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An Inquiry into Lorentzian Polynomials

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Abstract

In combinatorics, it is often desirable to show that a sequence is unimodal. One method of establishing this is by proving the stronger yet easier-to-prove condition of being log-concave, or even ultra-log-concave. In 2019, Petter Brändén and June Huh introduced the concept of Lorentzian polynomials, an exciting new tool which can help show that ultra-log-concavity holds in specific cases. My thesis investigates these Lorentzian polynomials, asking in which situations they are broadly useful. It covers topics such as matroid theory, discrete convexity, and Mason's conjecture, a long-standing open problem in matroid theory. In addition, we discuss interesting applications to known combinatorial objects and possible future paths for discovery.

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And it may be a slight bit corny, but dear reader, I do thank you as well. You have decided to trust me with introducing you to or giving you a new perspective regarding a new and interesting topic, and I will not be taking that for granted. I hope you find this mathematics as interesting as I did when I first encountered it.

Chapter 1

Introduction

Lorentzian polynomials are currently some of the most exciting objects in the world of combinatorics. My hope is that, through this thesis, we can take an informative journey through the world of these fascinating objects together. I present a quick outline of what this thesis entails, chapter by chapter, and a quick note on how it is written in the end.

We begin with chapter 2, in which we introduce the concept of **unimodality** (Definition 1), a highly desirable result to obtain in a combinatorial setting. We describe two related concepts, **log-concavity** (Definition 2) and **ultra-log-concavity** (Definition 3), and provide some examples as to illustrate its importance (Theorems 2, 4). We also introduce an idea of **multivariable log-concavity** (Definition 7).

In chapter 3, we introduce the main subjects of the thesis, **Lorentzian polynomials** (Definition 11). After a brief history of their prominence, we introduce a recursive definition, along with reintroducing some important objects, such as the **Hessian** (Definition 10), and provide several examples (Example 7). We relate this back to ultra-log-concavity, and provide a broad motivation for an alternative definition (Example **??**).

In chapter 4, we present an alternative characterisation for Lorentzian polynomials. We present some important ideas, such as the **support** of a polynomial (Definition 12) and ***M*-convexity** (Definition 13), which are used in this equivalent characterisation (Theorem 7). We then provide some examples to help show the utility of this characterisation (Example 13).

In chapter 5, we introduce two definitions of **matroids**, algebraic objects closely related to the prior characterisation (Definitions 14 and 15). We give several examples to help illustrate their utility (Example 16). We end by sketching out how the language of matroids can be used to prove **Mason's**

conjecture, a long-standing open problem in matroid theory (Theorem 9).

In chapter 6, we briefly mention **denormalised Lorentzian polynomials**, as a possible way to apply Lorentzian polynomials to non-matroidal situations, and provide a link to a pre-print which may do so.

Dear reader, there are many things I love about mathematics, but the way it is typically written is not one of them. Mathematical texts are almost always designed with more care for the mathematics inside of them than for the reader of these ideas. It is paradoxical, as the two cannot truly be separated. This is a goal I wish to address in this thesis. In addition to learning about a fascinating combinatorial idea, I hope that you, reader, feel invited by the mathematics within this text. I hope that I express my passion for it in a way such that you, too, can feel that this work is accessible and maybe even be inspired to think about it yourself!

Chapter 2

Unimodality and Log-Concavity

Let us begin, as we often do, with a definition.

Definition 1 (Unimodality, (Stanley, 2011))**.** A sequence $\{a_k\}$ of length $n + 1$ is *unimodal* if there exists a value $0 < k < n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_{n-1} \geq a_n.$$

This behaviour is foundational in combinatorics. There are several ways we can conceptualise why. Of course, we may be satisfied with the explanation that unimodality is important for the sake of unimodality, in the same sense that in algebra, finding structure is important for the sake of finding structure. However, unimodality can also be interesting for reasons related to statistical analysis of sequences. One may note that, for statistical results such as the local limit theorem, which shows when a distribution approximates the normal distribution, it may be helpful for said distribution to be unimodal (Bender, 1973). And, in general, many sequences we care about happen to be unimodal. Some examples follow.

Example 1 (Binomial Coeffcients (Brändén et al., 2015))**.** The sequence of binomial coefficients for a fixed $n \in \mathbb{Z}^+$ and iterating k, that is, the sequence such that $a_k = \binom{n}{k}$ for $0 \leq n \leq k$, is unimodal. These quantify all sorts of useful phenomena, including how to pick a subset of k distinct objects from a set of n objects.

Example 2 (Symmetric Group on n Letters)**.** Permutations of the symmetric group on $n \in \mathbb{Z}^+$ letters, S_n, with exactly $k \leq n$ disjoint cycles in their decomposition, are unimodal. An example of a permutation on 5 letters with 2 disjoint cycles in its composition would be (124)(35), that is, a function sending the list (a, b, c, d, e) to the list (d, a, e, b, c). This we will prove later.

Example 3 (Stirling Numbers of the Second Kind (Sibuya, 1988))**.** The Stirling numbers of the second kind for a fixed $n \in \mathbb{Z}^+$ and an iterating k, that is, the sequence such that $a_k = \left\{ {n \atop k} \right\}$ for $1 \leq n \leq k$, are unimodal. The value $\left\{ {n \atop k} \right\}$ quantifies the number of ways to partition an n element set into k distinct subsets.

What may be helpful to note about some of these sequences is that they do not necessarily have nice closed forms. In particular, it is somewhat difficult to use a generating function to find Stirling numbers of the second kind, as doing so involves a somewhat nasty generating function (Boyadzhiev, 2012). As such, simply knowing that any such sequence will be unimodal helps capture some of the behaviour we wish to understand about it without actually needing to calculate any such sequence.

Although it is sometimes done, it is rare that one proves something is unimodal directly. Instead, it is often advantageous to prove something slightly stronger, after which we can show that a sequence is unimodal. In particular, we can prove that a sequence is *log-concave*.

Definition 2 (Log-Concavity (Stanley, 2011))**.** We call a sequence $\{a_k\}$ *log-concave* if, for every k with $0 < k < n$, it is the case that

$$a_k^2 \geq a_{k-1}a_{k+1}.$$

Let us firstly show that this statement is indeed stronger than unimodality, through two quick propositions.

Theorem 1. Log-concavity is a stronger condition than unimodality. That is, log-concavity implies unimodality, but the converse is not true.

