Discrete Analogues of the Poincaré-Hopf Theorem

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Discrete Analogues of the
Poincaré-Hopf Theorem

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My thesis unpacks the relationship between two discrete formulations of the Poincaré-Hopf index theorem. Chapter 1 introduces necessary definitions. Chapter 2 describes the discrete analogs and their differences. Chapter 3 contains a proof that one analog implies the other and chapter 4 contains a proof that the Poincaré-Hopf theorem implies the discrete analogs. Finally, chapter 5 presents still open questions and further research directions.
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Chapter 1

Introduction

This thesis discusses two different discrete analogues of the Poincaré-Hopf Theorem, which is a powerful theorem that relates analysis, topology, and combinatorics. We’ll state it more carefully soon, but roughly speaking, the Poincaré-Hopf Theorem says that the sum of the indices of the critical points of a vector field on a manifold is equal to the manifold’s Euler characteristic. We will define the terminology in the previous sentence in next section, starting with the Euler characteristic.

1.1 Definitions

The Euler characteristic is a topological invariant of a manifold that can be computed combinatorially or analytically. It is always an integer value, as will be evident from both definitions. Table 1.1 shows the Euler characteristics of some common manifolds.

A $d$-manifold is a surface that looks locally like $\mathbb{R}^d$. The disk in Table 1.1 is a 2-manifold (with boundary) whereas the 2-sphere (or 2-ball) and torus are 2-manifolds without boundary. From now on, we will assume all manifolds are two-dimensional without boundary.

We will also want to talk about manifolds without holes, especially as part of a given 2-manifold. A $d$-cell (or just cell if the dimension is clear) is a topological space that can be continuously deformed into a $d$-ball via stretching or twisting but not tearing.
Table 1.1  Euler characteristics of several manifolds. The disk and sphere are pictured on a reference plane to disambiguate them.

1.1.1 Euler Characteristic

Consider a 2-dimensional cell complex \( K = (V, E, F) \) with \( v = |V| \) vertices, \( e = |E| \) edges, and \( f = |F| \) faces. We define the **Euler characteristic** of \( K \) by

\[
\chi(G) := v - e + f.
\]

This definition will be useful very shortly.

We can compute the Euler Characteristic of a 2-manifold by subdividing its surface into polygonal 2-cells. It turns out this is always possible for a 2-manifold. As an interesting side note, it is not always possible for other values of \( d \) to subdivide the surface of a \( d \)-manifold into \( d \)-cells. Specifically, it is possible for \( d \leq 3 \), but in the 1980’s, a sequence of papers showed it is not possible for \( d \geq 4 \). [Manolescu (2016)] provides more detail. We provide this result only as a fun math fact; it will not be used in this thesis.

Euler’s surprising statement is that the Euler characteristic of a manifold
does not depend on the particular cell decomposition. Thus, it makes sense to also define

$$\chi(M) := \chi(G),$$

where $M$ is a manifold and $G$ is any decomposition of the surface of $M$ into polygons. [Henle(1979)] contains a proof that the Euler characteristic is invariant under different cell decompositions.

### 1.1.2 Index of a Critical Point

The Euler characteristic of a manifold can also be computed as the sum of the indices of critical points for a vector field embedded on its surface. We define the index of critical points in this section and then state this formally as the Poincaré-Hopf theorem. A proof can be found in [Milnor and Weaver(1997)].

A vector field on a $d$-manifold $M$ is a function $\mathcal{V}$ mapping each point on a manifold to a $d$-dimensional tangent vector at that point. A vector field is called continuous if the function $\mathcal{V}$ is continuous. Informally, this means two points close together on the surface of the manifold will have tangent vectors whose magnitude and direction are also close together. For example, a vector field of wind could map each point on the surface of the Earth to a vector parallel to the ground and pointing in the direction of the wind whose magnitude is the speed of the wind you would feel standing at that point. At some points, the wind might completely die off. A point at which the magnitude of the vector field is zero is called a critical point. The direction of a vector field may change at a critical point. For example, suppose you pointed two high-powered fans at each other. Where the streams meet, the wind vector field will quickly die down to nothing before pointing in the opposite direction.

The index of a point $p$ intuitively is the number of times the direction of the vector field makes a complete rotation (with respect to a fixed reference vector) as you traverse a small loop around $p$ counterclockwise. See [Henle(1979)] for a more formal definition, which we will not need. The index of a point $p$ in a vector field $\mathcal{V}$ on a manifold $M$ is denoted $\text{Ind}(p)$, where $\mathcal{V}$ and $M$ will be clear from context. It might be easiest to understand the definition of index from examples. We compute the index of several critical points below. Note the index is always 0 for noncritical points because the direction of the vector field does not change in a small loop around the point.
Figure 1.1  Four examples of critical points and their indices
1.2 The Poincaré-Hopf Theorem

We can now state the continuous version of the Poincaré-Hopf Theorem.

**Theorem 1.2.1** (Poincaré-Hopf). Suppose \( \mathcal{V} \) is a continuous vector field on a manifold \( M \) with a finite set of critical points \( P \). Then

\[
\sum_{p \in P} \text{Ind}(p) = \chi(M).
\]

Notice this implies that the left-hand side sum is well-defined. That is, the sum of the indices of critical points is independent of the vector field we choose. The main takeaway from this theorem is that the Euler characteristic relates many fields of math (combinatorics, analysis, and topology) through the Poincaré-Hopf theorem.

