lesson entitled (something along the lines of) *Solving Equations with Subtraction* will prompt the student to subtract one from both sides of the equation, at which point they will be left with the statement

$$x = 2,$$

which can be interpreted as the easiest of all the statements to determine the value that makes the statement true. (Sometimes, probably more often than not, the achievement is celebrated by surrounding this, the simplest of the statements, with a big red box.) Progressively, the ability to determine the value that makes the statement true (2) becomes easier ($3x+3=9$) and easier ($x+1=3$) and easier ($x=2$) to determine. Ultimately, students are taught a variety of lessons (e.g., solving equations by addition, subtraction, multiplication, division and many others) so that they are able to reveal the solution (e.g., $x=2$) to the equations (e.g., $3x+3=9$) that they are asked to solve.

2. The Concealing

If the task of the student is to reveal solutions, then clearly it is the role of the teacher to conceal the solutions. Given the embarrassment of riches, that is, the massive quantity of concealed solutions available at a math teacher's fingertips (e.g., the curriculum, textbooks, old tests and quizzes, lesson plans, the Internet), we may forget how these concealed solutions came to be. Let us remind ourselves.

A concealed solution has humble beginnings. For instance, let us start with a number that we would like to be the solution to an equation. Let us suppose (to continue with our example, albeit in a different direction) the decision is made that the answer to the equation will be 2 (after all, whole numbers seem to dominate the answer keys in the back of school math textbooks). Next, supposing, say, that the class is not yet comfortable with using other letters of the alphabet, we adopt the use of the letter x for our statement. Of course, the statement $x=2$ is, simply, too easy: the student will be able to determine the value that makes the statement true (they will simply be able to look at it and see that the value that makes the statement true is 2). As such, let the concealing begin.

How the concealing occurs, by and large, is up to the teacher and at which point they would like the students to begin revealing the answer. Continuing with our example, rather than showing the students the statement

$$x=2,$$

perhaps the teacher decides to make the solution a little more difficult to determine and, accordingly, considers the statement

$$x + 1 = 2 + 1.$$

Thinking about it a little more, the teacher realizes that the "$2 + 1$" on the right side of the equation is a dead giveaway to the concealing that is taking place and, as a result, she or he decides to add the 2 and the 1 so that the solution is further concealed, that is,

$$x + 1 = 3.$$

Stopping here is appropriate, if the goal was to get students to learn *Solving Equations with/via/by Subtraction* (or however the section was titled in the text book). However, if the students have already had that particular lesson, then more concealing is required.

Carrying on with our example, the equation $x + 1 = 3$ (a further concealing of $x+1 = 2+1$, which is a concealing of $x = 2$) can be even further concealed, that is, made more difficult (for students) to determine the value that makes the given statement true by, for example, multiplying both sides by 3. Thus, $x + 1 = 3$ becomes

$$3(x + 1) = 3 \cdot 3,$$

and (depending on the degree of concealing that the teacher is interested in) the factored terms on one, the other, or both sides of the equation can be distributed. Progressively, the ability to determine the value that makes the statement true (2) becomes harder ($x = 2$) and harder ($x+1 = 3$) and harder ($3x + 3 = 9$) to determine. Ultimately, the teacher conceals the solution to the point at which they want their students to start revealing the solution. See Figure 1.

3. Make-Work Project Perspective

In establishing the role of (1) the student, which is to reveal the solution (as necessary) so that they may determine the value that makes a given statement true, and (2) the teacher, which is to conceal the solution so that the students are unable (for a certain period of time) to determine the value that makes the statement true, it becomes clear that solving equations is, in essence, a make-work project for math teachers and students.

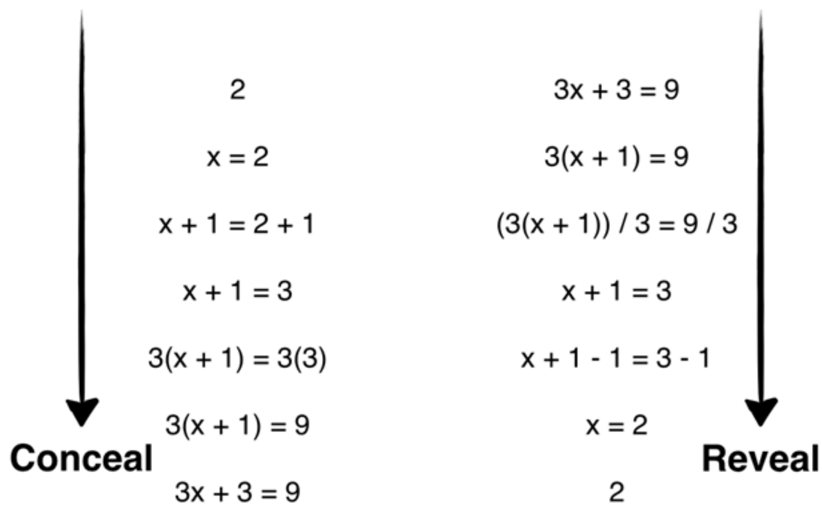

Figure 1: Solving equations: concealing and revealing.