Proof. We firstly prove that log-concavity implies unimodality using the contrapositive. If we can show that a sequence $\{a_k\}$ that is not unimodal is also not log-concave, then this statement will be proven. Note that, were a sequence $\{a_k\}$ to not be unimodal, there would be some k in the sequence for which $a_{k-1} > a_k$ and $a_k < a_{k+1}$. Colloquially, this would mean that there would be at least one "dip" and thus at least two "peaks" in the sequence. Combining these two statements, we would see that that $a_k^2 < a_{k-1}a_{k+1}$, implying that the sequence is, indeed, not log-concave. Thus, we have proven that if a sequence is log-concave, then it is unimodal.

We then show that a sequence that is unimodal is not necessarily log-concave through simple counterexample. Consider the sequence $1, 5, 3, 2$. This is clearly unimodal, as we see $a_1 = 5$ to be its only peak. However, note that $a_2^2 < a_1 \cdot a_3$; that is, $3^2 < 5 \cdot 2$. As such, this is a sequence that is unimodal, but not log-concave.

Thus, taken together, these imply that log-concavity is a stronger condition than unimodality. □

We have seen several examples of log-concave sequences already; in fact, all the sequences we have mentioned up to this point we know to be unimodal because we know them to be log-concave.

There are several ways we can choose to establish log-concavity. We can first choose to do so using the technique of injection. As a broad idea of how this argument may work, we note that, we often can construct sets based on each element of the sequence, such that $a_i = |S_i|$. This is especially true if our sequence is firmly based in a known combinatorial object. If we can find an injection from the pairs of sets $S_{i-1} \times S_{i+1}$ into $S_i \times S_i$ that works for every $0 < i < n$, then it will automatically be the case that $a_i^2 \geq a_{i-1}a_{i+1}$ for each $0 < i < n$. This is broadly the most direct, and thus the most desirable, way to prove that something is generally log-concave.

As an example, let us show that this works for the binomial coefficients.

Theorem 2. The sequence of binomial coefficients, as described in Example 1, is log-concave.

Proof. We firstly establish which sets we will be working with. Given n, for any a_k, we associate it with a set S_k such that $a_k = |S_k|$. In particular, we consider the set of subsets of $\{1, 2, \cdots, n\}$ with k elements; we note that these are counted exactly by $\binom{n}{k}$. Thus, a prototypical element $(s_\alpha, s_\beta) \in S_\alpha \times S_\beta$ will be pairs of subsets of $\{1, 2, \cdots, n\}$, the first with α elements and the second with β elements.

Now that we've established the sets we will be using to be counted by each element of a_k, we now show the injection from $S_{k-1} \times S_{k+1}$ to $S_k \times S_k$. For each element of the set $(s_{k+1}, s_{k-1}) \in S_{k+1} \times S_{k-1}$, consider the set $I = s_{k+1} \backslash (s_{k+1} \cap s_{k-1})$. This is exactly the set consisting of all elements in s_{k+1} that are not in s_{k-1}. Also note that this set cannot be empty, as there are at least two elements in s_{k+1} that are not in s_{k-1}, due to the size of the sets. Thus, we pick the smallest element $e \in I$ and add it to the subset s_{k-1} while subtracting it from s_{k+1} to get two sets of size k. That is, more explicitly, we send the element (s_{k+1}, s_{k-1}) to the element $(s_{k+1} \backslash \{e\}, s_{k-1} \cup \{e\})$. We note, then, that this process is an injection; if $f(s_{k+1} \times s_{k-1}) = (s_k \times s'_k)$, then to reconstruct the original element of our set, we find the smallest element in s'_k that's not in s_k, add it to s_k and remove it from s'_k. This will always be possible if the sets were generated in this manner, and the answer of $s_{k+1} \times s_{k-1}$ will be unique. As such, this is an injection. □

It is also the case that one can prove that a finite sequence is log-concave if its generating polynomial has all of its roots real and negative; this is how one proves, for example, that elements of S_n with exactly k disjoint cycles in their decomposition are log-concave. Let us firstly lay out a proof showing that, indeed, these functions are log-concave, and then show their log-concavity, and hence, their unimodality.

Theorem 3 ((Brändén et al., 2015))**.** Given a finite sequence $\{a_k\}$ of length n with non-negative coefficients, if the associated polynomial

$$A(x) = \sum_{k=0}^{n} a_k x^k$$

has all roots real and negative, then the sequence $\{a_k\}$ is log-concave.

Proof. This proof proceeds through induction on n.

We firstly note two base cases. If $n = 1$, then the sequence is automatically log-concave, as the sequence is of the form $a_0 + a_1 x$, and as such, has no middle terms we need to worry about satisfying the log-concavity condition.

If $n = 2$, note that the condition of $a_0 + a_1 x + a_2 x^2$ being real rooted, by utilising the discriminant, is the condition that $a_1^2 - 4a_0 a_2 \geq 0$. This implies log-concavity, as, since all the terms are positive,

$$a_1^2 \geq 4a_0 a_2 \geq a_0 a_2.$$

Now we perform our inductive step. Suppose that, for some fixed $n \in \mathbb{Z}^+$ and $c \in \mathbb{R}^+$, we have that the polynomial $A(x) = (x + c)B(x)$, where $B(x) = b_n x^n + \cdots + b_1 x + b_0$, has roots all real and negative. We know, then, that $B(x)$ also has roots only real and negative, as it has the same roots as $A(x)$, save $-c$. Thus, $B(x)$ is log-concave, meaning $b_k^2 \geq b_{k-1} b_{k+1}$ for all $0 < k < n$. Since we know that $A(x) = (x + c)B(x)$, performing this multiplication tells us that

- $a_0 = cb_0$
- $a_{n+1} = b_n$
- $a_k = b_{k-1} + cb_k$ otherwise (that is, for $0 < k < n + 1$