A well-known corollary of [1.2.1] is the hairy ball theorem. Take \( M \) to be the 2-sphere, which has index 2 (from [1.1]). It follows from [1.2.1] that any vector field on \( M \) with a finite set of critical points must have at least one critical point. Informally, there is no way for wind to flow on the Earth so that anywhere a leaf falls, it will be blown in the wind.

In the next chapter, we will explain two discrete analogues of the Poincaré-Hopf theorem.
Chapter 2

Discrete Analogues of the Poincaré-Hopf Theorem

2.1 Glass’ Discrete Analogue

The first formulation of Poincaré-Hopf for graphs that we will see is due to Glass [1973]. It applies to a directed graph embedded on a 2-manifold so that it subdivides the surface of the manifold into 2-cells.

2.1.1 Statement of Glass’ Theorem

We are interested in the index at critical points of the embedded graph, which we will say are vertices and faces. Later, we will discuss the completion of a Glass object, which will provide intuition for why we allow critical points on only vertices and faces and not edges.

We first define the index at a vertex. Suppose a vertex $v$ has $n$ neighbors and $l = (v_1, v_2, ..., v_n, v_1)$ is a list of them ordered by counterclockwise rotation about $v$. We say there is a direction change between consecutive vertices $v_i$ and $v_{i+1}$ in this list if the edge $vv_i$ is directed in (respectively, out) and the edge $vv_{i+1}$ is directed out (respectively, in). Define $d_v$ to be the number of direction changes between consecutive vertices in $l$. Then the index of $v$

$$\text{Ind}_{\text{Glass}}(v) = 1 - \frac{d_v}{2}.$$ 

Next we define the index of a cell $C$. Suppose $C$ is bounded by $n$ vertices and $(v_1, v_2, ..., v_n, v_1)$ is a list of them ordered by counterclockwise rotation.
Discrete Analogues of the Poincaré-Hopf Theorem

In this case, we say there is a direction change at $v_i$ if $v_{i-1}v_i$ and $v_iv_{i+1}$ are both directed towards $v_i$ or are both directed away from $v_i$. That is, the edges at $v_i$ either both point in or both point out. Let $d_c$ be the number of vertices $v_1, \ldots, v_n$ at which there is a direction change. Then the index of $C$

$$\text{Ind}_{\text{Glass}}(C) = 1 - \frac{d_c}{2}.$$ 

Note both of these indices must be integers. We are now ready to state Glass’ discrete analogue of the Poincaré-Hopf Theorem.

**Theorem 2.1.1** (Glass [1973]). Suppose $G = (V, E)$ is a directed graph embedded on a manifold $M$ so that each face in its face set $F$ is a topological disk. Then

$$\sum_{v \in V} \text{Ind}_{\text{Glass}}(v) + \sum_{f \in F} \text{Ind}_{\text{Glass}}(f) = \chi(M).$$

2.2 Knill’s Discrete Analogue

The second formulation of Poincaré-Hopf for graphs that we will see is due to [Knill, 2019]. It is more general than we will see. In particular, it holds for manifolds in arbitrary dimension. We will restrict our attention to what Knill says for 2-manifolds.

Suppose $G = (V, E)$ is a graph embedded on a manifold $M$ so that each face in its face set $F$ is a topological disk. Define a dimension function $H : X \to \mathbb{Z}$, where $X$ is the set of all vertices, edges, and faces:

$$H(x) = \begin{cases} 1 & \text{x is a vertex or face} \\ -1 & \text{x is an edge} \end{cases}$$

Also define a direction function $F : X \to V$ such that $F(x) \in x$. This means that vertices are directed towards themselves, edges to one of their endpoints, and faces to one of their bounding vertices.

Then we will define the Knill index of a vertex

$$\text{Ind}_{\text{Knill}}(v) = \sum_{x \in F^{-1}(v)} H(x)$$

**Theorem 2.2.1** (Knill [2019]). Let $M, G, \text{Ind}_{\text{Knill}}(\cdot)$ as defined above. Then

$$\sum_{v \in V} \text{Ind}_{\text{Knill}}(v) = \chi(M).$$
Notice that this is not very surprising in the case of directed graphs. Each vertex, edge, and face is directed toward exactly one vertex and vertices and faces are added by $H$ while edges are subtracted. It follows that

$$\sum_{v \in V} \text{Ind}_{\text{Knill}}(v) = |V| - |E| + |F|,$$

which is our combinatorial definition of Euler characteristic.

2.3 Comparing Knill and Glass’ Formulations

To illustrate the difference between Glass’ and Knill’s formulations, consider the following example. The left panel shows the indices of each vertex and cell of the complex as computed using Glass’ definition of index; the right panel shows the indices of each vertex as computed using Knill’s definition of index. Notice that the numbers do not correspond. Even worse, vertices that have the same Glass index do not necessarily have the same Knill index. For example, notice vertices $u$ and $v$ both have Glass index 0. However, because vertex $u$ has a face oriented towards it and vertex $v$ does not, they have different Knill indices.