Let me explain further. As established, math teachers take a predetermined value that makes a particular statement true, and then, through concealment, make it harder and harder and harder and harder for their students to determine the value that makes the statement true. However, and now adopting a make-work project perspective, math teachers do so with the explicit purpose of teaching their students how to reveal the solution that they themselves have concealed. Alternatively stated, math teachers conceal the solution to an equation with the explicit purpose of then teaching their students a variety of lessons (for instance, solving equations by adding, subtracting, multiplying, and dividing and through other means), which are specifically designed to teach procedures (often, for months on end) that make it easier and easier and easier for students to determine the value that makes a given statement true… A make-work project for math students and teachers.

4. Make-Work Project Parlance

Analogously speaking, math teachers dig hole, after hole, after hole, after hole in their math class, but keep in mind that they do so with the explicit purpose of teaching students how to fill and them having them fill each of the holes that they dug. Further, math teachers, in order to be able to spend month after month after month providing their students with the necessary procedures to reveal their concealed solutions, must dig deeper and deeper and deeper. To illustrate this point, we now turn to what happens once students are comfortable with solving linear equations.

Students, once they are able to solve linear equations, are typically then faced with quadratic equations. Considering a standard approach to solving quadratic equations in school mathematics, that is, taking the quadratic and, in essence, turning it into two linear equations, the hole dug for quadratic equations is deeper than the hole dug for linear equations; or, stepping back from our hole digging parlance, but only for a moment, quadratic equations require further concealing on the part of the teacher and further revealing on the part of the students. While keeping in mind that, for the most part, math teachers have a treasure trove of pre-concealed solutions in the form of quadratic equations to choose from, we will visit the process of concealing the solution for a quadratic equation.

For the sake of argument, let us assume that a math teacher is interested, this time, in two distinct solutions: 2 and -3. Sticking with the infamous letter x, we now have $x = 2$ and $x = -3$. At this point, it is very easy (too easy) to determine the values that make the statements true. Let the concealing begin. Knowing full well that a lesson on the zero product property will be taught in the not too distant future (or, perhaps, that it has just been taught), $x = 2$, now $x - 2 = 0$, and $x = -3$, now $x + 3 = 0$, are multiplied so that we now have $(x - 2)(x + 3) = 0$. However, even at this point, if the students are familiar with the zero product property, the solutions will be quite easy to determine. As such (knowing full well that the class has already spent a number of months on factoring) $(x - 2)(x + 3) = 0$ becomes $x^2 + 3x - 2x - 2 \cdot 3 = 0$ which, given the students' proficiency in factoring, is then concealed a step further to become $x^2 + 1x - 6 = 0$. The solutions of 2 and -3, although they could, if necessary, be concealed further (e.g., by multiplying both sides by some factor) have been sufficiently concealed for students at the beginning of a unit on solving quadratics. Time for the revealing.

As mentioned earlier, the depth of the hole associated with a quadratic equation is deeper than that of the hole associated with a linear equation. A student filling the quadratic hole (that the teacher dug) back up with dirt will, at some point, hit a particular depth with which they are familiar—that of the linear equation. Importantly, in order for the students to be able to fill the quadratic hole back up to the depth of a linear hole, which is where the teacher knows that they will be able to fill the entire hole back up, the teacher must equip her or his students with the tools required (e.g., factoring and zero product property) to get the quadratic hole (which is deeper than the linear) filled back to the linear depth. Thus, the teacher spends time working with students to train them how to factor quadratics and what the zero-product property is (partly) because they very well know that they want the students to be able to fill this new, deeper hole that they have dug. The depth-of-hole argument, of course, could be extended further to solving cubics, quartics or other equations encountered in the school math classroom.