We firstly note that for $2 \leq k \leq n-1$, it is the case that, since

$$b_k^2 \geq b_{k-1}b_{k+1},$$

multiplying both sides by b_{k-1} gives us

$$b_k^2 b_{k-1} \geq b_{k-1}^2 b_{k+1} \tag{2.1}$$

and similarly, since

$$b_{k-1}^2 \geq b_k b_{k-2},$$

multiplying both sides by b_{k+1} gives us

$$b_{k-1}^2 b_{k+1} \geq b_{k+1} b_k b_{k-2}. \tag{2.2}$$

Combining equations 2.1 and 2.2 gives us that

$$b_k^2 b_{k-1} \geq b_{k+1} b_k b_{k-2},$$

or more simply,

$$b_k b_{k-1} \geq b_{k+1} b_{k-2}. \tag{2.3}$$

Thus, we see that

$$\begin{aligned} a_k^2 - a_{k+1}a_{k-1} &= (b_{k-1} + cb_k)^2 - (b_k + cb_{k+1})(b_{k-2} + cb_{k-1}) \\ &= (b_{k-1}^2 - b_k b_{k-2}) + c(b_k b_{k-1} - b_{k+1} b_{k-2}) + c^2(b_k^2 - b_{k+1} b_{k-1}). \end{aligned}$$

Note that the three terms which make up this sum are all nonnegative. The first term $(b_{k-1}^2 - b_k b_{k-2})$ is as such because of our log-concavity condition. Similarly, the third term $c^2(b_k^2 - b_{k+1}b_{k-1})$ is also true due to the log-concavity condition, as the index has shifted but otherwise all parts of the quanitity are nonnegative. And finally, the term $c(b_k b_{k-1} - b_{k+1} b_{k-2})$ is nonnegative due to 2.3 above. As such, this is indeed nonnegative always. Finally, one can check edge cases.

Thus, if this is true for k, it must also be true for $k+1$, meaning our inductive step holds.

As such, this proof is shown. □

Let us now show that the sequence of elements of S_n with exactly k disjoint cycles in their decomposition are log-concave.

Theorem 4 (The Symmetric Group on n Letters is Log-Concave)**.** The sequence of the symmetric group on n letters, as described in Example 2, is log-concave.

Proof. We use an inductive argument to firstly prove that $\sum_{k=0}^{n} a_k x^k = x(x+1)(x+2)\cdots(x+n-1)$. Note that, for fixed n, it holds for $k=0$, as there are no disjoint cycle decompositions. We note that, for any

$$f_{n+1}(x) = x f_n(x) + n f_n(x),$$

and as such,

$$[x^k]f_{n+1}(x) = [x^k]x f_n(x) + [x^k]n f_n(x),$$

meaning

$$[x^k]f_{n+1}(x) = a_{k-1} + n \cdot a_k.$$

This is true, as the first term represents putting $n+1$ in its own cycle, while the second represents putting it in an existing cycle, after one of the n existing numbers–there are n ways to do this.

We note that

$$(x+1)(x+2)\cdots(x+n-1)$$

only has negative real roots, and as such, is log-concave. Similarly,

$$x(x+1)(x+2)\cdots(x+n-1)$$

must too be log-concave, as $a_{k-1}a_{k+1} \leq a_k^2$ still holds, just for a different indexing. The only term we need to check is that of a_1, and since $a_0 = 0$, $a_0 a_2 \leq a_1^2$. So this is indeed a log-concave sequence. □

There is, in fact, an even stronger condition than log-concavity that we often encounter. Let us define it below.

Definition 3 (Ultra Log-Concavity (Anari et al., 2018))**.** We denote a sequence $\{a_k\}$ as *ultra log-concave* if, for every k with $0 < k < n$, it is the case that

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}.$$

This is a stronger case even than log-concavity; let us now show this.

Theorem 5. Ultra log-concavity is a stronger condition than log-concavity. That is, ultra log-concavity implies log-concavity, but the converse is not true.

Proof. We firstly show that ultra log-concavity implies log-concavity. Note that we can rewrite the condition for ultra log-concavity in terms of factorial coefficients, that is, show that

$$\frac{a_k^2}{\frac{n!^2}{k!^2(n-k)!^2}} \geq \frac{a_{k-1}}{\frac{n!}{(k-1)!(n-(k-1))!}} \frac{a_{k+1}}{\frac{n!}{(k+1)!(n-(k+1))!}}.$$

Multiplying both sides by $\frac{n!^2}{k!^2(n-k)!^2}$ thus gives us that

$$a_k^2 \geq \frac{(k+1)\cdot(n-(k-1))}{k\cdot(n-k)} a_{k-1}a_{k+1}$$

We note that, since $k+1 > k$ and $n-(k-1) > k$, the fraction $\frac{(k+1)\cdot(n-(k-1))}{k\cdot(n-k)} > 1$, and as such, we see that it must be the case that

$$a_k^2 \geq a_{k-1}a_{k+1},$$

which implies log-concavity.

Next, we show that log-concavity does not always imply ultra log-concavity. Consider the sequence $1, 4, 3, 2$. It can be checked that this sequence is log-concave. However, note that

$$\frac{a_2^2}{\binom{3}{2}^2} < \frac{a_1}{\binom{3}{1}} \frac{a_3}{\binom{3}{3}},$$

as

$$\frac{9}{9} < \frac{8}{3}.$$

Thus, it is not always the case that a log-concave sequence is ultra log-concave.

As such, it must be the case that, in general, ultra log-concavity is indeed stronger than log-concavity. □

Example 4. An example of a sequence that is ultra log-concave is that of binomial coefficients outlined in Example 1. Note that, in all cases, the inequality reduces to the statement that $1 \geq 1$, which is of course quite true.

We finally introduce an idea of multivariable log-concavity, one which will come into play later as we try to apply the ideas of these Lorentzian polynomials. We must firstly define a notation that helps us simplify our definitions.

Definition 4 (Δ_n^d (Matherne et al., 2022))**.** We write that the n-dimensional vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with all $\alpha_i \geq 0$ is in the set Δ_n^d if $\sum_{i=1}^n \alpha_i = d$.

Definition 5 ($\mathbf{x}^\alpha$ (Matherne et al., 2022))**.** We also write, for any $\alpha \in \Delta_n^d$ and for the n-dimensional vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, the notation $\mathbf{x}^\alpha$ to represent the monomial $\prod_{i=1}^n x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 6 (Multivariable Factorial (Matherne et al., 2022))**.** We finally define $\alpha! = \prod_{i=1}^n \alpha_i$, and e_i to be ith standard basis vector, with 1 in the ith position and 0 everywhere else.

We now introduce the idea of multivariable log-concavity.