![Figure 2.1](image.png)

**Figure 2.1** This example shows that the Knill and Glass indices of vertices are not equal in general. Glass indices are shown at left figure and Knill indices at right, both in red circles. The red dotted arrows show the direction of the faces, and the small orange arrows show that each vertex is directed to itself. Both are used only to compute the Knill index of the cell. The cycle is embedded on a sphere so that the inner and outer regions are both cells.
Chapter 3

Knill’s Theorem Implies Glass’ Theorem

This chapter shows a correspondence between Glass’ and Knill’s formulations of the Poincaré-Hopf theorem on graphs. Specifically, we will show how to convert a Glass object to a Knill object while preserving the indices of critical points in a specific sense, which will show that Knill’s Theorem implies Glass’ Theorem.

3.1 Discrete Analogues of Critical Points

Poincaré-Hopf-type theorems are statements of the form

\[ \sum \text{Ind}(P) = \chi(M) \]

Notice that if we consider the left-hand side of both Glass’ and Knill’s formulations of Poincaré-Hopf in this light, we see that for Glass, critical points can be either faces or vertices of the graph whereas for Knill, they can only be vertices.

**Theorem 3.1.1 (Glass [1973])**. Suppose \( G = (V, E) \) is a directed graph embedded on a manifold \( M \) so that each face in its face set \( F \) is a topological disk. Then

\[ \sum_{v \in V} \text{Ind}_{\text{Glass}}(v) + \sum_{f \in F} \text{Ind}_{\text{Glass}}(f) = \chi(M). \]
Theorem 3.1.2 (Knill [2019]). Let $M, G, \text{Ind}_{\text{Knill}}(\cdot)$ as defined in section 2.2. Then
\[
\sum_{v \in V} \text{Ind}_{\text{Knill}}(v) = \chi(M).
\]

We will resolve this discrepancy in critical points by inserting a vertex on each face of a Glass object, along with new edges, and orienting the new 0-1- and 2-cells thus formed in a specific way.

3.2 Converting a Glass Object to a Knill Object

We will show how to convert a complex $G$ as in Glass’ theorem to one $K$ as in Knill’s theorem such that indices of vertices are invariant (where we use Glass’ definition of index for $G$ and Knill’s definition for $K$) and indices of faces in $G$ become indices of corresponding vertices in $K$.

As discussed in the last section, since Glass allows for critical points to lie within faces of a graph, in general, we must add an additional vertex in each face when going from a Glass object to a Knill object to allow for the possibility that the face is a critical point. This is the first step in converting. We will also add new edges between the new vertex and each other vertex in the cell. Then we must choose a direction for the new edges and also orient the vertices and faces of the graph.

The vertices will be oriented toward themselves. To orient the new faces and edges, we will need a new definition.

Fix a particular face $f$ of $G$. Denote the graph of edges and vertices on its boundary by $B = (V, E)$. Notice $B$ is a cycle, so each vertex in $V$ must have exactly two neighbors in $E$. Still with reference to a particular face $f$ with boundary $B = (V, E)$, we will call a vertex $v \in V$ a sink for $f$ if both edges in $E$ that are incident to $v$ are directed towards $v$.

Now, choose to direct each new edge and face to a sink vertex if it is incident to one and to the new vertex otherwise.

To summarize, here is our algorithm to convert from a Glass complex to a Knill complex.
Algorithm 1 Conversion algorithm from Glass to Knill

for every vertex \( v \) in \( G \) do
    Direct the vertex towards itself
end for

for each face \( f \) of \( G \) with boundary \( B = (V, E) \) do
    Add a new vertex \( v_f \) embedded inside of \( f \) whose neighbor set is \( V \)
    for \( v \in V \) do
        if \( v \) is a sink for \( f \) then
            Direct \( vv_f \) towards \( v \)
            Direct all new faces incident to \( v \) towards \( v \)
        else
            Direct \( vv_f \) towards \( v_f \)
        end if
    end for
    for each remaining face do
        Direct the face towards \( v_f \)
    end for
end for

Here are some properties of \( G \) and \( K \).

Theorem 3.2.1. 1. \( K \) is a triangulation.

2. For any vertex \( v \in G \), \( Ind_{Glass}(v) = Ind_{Knill}(v) \).

3. For any new vertex \( v_f \in K \setminus G \) embedded in the face \( f \), \( Ind_{Knill}(v) = Ind_{Glass}(f) \).

Proof. 1. We subdivided each existing face into triangular faces by adding a vertex inside and adding edges between it and every vertex on the boundary of the face.

2. Recall the Glass index of a vertex \( v \),

\[
Ind_{Glass}(v) = 1 - \frac{\# \text{ dir changes}}{2}.
\]

We will partition the index of \( v \) into the first term, 1, plus the contribution to the second term of each face incident with \( v \). This contribution is \( \frac{\# \text{ dir changes within the face}}{2} \).
Now, we will partition the Knill index of \( v \) into the contribution of the vertex being directed towards itself, which is 1, plus the contributions of faces and edges incident to \( v \). Each triangular face in \( K \) is contained in exactly one cell in \( G \), so its contribution to the index of \( v \) in \( K \) does not change when partitioning by cells. Similarly, the edge between \( v \) and the new vertex is contained within one face. However, the boundary edges of each face in \( G \) are shared between two faces (since we have assumed all manifolds are without boundary), so they will each contribute half their value to the Knill index when partitioning by faces.