In extending the depth-of-hole argument beyond the current progression (from linear to quadratic to cubic to quartic equations), one can see that school mathematics, generally speaking, is a make-work project for math teachers and students. More specifically, throughout their schooling, students fill up a variety of different holes, holes that their teachers have dug with the explicit purpose of teaching their students how to fill the holes that they have dug for them. As a student proceeds from one school year to the next, the holes keep getting deeper and deeper and deeper. In fact, if one were to organize the different types of equations that students are asked to solve in school mathematics (e.g., linear, linear with two unknowns, quadratic, rational, exponential, logarithmic, and trigonometric equations, as well as equations involving radicals, absolute values, and/or fractions, to name a few) one would find a correlation between the number of steps involved in solving a given equation and the grade in which students are asked to solve that particular equation. In general, the more concealing and revealing involved in determining the value that makes a statement true, the further a student is in their school mathematics career. In the end, everything changes, yet everything stays the same.

5. An Arms Race

Solving equations, as a make-work project for math students and teachers, eventually turns into an arms race to calculus (where the most concealing and revealing takes place). This arms race, between the teachers and the students, turns math into the continuous presentation of more and more procedures, algorithms, and calculations by the teacher, presented so that students may fill up deeper and deeper holes that their teachers have dug. If neither side blinks, both the arms race and the make-work project continue enabling students to make their way to calculus class at university, which, to most, is considered "success". However, if at some point in time a student does not grasp a particular procedure or algorithm, then they will be unable to fill a given hole back up to a familiar depth (e.g., unable to get the hole back up to quadratic or linear depths) and, as a result, may find themselves stuck at the bottom of ever deeper and deeper holes—where it may be very, very dark indeed.

6. Future Teachers Of Mathematics

There is no doubt that the picture I have painted, thus far, is rather dire. However, this picture does present some essential truths and, because of this point, I make it my business to share my bent perspective with the (approximately 150) future elementary, middle and high school math teachers that I teach each year. It is an interesting moment to experience in my classroom: the moment right after I have told a room full of future math teachers that their school mathematics experience was, in essence, a make-work project, and that they, too, upon becoming future math teachers, will begin the work of digging holes with the explicit purpose of teaching students how and then having students fill the very holes that they dug... thus, perpetuating the make-work project for future generations. Believe it or not, though, I do not present my perspective on solving equations to demoralize future math teachers. It just feels that way at the moment. In fact, there are a number of reasons why I share my make-work project perspective with them.

Future math teachers (at least, a number of the ones that I've worked with over the years, and through no fault of their own), have an *uber-procedural* understanding of solving equations. Before we, as a class, discuss the make-work project perspective, I ask them: "What are you actually doing when you solve an equation?" Their responses are, most often, some version of "You're balancing a teeter-totter" or "You're doing the same thing to both sides". Other responses include "You're solving for x" or "You're isolating the variable". When I prod further, their procedural understanding of solving equations naturally reveals itself. For example, they explain to me how they were always bothered when their teachers tried to teach two sections in one day—even if the sections were, say, solving equations by addition and solving equations by subtraction—or if the "x" was on the right side of the equals sign. As another example, becoming quite candid, they explain that although they never really understood why, they were repeatedly asked to check their answer when they were done. Others reveal that they never really knew where all the questions in the textbook and tests and quizzes came from. The list of stories goes on and on and on. (My personal favourite: "I was told that when you drag a number from one side of the equation to the other that magic pixie dust gets sprinkled on it and changes the sign from a positive to a negative or a negative to a positive."). One response that I never (read: very rarely) hear, though, is that "To solve an equation is to determine the value(s) that make the statement true". Based on the responses that I have heard and the ones that I have not heard, the notion of solving equations as a make-work project for math teachers and students seems, at least directly after hearing the argument, more than a bit deflating. But, a few moments later, students come to realize that this new perspective on solving equations is designed to help them, as future math teachers, resolve the misconceptions/preconceptions that they have about solving equations.