Definition 7 (Multivariable Log-Concavity (Matherne et al., 2022))**.** We consider a polynomial $f = \sum_{\alpha \in \Delta_n^d} c_\alpha \mathbf{x}^\alpha$, with each c_α some non-negative integer. We understand this to possess **multivariable log-concavity** when

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha+e_j-e_i}$$

for all $i, j \in \{1, \cdots n\}$ and $\alpha \in \Delta_n^d$.

We can think of this informally as being indicative of log-concavity "in all directions," as if we fix i and j, this becomes a similar idea to log concavity in one direction.

Remark. Note that this is generally true when

$$(\alpha!)^2 c_\alpha^2 \geq (\alpha + e_i - e_j)!(\alpha + e_j - e_i)! \cdot c_{\alpha+e_i-e_j} c_{\alpha+e_j-e_i},$$

again, for all $i, j \in \{1, \cdots n\}$ and $\alpha \in \Delta_n^d$. This mirrors, in some sense, the same idea as that of ultra log-concavity. In fact, one can prove that this implies $c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha+e_j-e_i}$ (Matherne et al., 2022)).

We now move on to describing the central objects of this paper.

Chapter 3

A First Pass at Defining Lorentzian Polynomials

In this section, we take a first look at Lorentzian polynomials, defining them recursively as Brändén and Huh do in their paper. We also introduce a couple of clarifying examples.

In 2019, mathematicians Petter Brändén and June Huh wrote a paper entitled "Lorentzian Polynomials" (Brändén and Huh, 2020). This paper caused a lot of buzz within the mathematics community, as people were fascinated by how this class of polynomials could be used to establish log-concavity for various combinatorial objects. June Huh even won both a Fields medal and a MacArthur Genius Grant, in part, for his work surrounding these mathematical structures. In short, these Lorentzian polynomials happen to have had a clear impact on mathematics, so it is important to be able to understand them.

So how do we define Lorentzian polynomials? To do so, we use the same recursive formulation laid out by Brändén and Huh in their original paper.

We first start by laying out some notation.

Definition 8 (H_n^d, (Brändén and Huh, 2020))**.** For nonnegative integers n and d, we write H_n^d as the set of degree d homogenous polynomials in $\mathbb{R}[x_1, \cdots, x_n]$.

Definition 9 (P_n^d, (Brändén and Huh, 2020))**.** We additionally denote by P_n^d the subset of H_n^d containing polynomials with all positive coefficients.

Let us show a quick example.

Example 5 (Polynomial in P_n^d)**.** We see that

$$6x_1^3 + 8x_1^2x_2 + 3x_1x_2^2 + x_2^3 \in P_2^3.$$

Remark. We can define a natural topology on H_n^d for any fixed $n, d \in \mathbb{Z}_{\geq 0}$ by using the Euclidean norm on the coefficients considered as a vector.

We now introduce the Hessian.

Definition 10 (Hessian (Brändén and Huh, 2020))**.** The Hessian of any function $f \in \mathbb{R}[x_1, \cdots, x_n]$ is the symmetric matrix

$$\mathcal{H}_f = \left[\partial_i \partial_j f\right]_{i,j=1}^n,$$

where ∂_i represents the derivative $\frac{\partial}{\partial x_i}$.

We provide an example below.

Example 6. The Hessian of the polynomial $6x_1^2 + 8x_1x_2 + 3x_2^2$ is $\begin{bmatrix} 12 & 8 \\ 8 & 6 \end{bmatrix}$.

Remark. We used a polynomial of degree 2 in our above example, as this will always give us a matrix with no indeterminates. This is the type of Hessian we eventually plan on calculating most often.

Now, we delve into Lorentzian polynomials themselves, which are defined recursively.

Definition 11 ((Strictly) Lorentzian Polynomials (Brändén and Huh, 2020))**.** Let us firstly define the set of *strictly Lorentzian* polynomials $\mathring{L}_n^d$. We set $\mathring{L}_n^0 = P_n^0$ and $\mathring{L}_n^1 = P_n^1$. We also set

$$\mathring{L}_n^2 = \left\{f \in P_n^2 \mid \mathcal{H}_f \text{ is invertible and has exactly one positive eigenvalue}\right\}.$$

And finally, for $d > 2$, we write

$$\mathring{L}_n^d = \left\{f \in P_n^d \mid \partial_i f \in \mathring{L}_n^{d-1} \text{ for all } i \in \{1, 2, \ldots, n\}\right\}.$$

These polynomials are *strictly Lorentzian*.

We then consider the closure of a set of strictly Lorentzian polynomials under the topology discussed earlier. These are precisely the set of *Lorentzian* polynomials L_n^d.

It may help to have some examples to clarify these definitions. Let us firstly show a strictly Lorentzian polynomial, then some examples where it is not the case that a polynomial is strictly Lorentzian, even though they are within P_n^d. Showing that a polynomial is Lorentzian but not strictly Lorentzian is somewhat unclear at the moment, and as such, we will leave it for a future chapter.

Example 7 (Polynomial in P_n^d)**.** We see that $6x_1^2 + 8x_1x_2 + x_2^2 \in P_2^2$ is a strictly Lorentzian polynomial. It is of degree 2, and as such, we need only examine its Hessian. We see that said matrix is

$$\begin{bmatrix} 12 & 8 \\ 8 & 2 \end{bmatrix}.$$

It has a determinant of -40, and is thus invertible. Additionally, its eigenvalues are $7 \pm \sqrt{89}$, meaning that it has exactly one positive eigenvalue. As such, it is a Lorentzian polynomial by Definition 11.

Example 8 (Non-Strictly Lorentzian Polynomial: Eigenvalues)**.** The polynomial $6x_1^2 + 2x_1x_2 + 8x_2^2 \in P_2^2$ is not Lorentzian; to see this, simply consider its Hessian, $\begin{bmatrix} 12 & 2 \\ 2 & 18 \end{bmatrix}$. A quick calculation will show that it has eigenvalues of $2(7 + \sqrt{2})$ and $2(7 - \sqrt{2})$, both of which are positive. Thus, it has more than one positive eigenvalue, and therefore is not Lorentzian.

Example 9 (Non-Strictly Lorentzian Polynomial: Invertibility)**.** Consider the polynomial $x_1^3 + x_2^3$. We note that its partials with respect to both variables are $3x_1^2$ and $3x_2^2$, respectively. Note that the Hessian of both of these terms $\begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$. and $\begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}$, respectively, each of which have exactly one positive eigenvalue. However, these matrices are not invertible, and as such, this is not a Lorentzian polynomial.