We show that each face contributes the same amount to the Glass and Knill indices of \( v \) (where the contribution of a face includes the contribution of the edges on the boundary of the face which are also incident to \( v \)). Because the only other term in the index of \( v \) is 1 in both the Glass and Knill index formulas, it follows that the index of \( v \) is the same in \( G \) and \( K \).

We consider different cases. Recall that with reference to a particular face \( f \) with boundary \( B = (V, E) \), we call a vertex \( v \in V \) a sink if both edges in \( E \) that are incident to \( v \) are directed towards \( v \). Similarly, we call \( v \) a source if both such edges are directed away from \( v \). Finally, we call \( v \) a buffer if one of the edges is directed towards \( v \) and the other, away from \( v \).

- If \( v \) is a sink, the number of direction changes within \( f \) is 0, because both edges on the boundary of \( f \) point towards \( v \) (the edge \( vv_f \) is not present in \( G \), only in \( K \), so it cannot contribute direction changes). Hence, the contribution of \( f \) to the Glass index of \( v \) is 0. In \( K \), the two boundary edges in \( E \) incident to \( v \) and the edge between \( v \) and the new vertex are all directed towards \( v \). So the boundary edges contribute \(-\frac{1}{2}\) (half their value) and the edge \( vv_f \) contributes \(-1\) to the Knill index of \( v \) within \( C \). The total edge contribution of \( f \) is therefore \(-2\). The two triangular faces within \( f \) that share the edge \( vv_f \) are also both directed towards \( v \). Therefore, the contribution of \( f \) from the orientation of \( f \) alone (and not also the edges on the boundary of \( f \)) to the Knill index of \( v \) is 2. So the total contribution of \( f \) to the Knill index of \( v \) in \( K \) is the sum of the contributions of edges and faces, \(-2 + 2 = 0\), which equals the contribution of \( f \) to the Glass index of \( v \) in \( G \), as desired.
• If $v$ is a source, the number of direction changes within $f$ is again 0, because both edges on the boundary of $f$ point away from $v$. Hence, $f$ contributes 0 to Glass index of $v$. Now in $K$, the two boundary edges in $E$ incident to $v$, the edge between $v$ and the new vertex, and the triangular faces within $f$ incident to $v$ are all directed away from $v$. So edges and faces on the boundary of $f$ contribute 0 to the index of $v$, and hence the total contribution of $f$ to the Knill index of $v$ is also 0. It follows that $f$ contributes the same to the Knill index of $v$ in $K$ as to the Glass index of $v$ in $G$.

• The last case is when $v$ is a buffer. In this case, the number of direction changes within $f$ is 1, Hence, the contribution of $f$ to the Glass index of $v$ is $-\frac{1}{2}$. In $K$, the new edge between $v$ and the new vertex and the new triangular faces incident to $v$ are all directed away from $v$, as is one of the edges on the boundary. The other boundary edge is directed towards $v$ and thus contributes $-\frac{1}{2}$ to the Knill index of $v$ in $f$. So in all cases, the Glass index of $v$ in $G$ equals the Knill index of $v$ in $K$.

3. Let $n$ be the number of vertices of $f$ and $s$ be the number of sinks for $f$. We will show that $\text{Ind}_{\text{Knill}}(v) = \text{Ind}_{\text{Glass}}(f) = 1 - s$.

• $\text{Ind}_{\text{Knill}}(v) = 1 - s$. Notice that we cannot have two adjacent sinks for $f$ because the edge between them would need to be directed toward one or the other. So, each sink has exactly two faces directed towards it. All other $n - 2s$ faces are directed toward $v$. Each sink has exactly one edge directed toward it and the other $n - s$ edges are directed toward $v$. So

$$\text{Ind}_{\text{Knill}}(v) = 1 + (n - 2s) - (n - 2) = 1 - s.$$  

• $\text{Ind}_{\text{Glass}}(f) = 1 - s$. Notice that the number of direction changes around a face is twice the number of sinks. To see this, consider traversing the boundary of $f$ starting on an arbitrary edge. Sink vertices correspond to the direction of edges changing from with the direction of traversal to against it. Similarly, passing a source vertex flips the direction of edges from against our traversal to with it. This means that source and sink vertices must alternate on the boundary of $f$, and the total number of direction changes is the number of source vertices plus the number of sink vertices. Because we stop counting direction changes once we have reached
the edge we started on, the direction must return to what we started with. For this to happen, there must be the same number of source vertices as sink vertices. It follows that the total number of direction changes along the boundary of $f$ is $2s$, so

$$\text{Ind}_{\text{Glass}}(f) = \frac{2s}{2} = 1 - s.$$  

\[ \square \]