One of the keys to establishing a positive spin to the make-work project perspective is not to focus on the analogy of digging a hole just to teach students how and then having them fill the hole back up, but, rather, to rely on the notion that to solve an equation in school mathematics (yes, in non-school mathematics equations are not necessarily solved) is to determine the value that makes the statement true. In fact, and this is the point that I stress to my students, whether one is solving linear, linear with two unknowns, quadratic, rational, exponential, logarithmic or trigonometric equations or equations involving radicals, absolute values or fractions, in each case one is doing the exact same thing: determining the value that makes the statement true. Once this common thread has been presented, the students begin to resolve other issues that they had with respect to solving equations. For example, they begin to understand, from a "trees" perspective, that solving a linear equation is different than solving a quadratic; however, from a "forest" (for the "trees") perspective, solving a linear equation is just like solving a quadratic. While the procedures and the particular details are different for solving linear and quadratic equations, in both cases, one is determining the value(s) that make the statement true. All of a sudden, the notion of working on two sections in one day does not seem as disparate or daunting as it had seemed before. Further, realizing that checking your answer is designed to determine whether or not one has made a clerical error "along the way" while solving the equation (after all, the value that makes the statement true for $3x+3 = 9$ is the same as it is for $x+1 = 3$, which is the same as it is for $x = 2$), that is, to make sure the revealing went ok, you throw it into the concealed version to make sure it works. Involving future math teachers in the process of concealing a few solutions, that is, in digging a few holes of their own, has also proved to be quite impactful (at least, based on their Tweets after class). The biggest advantage, however, of getting future math teachers to realize that, yes, there is indeed a connection between all the (seemingly) disparate holes that they were forced to fill during their schooling is that it helps combat the Lima Syndrome in their future math classrooms.

7. The Lima Syndrome

Once you begin looking for it, you soon realize that the Lima syndrome is present in many math classrooms. The Lima syndrome, the inverse of the Stockholm syndrome, is where abductors develop sympathy for their hostages. In my (new) analogy, the teachers are the abductors (please take this in the spirit in which it is intended) and students are the hostages (again, please take this in the spirit in which it is intended). In this analogy, it is the abductor that forces the hostage to fill up the hole that they have dug. (I guess, in a perfect analogy, the abductor would have the hostage both dig and fill the hole, but, in the interest of consistency, it will suffice that the abductor does not dig the hole, rather utilizing pre-dug holes [e.g., textbooks] with the explicit purpose of teaching the hostages how to fill the holes back up and then having them do so.) However, as one might expect, through the process of teaching hostages how to fill a certain type of hole (e.g., linear, quadratic) and then having the hostages fill the holes, a relationship is formed between the abductor and the hostage. Said relationship is not immune from (sympathetic) feelings when, for instance, a hostage is unable to fill a hole back up to the surface because they have not quite grasped the integral zero theorem and, thus, are unable to fill the hole back up to the quadratic level, which is where they would then be able to fill the hole back up to the linear level, a point from which they could easily reach the surface. Instead, they are stuck. Stuck in a hole. Some hostages could be stuck in a deep, dark hole for a very long time.

Given their shared history (the abductors, after all, were once hostages themselves) and, perhaps, due to sympathetic feelings towards the hostage that is stuck in a deep, dark hole with no way of getting out, the abductor throws the hostage a lifeline. For example, "when you drag a number from one side of the equation to the other all you have to do is change the plus sign to minus or minus sign to a plus sign". Or, overheard near a deeper hole, "Look, all you really need to remember for solving inverse equations is to swap the x's and the y's, solve for x and, finally, swap back the x's and y's". Unfortunately, what may initially seem like a lifeline could cause irreparable damage. Too many disparate lifelines offered to the student eventually (perhaps inevitably) morph into a house of cards. To combat the development of a disparate, uber-procedural understanding of solving equations in the school mathematics classroom, one can show future math teachers that there is, in fact, a connection between all those (seemingly) disparate topics found in the math class by explaining to them that solving equations is a make-work project for math teachers and students. (To date, I have always recommended that they do not share the make-work-project perspective with their future math students.)

8. A Potential New Response To An Infamous Question

I remember, like it was yesterday, the first time that I was ever asked the "When are we ever going to *need* this" question, which is more likely to arise with certain topics than others. For me, it came up during one of my very first few lectures involving the integral zero theorem and synthetic division. Of course, as a young math teacher, I tried my best to espouse the virtues of mathematics and how it is a particular lens with which to view the world (blah, blah, blah) in my answer. My answers to this infamous question, if I were still teaching high school mathematics, would have definitely become shorter and more terse. I picture myself ten years into my teaching career, being asked "When are we ever going to need synthetic division and the integral zero theorem?" and my response. "Because it's gonna be on the test we have two days from now", I hear myself reply. A little harsh, I agree. With that said, not as harsh as: "Well, I teach the integral zero theorem and synthetic division because these are the tools that you'll need in order for you to fill this new and ever deeper hole that I dug with the explicit purpose of you filling it back up again back to the depth of a quadratic hole, which you are familiar with, which you can then fill back up to the depth of a linear hole, which you can then fill back up to where we started... Which, by the way, is part of a grand make-work project for you and me that, if it were not affectionately known as 'solving equations in the school math class', would call into question my role as a math teacher, your role as a math student, our homework, our quizzes, our tests and, essentially, the teaching and learning of mathematics as a *hole*."