It may seem a bit unclear as to why Lorentzian polynomials are related to unimodality and log concavity at all. However, an interesting fact can help us further elucidate this.

Theorem 6 (Generalised Bivariate Polynomial (Brändén and Huh, 2020))**.** Consider the generalised bivariate polynomial

$$f(x_1, x_2) = \sum_{k=0}^{n} a_k x_1^k x_2^{n-k}.$$

with positive coefficients, that is, such that $f(x_1, x_2) \in P_2^n$. In fact, f is Lorentzian if and only if the sequence $\{a_k\}$ satisfies

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}.$$

This condition is precisely that of ultra-log-concavity! Thus we see that, in this case, being Lorentzian and satisfying log-concavity are inherently linked.

We will right now state this without proof, as a proof of this is more clear using a definition we introduce in the next chapter.

Now, although the recursive definition is fundamental to our understanding, it is not always the most useful. As with any recursive definition, as we get in some sense larger (in this case as our degree increases), the recursive definition becomes somewhat unwieldy. As such, we may wish to introduce an equivalent definition.

Chapter 4

M-Convexity Characterization of Lorentzian Polynomials

In this section, we introduce an equivalent definition of Lorentzian polynomials.

We firstly recall that it would be quite time-consuming to check that a polynomial is Lorentzian using the recursive definition each time. However, there is indeed a way to understand if a polynomial is Lorentzian without actually referring to the recursive definition. To understand this, we need to firstly understand two other definitions.

Definition 12 (Support (Brändén, 2020))**.** For a homogenous multivariable polynomial $f = \sum_{\alpha \in \Delta_n^d} c_\alpha \mathbf{x}^\alpha$, we define the support of such a polynomial as

$$\text{supp}(f) = \{\alpha \in \Delta_n^d \text{ such that } c_\alpha \neq 0\}.$$

Informally, we can understand this as the vector coefficients such that the term in the polynomial is non-zero. While this definition can be generalised to non-homogenous polynomials, we will be using it in relation to those that are homogenous, and as such, will keep the definition specific as of right now.

Definition 13 (M-convexity (Brändén, 2020))**.** Consider a collection of vectors $J \subseteq \mathbb{Z}_{\geq 0}^n$. We call such a collection *M-convex* if, for any two vectors $\alpha, \beta \in J$ such that $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, if for some $i \in \{1, 2, \cdots, n\}$, $\alpha_i > \beta_i$, then it is the case that there is some $j \in \{1, 2, \cdots, n\}$ such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$.

Let us examine what sort of collections are M convex.

Example 10 (M-convex and non-M-convex sets (Brändén, 2020))**.** Consider the collection $\{(0,0), (-1,1), (-2,2)\}$. One can check, through pairwise analysis, that indeed this set is M-convex. However, the similar set $\{(0,0), (-2,2)\}$ is not M-convex, as, if we take $\alpha = (0,0)$ and $\beta = (-2,2)$, and $\alpha_1 > \beta_1$ and there exists a j, namely, $j = 2$, such that $\alpha_j < \beta_j$, the element $\alpha - (1,0) + (0,1) = (-1,1)$ is not in our original set.

Example 11 (A Useful M-convex Set (Brändén, 2020))**.** We firstly define an operation on elements $\alpha \in \mathbb{Z}_{\geq 0}{}^n$, that is, vectors with integer components Consider the set $\Delta_n^d \subseteq \mathbb{Z}_{\geq 0}{}^n$, which sometimes referred to as the *d-th discrete simplex*. We define elements of Δ_n^d as all $\alpha \in \mathbb{Z}_{\geq 0}{}^n$ such that $|\alpha|_1 = d$.

Remark. It is the case that the M in M-convexity is related to matroids, a type of well-studied algebraic object. We will discuss these objects in the next chapter.

We can now move on to our equivalent characterisation of Lorentzian polynomials.

Theorem 7 (An Equivalent Characterisation of Lorentzian Polynomials (Brändén and Huh, 2020))**.** A polynomial in H_n^d with non-negative coefficients is Lorentzian if and only if

1. Its support is M-convex.
2. The Hessian of $\partial_{i_1}\partial_{i_2}\ldots\partial_{i_{d-2}}f$ has at most one positive eigenvalue for all $1 \leq i_1, i_2, \ldots, i_{d-2} \leq n$.

The proof of this theorem is difficult and involves mathematics outside of the scope of this thesis. As such, we will not be proving this statement. However, we can perhaps better understand how we utilise it in order to check if a polynomial is Lorentzian.

In using this characterisation to check if a polynomial is Lorentzian, we firstly find that the coefficients of our multivariable objects understood as vectors form a collection that is M-convex, and then show that, no matter how we "reduce" our original vector down to a quadratic, we still get a Hessian with at most one positive eigenvalue. This is a relatively straightforward process, and as such, is the one we tend to use when deciding whether a polynomial is Lorentzian. It may retain a slight bit of that recursive nature, in that we need to check all of the partials given each pair of indeterminates, but that requires analysing only $\binom{n}{d-2}$ Hessians, as opposed to $n!$ Hessians, and so is broadly much more scalable than using the prior characterisation.

Let us maybe find a way to understand this idea utilising some good examples.

Example 12 (Lorentzian Polynomial)**.** We recall the polynomial $6x_1^2 + 8x_1x_2 + x_2^2 \in P_2^2$ from Example 7 is Lorentzian. Let's see how we can prove this using Theorem 7 instead. We note that the support of this polynomial, $\{(2,0),(1,1),(0,2)\}$, forms an M-convex set. Additionally, we've seen already that the Hessian of this polynomial,

$$\begin{bmatrix} 12 & 8 \\ 8 & 2 \end{bmatrix},$$

has eigenvalues of $7 \pm \sqrt{89}$, meaning that it has at most one positive eigenvalue. As such, it can also be seen to be Lorentzian through this method.