### 3.3 Knill’s Theorem Implies Glass’ Theorem

We now have done most of the work to show the main result of this chapter, that Knill’s Theorem implies Glass’ Theorem. To see why, suppose we have a Glass object $G = (V, E)$ embedded on $M$ with face set $F$ so that each face $f \in F$ is a topological disk. Then if $K = (V_K, E_K, F_K)$ is the Knill object obtained from the conversion process in algorithm\[1\]

$$\sum_{v \in V} \text{Ind}_{\text{Glass}}(v) + \sum_{f \in F} \text{Ind}_{\text{Glass}}(f) = \sum_{v \in V} \text{Ind}_{\text{Knill}}(v) + \sum_{f \in F} \text{Ind}_{\text{Knill}}(v_f)$$

$$= \sum_{v \in V_K} \text{Ind}_{\text{Knill}}(v)$$

$$= \chi(M) \text{ (by 2.2.1, Knill’s Theorem).}$$

This shows that Knill’s Theorem implies Glass’ Theorem.
Chapter 4

Poincaré-Hopf Theorem Implies Knill’s Theorem

In this chapter, we describe a method of completing a discrete vector field to a continuous one. Suppose $K$ is a Knill object, as defined in section 2.2. We will define a vector field on $M$ with the following properties:

- The magnitude of the vector field is zero at the vertices of the graph and nonzero elsewhere. Thus, $\mathcal{V}$ preserves critical points.

- The index of each critical point in the completion $\mathcal{V}$ equals the discrete index of the vertex.

This shows that the Poincaré-Hopf Theorem implies Knill’s discrete analogue.

4.1 Completion Sketch

In this section, we will sketch a vector field completion. This will be enough to see intuitively (and to rigorously prove in 4.2) that the index of vertices is preserved in the completion. That is, the Knill index of a vertex equals the index of that point in $\nu$. We then will demonstrate an explicit vector field in 4.3 that shows a topologically equivalent completion is possible and prove in 4.3.3 that the vertices of the cell complex are the only critical points in this vector field.

To show that the index of each critical point is preserved, we will show something stronger: the contribution of each cell to the index of the critical point is preserved. For this, we will need a to consider how much each
triangle containing a critical point contributes to its index. We will define this contribution as the cell index of \( v \) (in a cell \( C \)). Because we have different definitions of index in discrete and continuous settings, we need to define cell index in both cases. We will set up these definitions in discrete and continuous settings in the next two subsections, respectively.

### 4.1.1 Knill Cell Index

We will now formally define the Knill cell index of a vertex \( v \) in a cell \( C \). Let \( X \) be the set of all vertices, edges, and faces of a 2-dimensional simplex. Recall we previously defined an orientation function \( F : X \to V \) where \( F(x) \in x \) and a dimension function \( H : X \to \{-1, 1\} \) by

\[
H(x) = \begin{cases} 
1 & \text{if } x \text{ is a vertex or a face} \\
-1 & \text{if } x \text{ is an edge}
\end{cases}
\]

We now define \( H' : X \to \mathbb{Z} \) by

\[
H'(x) = \begin{cases} 
1 & \text{if } x \text{ is a face} \\
-\frac{1}{2} & \text{if } x \text{ is an edge}
\end{cases}
\]

Furthermore, we define the Knill cell index of \( v \) in \( C \) as

\[
\text{Ind}_{\text{Knill}}(v, C) = \sum_{x \in F^{-1}(v) \cap C} H'(x).
\]

We will relate the cell index of \( v \) to the index of \( v \). First, we partition the sum in the definition of cell index by dimension.

\[
\text{Ind}_{\text{Knill}}(v, C) = \sum_{x \in F^{-1}(v) \cap C} H'(x)
\]

\[
= \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is an edge}} H'(x) + \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is a face}} H'(x)
\]

\[
= \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is an edge}} -\frac{1}{2} + \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is a face}} 1.
\]

We make some observations about what happens in each of the cases when \( x \) is an edge or a face.
Each edge $e$ is contained in exactly two faces $C_1$ and $C_2$. Say $F(e) = v$, that is, $e$ is directed to the vertex $v$. Then $C_1, C_2 ⊃ v$. Therefore, if we fix a vertex $v$, after a bit of staring it makes sense that

$$
\sum_{x \in F^{-1}(v), \ x \text{ is an edge}} -1 = \sum_{C \ni v, x \in F^{-1}(v) \cap C, \ x \text{ is an edge}} \frac{1}{2}.
$$

On the other hand, each face $C$ is contained in exactly one face, namely itself. Therefore,

$$
\sum_{x \in F^{-1}(v), \ x \text{ is a face}} 1 = \sum_{C \ni v, x \in F^{-1}(v) \cap C, \ x \text{ is a face}} 1.
$$

We can relate the Knill cell index to the Knill index by summing over all cells. Because the vertex $v$ must be directed towards itself, it also contributes 1 to the Knill index. Since $v$ is not counted in any of the cell indices, we will add a +1 term counting the contribution of $v$ to its own index.

$$
1 + \sum_{C \ni v} \text{Ind}_{Knill}(v) = 1 + \sum_{C \ni v} \left( \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is an edge}} -\frac{1}{2} + \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is a face}} 1 \right)
= 1 + \sum_{C \ni v} \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is an edge}} -\frac{1}{2} + \sum_{C \ni v} \sum_{x \in F^{-1}(v) \cap C, \ x \text{ is a face}} 1
= 1 + \sum_{x \in F^{-1}(v), \ x \text{ is an edge}} -1 + \sum_{x \in F^{-1}(v), \ x \text{ is a face}} 1
= \sum_{x \in F^{-1}(v)} H(x)
= \text{Ind}_{Knill}(v, C).
$$

We summarize this relation between the Knill index and the Knill cell index in the following theorem.