Example 13 (Polynomial that is M-Convex but fails Hessian Test (Matherne et al., 2022))**.** Consider the polynomial

$$f(x_1,x_2,x_3,x_4,x_5) = \sum_{\{i,j,k,l\}\subset\{1,2,3,4,5\}} 24x_ix_jx_kx_l + \\ \sum_{\{i,j,k\}\subset\{1,2,3,4,5\}} 4x_i^2x_jx_k + 4x_ix_j^2x_k + 4x_ix_jx_k^2 + \sum_{\{i,j\}\subset\{1,2,3,4,5\}} 2x_i^2x_j^2.$$

It can be shown that the support of this polynomial is M-convex. However, if we examine the particular Hessian $\partial_1\partial_2 f(x_1,x_2,x_3,x_4,x_5)$, we can see that it equals

$$\begin{bmatrix} 0 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 & 8 \\ 8 & 8 & 8 & 24 & 24 \\ 8 & 8 & 24 & 8 & 24 \\ 8 & 8 & 24 & 24 & 8 \end{bmatrix},$$

which has eigenvalues of $\{8(4+\sqrt{15}), 8(4-\sqrt{15}), -8, -16\}$. The first two of these are positive, and as such, this is not a Lorentzian polynomial.

Example 14 (Polynomial that is Lorentzian but not Strictly Lorentzian)**.** Consider the polynomial $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$. We firstly see that its Hessian is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Note that this matrix is not invertible, as its determinant is 0. As such, we cannot say that it is strictly Lorentzian using the recursive Definition 11. However, we see that its support,

$$\{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1)\},$$

is M-convex, and the eigenvalues of the Hessian are

$$\left\{\frac{1}{2}\left(1+\sqrt{17}\right),\frac{1}{2}\left(1-\sqrt{17}\right),-1,0\right\},$$

only the first of which is positive. As such, it has at most one positive eigenvalue and can be seen to be Lorentzian by Theorem 7.

As such, we see that, often, it is more clear to prove that a polynomial is Lorentzian using this alternate characterisation, rather than using the recursive definition.

Remark. Keep an eye out for a polynomial much like this one in the next section of our paper.

With this new characterisation in our back pocket, let us now prove Theorem **??** from the prior chapter.

Proof. We wish to show that the generalised bivariate polynomial

$$f(x_1,x_2)=\sum_{k=0}^{n} a_k x_1^k x_2^{n-k}.$$

with positive coefficients is Lorentzian if and only if the sequence $\{a_k\}$ satisfies

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}}\frac{a_{k+1}}{\binom{n}{k+1}}.$$

We note firstly that the support of this polynomial is always M-convex. Note that this polynomial's support takes the form

$$\operatorname{supp}(f)=\{(0,n),\ (1,n-1),\ \cdots,\ (n-1,1),\ (n,0)\}.$$

We now need show that this set satisfies Definition 13. We note that it must, as for any two distinct vectors $\alpha,\beta\in\operatorname{supp}(f)$ such that $\alpha=(\alpha_1,\alpha_2)$ and $\beta=(\beta_1,\beta_2)$, either $\alpha_1>\beta_1$ and $\alpha_2<\beta_2$ or $\alpha_2>\beta_2$ and $\alpha_1<\beta_1$. As such, we know that if there is some i such that $\alpha_i > \beta_i$, then there exists some j such that $\alpha_j < \beta_j$; since there are only two variables in each vector, it must be the other. Additionally, note that if $\alpha_i > \beta_i$ then $\alpha_i > 0$, meaning that $\alpha_j < n$. As such, the vector $(\alpha_i - 1, \alpha_j + 1) \in \operatorname{supp}(f)$, meaning that, indeed, this set is M-convex.

We now aim to prove our main result.

Firstly, let us show that, if $f(x_1, x_2)$ satisfies the second condition of Theorem 7, and is thus Lorentzian, then the sequence $\{a_k\}$ satisfies ultra log-concavity. Note that, for any sequence of partial derivatives, we will need to take $q-1$ partials of x_2, with $1 < q < n-1$ (we choose this in order to make the labelling clearer). The polynomial we will be taking the Hessian of will thus be of the form

$$\frac{(q-1)!(n-(q-1))!}{2!}a_{q-1}x_1^2 + \frac{q!(n-q)!}{1!\cdot 1!}a_q x_1 x_2 + \frac{(q+1)!(n-(q+1))!}{2!}a_{q+1}x_2^2.$$

This can be rewritten as

$$\frac{n!}{2}\left(\frac{a_{q-1}}{\binom{n}{q-1}}x_1^2 + \frac{2a_q}{\binom{n}{q}}x_1x_2 + \frac{a_{q+1}}{\binom{n}{q+1}}x_2^2\right).$$

Note that the hessian of this polynomial is

$$\mathcal{H}_f = \begin{bmatrix} n!\frac{a_{q-1}}{\binom{n}{q-1}} & n!\frac{a_q}{\binom{n}{q}} \\ n!\frac{a_q}{\binom{n}{q}} & n!\frac{a_{q+1}}{\binom{n}{q+1}} \end{bmatrix},$$

which has the same eigenvalues as the matrix

$$\frac{1}{n!}\mathcal{H}_f = \begin{bmatrix} \frac{a_{q-1}}{\binom{n}{q-1}} & \frac{a_q}{\binom{n}{q}} \\ \frac{a_q}{\binom{n}{q}} & \frac{a_{q+1}}{\binom{n}{q+1}} \end{bmatrix}.$$

We recall that this needs to have exactly one positive eigenvalue. We also remember that the determinant of a matrix is the product of its eigenvalues. As such, if there is exactly one positive eigenvalue $\det(\frac{1}{n!}\mathcal{H}_f) \leq 0$, and as such,

$$\frac{a_{q-1}}{\binom{n}{q-1}}\frac{a_{q+1}}{\binom{n}{q+1}} - \frac{a_{q-1}^2}{\binom{n}{q}^2} \leq 0.$$

This can more simply be expressed as

$$\frac{a_{q-1}^2}{\binom{n}{q}^2} \geq \frac{a_{q-1}}{\binom{n}{q-1}} \frac{a_{q+1}}{\binom{n}{q+1}},$$

which is precisely ultra-log concavity! As such, this direction is proven.

The backwards part of this proof left as an exercise to the reader, but is relatively similar to the forwards version of it. □

Chapter 5

Matroids and Their Relevance

In this section, we introduce the algebraic objects known as matroids, which will help us broadly understand our prior equivalent characterisation of Lorentzian polynomials in Theorem 7.

Firstly, I want to introduce the concept of matroids. Broadly, matroids are a way to formalise the idea of independence, as usually introduced in linear algebra, to other structures. It is perhaps useful to firstly introduce a motivating example before introducing formal definitions.

Example 15 (The Columns of a Matrix can form a Matroid)**.** Consider the columns of the following matrix considered over the field $\mathbb{R}$:

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} \end{array} \\ M = & \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

We could write the **power set** of these columns, that is, all possible subsets of the set of all columns $E = \{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\}$, as

$$2^E = \{\emptyset, \{\mathbf{v_1}\}, \{\mathbf{v_2}\}, \{\mathbf{v_3}\}, \{\mathbf{v_4}\}, \{\mathbf{v_1}, \mathbf{v_2}\}, \{\mathbf{v_1}, \mathbf{v_3}\}, \cdots, \{\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}, \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}\}.$$

For ease of notation, we will represent a set as a list of its elements, that is, we will write $\mathbf{v_1 v_3}$ to represent $\{\mathbf{v_1}, \mathbf{v_3}\}$.
Note that, when considered as vectors, the independent sets of the columns of this matrix are

$$\mathcal{I} = \{\emptyset, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_1 v_2}, \mathbf{v_1 v_3}, \mathbf{v_1 v_4}, \mathbf{v_2 v_3}, \mathbf{v_2 v_4}\}.$$

Note, too, that the maximal independent sets, or bases, are

$$\mathcal{B} = \{\mathbf{v_1v_2}, \mathbf{v_1v_3}, \mathbf{v_1v_4}, \mathbf{v_2v_3}, \mathbf{v_2v_4}\}.$$

Also note that the bases all span the space generated by the columns of M.

There are some interesting properties to note about $\mathcal{I}$ and $\mathcal{B}$. These are what will define for us matroids.

Definition 14 (Matroids (Independent Sets Definition)(Alderete 2021, Reiner 2005))**.** A **matroid** $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$, with E a finite set (also called the **ground set**) and $\mathcal{I} \subseteq 2^E$ a collection of subsets satisfying the the following three properties:

I1) $\varnothing \in \mathcal{I}$

I2) $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ implies $I_2 \in \mathcal{I}$

I3) $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ implies there exists $x \in I_2 - I_1$ such that $I_1 \cup x \in \mathcal{I}$. This is sometimes called the **exchange axiom**.

In this case, we call $\mathcal{I}$ the collection of **independent sets** of $\mathcal{M}$.

A fun exercise is to convince oneself that this property holds for Example 15. We can also define matroids in the following manner.

Definition 15 (Matroids (Bases Definition) (Reiner, 2005))**.** A **matroid** $\mathcal{M}$ is an ordered pair $(E, \mathcal{B})$, with E a finite set and $\mathcal{B} \subseteq 2^E$ a collection of subsets satisfying the the following two properties:

B1) $\mathcal{B} \neq \varnothing$.

B2) Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, there exists $y \in B_2 - B_1$ such that

$$(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}.$$

This is sometimes called the **exchange axiom for bases**.

In this case, we call $\mathcal{B}$ the collection of **bases** of $\mathcal{M}$.

Note that this definition implies that all sets in $\mathcal{B}$ are of the same size.

While we see that the columns of a matrix can form an underlying set for a matroid, we can also formulate other examples.

Example 16 (A Set of Vectors in a Vector Space Can Form a Matroid)**.** Consider the set of standard basis vectors E in $\mathbb{R}^n$, and consider all subsets I such that, for all $I \in 2^E$, all elements span a vector space of dimension $m \leq n$ (with $m, n \in \mathbb{Z}^+$). These can be used to form a matroid, where $\mathcal{I}$ is the collection of all subsets of at most m vectors in E and $\mathcal{B}$ is the collection of all bases for m-dimensional vector spaces.

Example 17 (A Connected Graph Can Form a Matroid (Reiner, 2005))**.** Consider a connected graph $G = (V, E)$, with V representing its vertices and E its edges. Now consider all subsets $I \in 2^E$ such that the edges I do not form cycles (for those more familiar with graph theory, this is a **forest** of edges). These can be used to form a matroid, where $\mathcal{I}$ is the collection of all subsets of E such that no edges form a cycle in E and $\mathcal{B}$ is the collection of all spanning trees of G.

At this point, matroids may seem an interesting diversion, with their relation to Lorentzian polynomials being slightly opaque. Let us clear this up by noting that, in some cases, there is an equivalence between the basis of a matroid and M-convexity.

Theorem 8 (Matroids and M-convexity (Brändén, 2020))**.** Let $\mathcal{B}$ be the set of all bases of a matroid with ground set E. Note that $B \in \mathcal{B}$ can be associated to an M-convex set in $\{0, 1\}^{|E|}$ (that is, an set of ordered strings of 0's and 1's) whose sum is $|B|$. Then the polynomial $f_{\mathcal{B}} \in \mathbb{R}[x_i \ : \ i \in E]$ given by

$$f_{\mathcal{B}} = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$$

is Lorentzian.

An example of an application of this result can be seen below.

Example 18 (Extension of Example 15)**.** Recall that, in Example 15, our ground set E was formed by the columns of the matrix

$$M = \begin{array}{c} \begin{array}{cccc} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

considered over $\mathbb{R}$. We also recall the basis set of the matroid was

$$\mathcal{B} = \{\mathbf{v_1 v_2}, \mathbf{v_1 v_3}, \mathbf{v_1 v_4}, \mathbf{v_2 v_3}, \mathbf{v_2 v_4}\}.$$

Thus, Theorem 8 states that the polynomial

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$$

is Lorentzian.

Let us show a more concrete example of this using a larger matroid.

Example 19 (Example Using a Matroid of Rank 3)**.** Consider a matroid $(E, \mathcal{B})$ whose ground set E is produced by the columns of the following matrix:

$$M' = \begin{array}{c} \begin{matrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

over $\mathbb{R}$. Note that

$$\mathcal{B} = \{\mathbf{v_1v_2v_3}, \mathbf{v_1v_3v_4}, \mathbf{v_2v_3v_4}\}.$$

Thus, the polynomial of the form

$$p(x) = x_1x_2x_3 + x_1x_3x_4 + x_2x_3x_4$$

is Lorentzian. We can also convince ourselves of this using Theorem 7.

Note that the support of this polynomial,

$$\{(1,1,1,0), (1,0,1,1), (0,1,1,1)\}$$

is indeed an M-convex set; we see that, pairwise, the exchange axiom holds.