**Theorem 4.1.1.** For any vertex $v$ in a 2-dimensional cell complex, the Knill indices of the vertex satisfy

$$
\text{Ind}_{Knill}(v) = 1 + \sum_{C \ni v} \text{Ind}_{Knill}(v, C).
$$
4.1.2 Cell Index

Recall we compute the index of a critical point \( p \) in a vector field \( \mathcal{V} \) by walking completely around \( p \) once counterclockwise (not including any other critical points than \( p \) in our loop) and counting the number of counterclockwise rotations \( n \) that the vector field \( \mathcal{V} \) makes with respect to our path. Then \( \text{Ind}(v) = 1 + n \).

Now suppose that \( p \) is a vertex of \( G \) embedded on a manifold. Let’s rename \( p \) to remind ourselves to think of it as a vertex in a cell complex. Set \( v = p \). Walking once clockwise around \( v \) involves walking once counterclockwise through each 2-cell incident to \( v \).

Let’s define the cell index of \( v \) in the 2-cell \( C \) in the vector field \( \mathcal{V} \) to be the number of rotations \( \mathcal{V} \) makes with respect to our path when we walk counterclockwise through the cell. Here, the direction “counterclockwise” is determined by the way we embed \( G \) on \( M \). Suppose the neighbors of \( v \) in the embedding are \( w_1 w_2 \ldots w_d \), ordered counterclockwise. Then a counterclockwise path through the cell containing edges \( vw_i \) and \( vw_{i+1} \) starts on the edge \( vw_i \) and ends on the edge \( vw_{i+1} \). Denote the cell index of \( v \) in \( C \) by \( \text{Ind}(v, C) \). Then

**Theorem 4.1.2.**

\[
\text{Ind}(v) = 1 + \sum_{C \ni v} \text{Ind}(v, C).
\]

Now that we have defined the cell index and the Knill cell index of a vertex, we will show that under the vector field \( \mathcal{V} \) that we have defined, the cell index of each critical point is equal to its Knill cell index.

4.2 Index is Preserved

In this section, we will show that the Knill index of a vertex is equal to its index in \( \mathcal{V} \). From 4.1.2 and 4.1.1, we have that

\[
\text{Ind}(v) = 1 + \sum_{C \ni v} \text{Ind}(v, C)
\]

and

\[
\text{Ind}_{\text{Knill}}(v) = 1 + \sum_{C \ni v} \text{Ind}_{\text{Knill}}(v, C).
\]

We will show that the Knill cell index of a vertex is equal to the cell index of that vertex in 4.2.1. It then follows that \( \text{Ind}(v) = \text{Ind}_{\text{Knill}}(v) \).
Theorem 4.2.1.
\[ \text{Ind}(v, C) = \text{Ind}_{\text{Knill}}(v, C) \]

Proof. We consider different cases depending on the orientation of the two edges and face of $C$ incident with $v$.

- For the first case, suppose these edges and face are both oriented towards $v$. Since the vector field $V$ is completed by linear interpolation, it will always point towards $v$. Therefore, the vector field will always point to our left as we walk from one edge to another, so it does not rotate at all with respect to a counterclockwise path through $C$. Therefore, $\text{Ind}(v, C) = 0$. On the other hand,

\[
\text{Ind}_{\text{Knill}}(v, C) = \sum_{x \in F^{-1}(v) \cap C, x \text{ is an edge}} -\frac{1}{2} + \sum_{x \in F^{-1}(v) \cap C, x \text{ is a face}} 1
\]

\[
= -\frac{1}{2} + -\frac{1}{2} + 1
\]

\[
= 0.
\]

Therefore, in this case, the result $\text{Ind}_{\text{Knill}}(v, C) = \text{Ind}(v, C)$ holds.

- For the second case, suppose these edges and face are both oriented away from $v$. Similarly, $V$ will always point away from $v$ so to our right as we walk from one edge to another. Therefore we have again that $\text{Ind}(v, C) = 0$. The Knill cell index

\[
\text{Ind}_{\text{Knill}}(v, C) = \sum_{x \in F^{-1}(v) \cap C, x \text{ is an edge}} -\frac{1}{2} + \sum_{x \in F^{-1}(v) \cap C, x \text{ is a face}} 1
\]

\[
= 0 + 0
\]

\[
= 0.
\]

So again we have $\text{Ind}_{\text{Knill}}(v, C) = \text{Ind}(v, C)$.

- Now suppose these edges and face are both oriented away from $v$. Similarly, $V$ will always point away from $v$ so to our right as we walk from one edge to another. Therefore we have again that $\text{Ind}(v, C) = 0$. 
The Knill cell index

\[
\text{Ind}_{\text{Knill}}(v, C) = \sum_{x \in F^{-1}(v) \cap C, \ x \ is \ an \ edge} -\frac{1}{2} + \sum_{x \in F^{-1}(v) \cap C, \ x \ is \ a \ face} 1
\]

\[
= 0 + 0
\]

\[
= 0.
\]

So again we have \(\text{Ind}_{\text{Knill}}(v, C) = \text{Ind}(v, C)\).

- The other cases are similar. The third case is when the edges are both oriented towards \(v\) and the face, away. The fourth case is when the edges are both oriented away from \(v\) and the face, towards. The fifth case is when exactly one of the edges is oriented towards \(v\) and the face is oriented away. The sixth and final case is when the face and exactly one of the edges are oriented towards \(v\).

\[\square\]

Thus, we have shown the index of each critical point in the completion \(\mathcal{V}\) equals the discrete index of the vertex.

### 4.3 Explicit Vector Field

We now show that it is possible to complete the vector field as in \[4.1\] so that the vector field is 0 at the critical points and nonzero elsewhere. We will define a vector field \(\mathcal{V}\) in barycentric coordinates.

#### 4.3.1 Barycentric Coordinates

We describe our vector field, not in Cartesian coordinates, but in barycentric coordinates. This will enable us to define a completion of a directed abstract simplicial complex without needing to specify how the complex is embedded on a manifold.

Barycentric coordinates specify the position of a point in a triangle in terms of the triangle’s vertices. Because a triangle is a convex shape, every point in a triangle can be written a convex combination of the triangle’s vertices. That is, if the vertices of a triangle \(\tau\) are \(A, B,\) and \(C\), then any \(x \in \tau\) can be written as \(x = aA + bB + cC\) where \(a, b, c > 0\) and \(a + b + c = 1\). The ordered triple \((a, b, c)\) gives the barycentric coordinates of \(x\).
Figure 4.1 shows the barycentric coordinates of five points within a triangle. We arbitrarily label the vertices of the triangle as $A$, $B$, and $C$. Because $A = 1A + 0B + 0C$, the barycentric coordinates of $A$ are $(1, 0, 0)$. The purple point on the edge of the $\triangle ABC$ is three-quarters of the way along the vector pointing from $A$ to $C$, so it is $\frac{1}{4}A + \frac{3}{4}C$. The purple point in the interior of $\triangle ABC$ is the midpoint of $\left(\frac{1}{4}, 0, \frac{3}{4}\right)$ and $\left(\frac{1}{4}, \frac{3}{4}, 0\right)$, so its coordinates are $\frac{1}{2}\left[\left(\frac{1}{4}, 0, \frac{3}{4}\right) + \left(\frac{1}{4}, \frac{3}{4}, 0\right)\right]$, which are $\left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}\right)$. Notice that for each of the five points shown, the sum of the coordinates is 1, the sum of the coefficients in a convex combination. This is because each point is a convex combination of $A$, $B$ and $C$.

Note that $\triangle ABC$ need not be an equilateral triangle. For our purposes, it will not matter what shape it is or even whether the edges are straight lines in $\mathbb{R}^3$. We can stretch and twist an equilateral triangle until its shape matches that of any embedding of a triangular cell that is simply connected. Because continuity is a topological property, stretching and twisting does not change whether a vector field defined in terms of barycentric coordinates is continuous. Also, a vector field in barycentric coordinates vanishes exactly at the vertices of the triangular 2-cells will vanish at exactly those vertices in
any isomorphism.

### 4.3.2 Vector Field Definition

We will define the completion of a vector field within each cell and show that it is well-defined along cell boundaries. Suppose $\Delta ABC$ is a 2-cell of the complex and without loss of generality, say that the face $\{A, B, C\}$ is oriented towards $A$. That is, $F(\{A, B, C\}) = A$. We will first define the value of the vector field $v: \sigma \to \mathbb{R}^3$ at six special points of $\Delta ABC$ and then fill in the rest.

At each vertex $A, B$ and $C$ of the triangular cell, set the value of the vector field to be $\hat{0}$. Then set the value of the midpoint $M_{XY}$ of each directed edge $\overrightarrow{XY}$ where $X, Y \in \{A, B, C\}$ to be $X - Y$. Finally, set the value of the centroid $Z$ of $\Delta ABC$ to be $A - \frac{1}{2}B - \frac{1}{2}C$. These values are shown in barycentric coordinates in an example in figure 4.2.

Now we will fill in the rest of the vector field. First, we will subdivide $\Delta ABC$ into six smaller triangular cells: $AM_{AB}Z, AM_{AC}Z, BM_{AB}Z, BM_{BC}Z, CM_{AC}Z,$ and $CM_{BC}Z$ as in 4.2. We refer to these smaller triangular cells as subcells. Any point $p \in \Delta ABC$ lies in at least one subcell. Suppose $p = (a, b, c)$ in barycentric coordinates for the subcell $xyz$. Define $V(p) = aV(x) + bV(y) + cV(z)$.

Notice $V$ is well-defined: if $p$ is on the boundary of two subcells (not necessarily contained within the same larger cell), then its barycentric coordinates will be nonzero only on the endpoints of the edge(s) containing $p$, which are shared between both subcells. Therefore, the vector field $V(p) = aV(x) + bV(y) + cV(z)$ will have one coefficient vanish, say $c$. The remaining four quantities $a, V(v), b,$ and $V(y)$ do not depend on which cell is chosen, so $v$ is well-defined.

Also, $V$ is continuous because it linearly interpolates between six isolated points.