We can then investigate the Hessians of this polynomial. Note that, since

$$\partial_1 p(x) = x_2x_3 + x_3x_4,$$

we can write that the Hessian of $\partial_1 p(x)$ takes the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which, when considering multiplicties, has eigenvalues of $\{\sqrt{2}, 0, 0, -\sqrt{2}\}$, exactly one of which is positive. Note that $\partial_2 p(x)$ and $\partial_4 p(x)$ are the same polynomial, up to a change of labelling, so they too will have this property. We lastly note that

$$\partial_3 \, p(x) = x_1 x_2 + x_1 x_4 + x_2 x_4,$$

and as such, we can write that the Hessian of $\partial_3 \, p(x)$ takes the form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

With respect to multiplicities, we note that this polynomial has eigenvalues of $\{2, 0, -1, -1\}$, exactly one of which is positive. As such, this polynomial is indeed Lorentzian.

This is a really cool result! In particular, this result is useful in the proof of a particular result in combinatorics known as Mason's conjecture.

Mason's conjecture comes in three forms, each stronger than the last. We will state these three theorems below, then sketch out how Theorem 8 can be used to help prove this fact.

Theorem 9 (Mason's Conjecture (Anari et al., 2018))**.** For a matroid $M = (E, \mathcal{I})$ such that $|E| = n$ and with $\mathcal{I}_k$ independent sets of size k, the following is true:

i) $\mathcal{I}_k^2 \geq \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$. Note that this is the condition for log-concavity, as stated in Definition 2.

ii) $\mathcal{I}_k^2 \geq \left(1 + \frac{1}{k}\right) \cdot \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$.

iii) $\mathcal{I}_k^2 \geq \left(1 + \frac{1}{k}\right) \cdot \left(1 + \frac{1}{n-k}\right) \cdot \mathcal{I}_{k-1} \cdot \mathcal{I}_{k+1}$. Note that, by multiplying both sides by $\frac{1}{\binom{n}{k}^2}$, we obtain the statement $\frac{\mathcal{I}_k^2}{\binom{n}{k}^2} \geq \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}} \cdot \frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}}$, which is the condition for ultra log-concavity, as stated in Definition 3.

These are all written in an increasing order of strength, that is, the second implies the first, and the third implies both the second and first. While first conjectured in the 1970's, even the weakest formulation of this theorem had not yet been proven until 2018 using techniques in Hodge theory, a method of analysing surfaces that is outside the scope of this paper (Adiprasito et al., 2018). The work that had been made in proving the strongest version of this theorem had been very particular and seemed to only confirm it for matroids of size $n \leq 11$ or $k \leq 5$ (Kahn and Neiman, 2011), both of which are quite stringent conditions.

However, it is the case that, by showing polynomials generated by matroids are in fact Lorentzian, the third and strongest form of this conjecture can be proven quite effectively. We will not prove this thoroughly in this paper, but will give a sketch of how it could be done, and allow the reader to fill in the details.

Sketch of Proof. (Anari et al., 2018)

Given a matroid $\mathcal{M}$, we define r as the rank of $\mathcal{M}$ and n the size of its ground set. We order the ground set E in some arbitrary manner. In doing this, we can associate each element of the ground set to a variable x_i, with $1 \leq i \leq n$, depending on its order. We then construct the set of polynomials

$$\mathcal{I}_k(x_1, \cdots, x_n) = \sum_{\substack{I \in \mathcal{I} \\ |I|=k}} \prod_{x_i \in I} x_i.$$

We note that, for this polynomial, each summand represents an element I of $\mathcal{I}$ of size k, with a variable being multiplied in the summand if and only if its associated ground set element is an element of I. Thus, it serves as a way to codify which elements are in each subset of the independent sets of the matroids. We then construct the polynomial

$$\mathcal{I}_{\mathcal{M}}(x_1, \cdots, x_n, x_0) = \sum_{k=0}^{r} \mathcal{I}_k(x_1, \cdots, x_n) x_0^{r-k}.$$

We claim this to be strictly Lorentzian; one can check this by noticing it is homogenous and that its support must form a ground set, due to statements I2 and I3 of Definition 14. We can check the Hessian condition by utilising something called the *contraction* of a matroid; although this will not be explained here, a quick idea can be found in the linked paper by Anari et al., which points to a more robust explanation in the textbook by Oxley.

By proving this, one can then see that this implies the polynomial

$$f_{\mathcal{M}}(y, z) = \sum_{k=0}^{r} |\mathcal{I}_k| y^k z^{r-k}$$

is Lorentzian, as we have simply identified the variable $x_1 \cdots x_n$ with y and the variable x_0 with z. This is of the form of Theorem 6, and as such, we see that this sequence is in fact ultra-log concave.

Chapter 6

A Possible Non-Matroidal Direction

I want to end this brief survey by quickly describing an interesting direction in which we could apply Lorentzian polynomials not just to matroidal sequences, but too to log-concave and ultra log-concave sequences that are non-matroidal.

We define what it means for a polynomial to be denormalized Lorentzian.

Definition 16 (Denormalized Lorentzian Polynomials (Brändén et al., 2022))**.** Consider a polynomial $p(\mathbf{x})$ such that $\mathbf{x} = (x_1, x_2, \cdots x_n)$ defined as

$$p(\mathbf{x}) = \sum_{\mu} p_{\mu} x^{\mu}$$

for some set of elements μ such that μ. We define its **normalisation** to be the polynomial

$$N[p](\mathbf{x}) = \sum_{\mu} p_{\mu} \frac{x^{\mu}}{\mu!},$$

with the expression $\mu!$ defined as in Definition 6. If our original $p(\mathbf{x})$ was homogeneous and with positive real coefficients, then, if $N[p](\mathbf{x})$ is a Lorentzian polynomial, we say that $p(\mathbf{x})$ is **denormalized Lorentzian**.

This definition is interesting, because, it seems that if we can prove a sequence leads to a denormalised Lorentzian polynomial, it is possible that it could lead to showing that the original sequence is simply log-concave. This is of course still quite helpful in showing unimodality, the first concept introduced in this paper.

I would recommend checking out the following preprint by Hafner et. al, which discusses a knot-theoretic object called the Alexander polynomial (Hafner et al., 2023). One needs not understand knot theory in general or the Alexander polynomial in particular to appreciate the approach their proof takes, but hopefully, this provides a good idea of where Lorentzian polynomials could be applied in a non-matroidal context.

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