### 4.3.3 The only critical points fall on vertices of the cell complex

By construction, $V(v) = 0$ for any vertex $v$ of the complex. We need to check that $V(p) \neq 0$ for any $p$ which is not a vertex of the complex.

Any point $p$ in a 2-cell of the complex lies in at least one subcell $xyz$ as defined in 4.3.2. Moreover, if we write $p = ax + by + cz$, then $V(p) = aV(x) + bV(y) + cV(z)$. That is, the value of the vector field at $p$ is a convex
We define $\mathcal{V}$ as above. At the vertices of a 2-cell (blue points), set $\mathcal{V} = 0$. At the midpoints of edges and the centroid of the cell (purple points), the value of $\mathcal{V}$ is determined by the orientations of the edges and face, respectively, as in section 4.3.2.

Combination of the values at $x, y,$ and $z$. In particular, $\mathcal{V}(p)$ is a linear combination of $\mathcal{V}(x), \mathcal{V}(y),$ and $\mathcal{V}(z)$.

At least one of $x, y,$ and $z$ is a vertex of the complex, say $x$. At least one of the remaining vertices, say $y$ is the centroid of $\Delta ABC$. Then $z$ is the midpoint of an edge of $\Delta ABC$. We can then say $\mathcal{V}(x) = (0, 0, 0), \mathcal{V}(y)$ has all coordinates nonzero, and $\mathcal{V}(z)$ has exactly one of its coordinates equal to zero. This means that $\mathcal{V}(y)$ and $\mathcal{V}(z)$ are linearly independent vectors. Suppose that

$$\mathcal{V}(p) = a \mathcal{V}(x) + b \mathcal{V}(y) + c \mathcal{V}(z)$$
$$= a \cdot 0 + b \mathcal{V}(y) + c \mathcal{V}(z)$$
$$= b \mathcal{V}(y) + c \mathcal{V}(z)$$
$$= (0, 0, 0).$$

Since $\mathcal{V}(y)$ and $\mathcal{V}(z)$ are linearly independent, this implies $b = c = 0$, so $p = x$ is a vertex of the complex. This shows
**Theorem 4.3.1.** The vector field $\mathcal{V}(p) = 0$ if and only if $p$ is a vertex of the complex.

### 4.4 The Poincaré-Hopf Theorem Implies Knill’s Theorem

This completion shows that the continuous Poincaré-Hopf theorem implies the discrete Poincaré-Hopf theorem. We conclude this chapter by recapping the known relationships between the Poincaré-Hopf theorem and Knill and Glass’ discrete analogues.

![Implication Graph](image)

**Figure 4.3** This implication graph summarizes results we have proven and open questions.

### 4.5 The Poincaré-Hopf Theorem Implies Glass’ Theorem

We showed in this chapter that the Poincare-Hopf Theorem implies Knill’s Theorem and in chapter 3 that Knill’s Theorem implies Glass’ Theorem. This means we get for free that the Poincare-Hopf Theorem implies Glass’ Theorem.

Geometrically, we can understand the process of completing a Glass discrete vector field as first converting the Glass object to a Knill object by inserting critical points on the faces, as in chapter 3 and then completing the vector field in the resulting Knill object as discussed in this chapter.

Converting a Glass object to a Knill object yields a triangulation. Because there are exactly three edges and one face for each triangle, we can enumerate all possible cells and their orientations in the Knill object and draw their completions.

There are four possible orientations of a triangular 2-cell and its lower-dimensional subcells, up to vertex symmetry. These are shown shown in figure 4.4. The completion defined in section 4.3 is topologically equivalent to that in figure 4.5.
Figure 4.4 There are four possible orientations of a 2-cell and its subcells in a triangulation, up to vertex symmetry.
Figure 4.5  These flow lines show a completion of each cell in figure 4.4. The completions are topologically equivalent to those in section 4.3.
Chapter 5

Further Questions

5.1 Other Implications

This thesis has related two discrete analogues of the Poincaré-Hopf Theorem to each other and to the Poincaré-Hopf Theorem. However, several directions in 4.3 have not been proven. For example, it seems likely that we might also be able to convert a Knill object to a Glass object so that the Knill index of every vertex is equal to its Glass index and every additional vertex and every face in the Glass object has index zero. This is in line with the flavor of chapters 3 and 4 and would show that Knill’s Theorem implies Glass’ Theorem.

Another more difficult direction is showing that either Knill or Glass’ discrete analogue implies the Poincaré-Hopf Theorem or even a shadow of the Poincaré-Hopf Theorem. It seems likely this implication would not hold. It would be interesting to identify why and in what cases the implication does not hold.

5.2 Higher Dimensions

Another direction for future research is generalizing chapter 4 results in this thesis to higher dimensions. That is, the Poincaré-Hopf Theorem and Knill’s Theorem both hold in arbitrary dimension. In what cases can we complete a Knill object to a continuous vector field on a manifold in a way that preserves the index of vertices and does not introduce new critical points?
5.3 Generalization of Discrete Analogues

Finally, a third direction is formulating a generalization of the Knill and Glass’ discrete analogues in which an arbitrary subset of the cells (vertices, edges, and faces) are allowed to be directed. This could be useful in applications in which only partial information about a direction function is known.
Bibliography


