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Synopsis

The infinite plays a significant role in many of Franz Kafka's stories. In this note we show that a version of the middle-thirds Cantor set construction appears in Kafka's *Great Wall of China*. His description of the Wall's construction mirrors an iterated system of functions whose limit set is the Cantor set; we present the relevant mathematical details and a close reading of the text of the story to illustrate this metaphor.

> *Two ideas—or rather two obsessions—pervade the work of Franz Kafka. The first is subordination, the second is infinity.*
>
> –Jorge Luis Borges

Judith Butler [1] called Franz Kafka's fiction the "poetics of nonarrival," and for good reason. Kafka's characters (or readers) are often waiting for something which never arrives, as in *An Imperial Message* [6], or are paralyzed by inaction, unable to begin, as in *Before the Law* [3]. In the former, a dying emperor whispers a message to a courier for delivery to you (the reader), but the messenger must first pass through the palace and its extensive network of rooms, then through the courtyards and the second palace containing the first, and so on. There appear to be an infinity of these barriers and even if he manages to clear all of them he will still have to get through the capital city and then across the country to reach you; Kafka assures us that it will "never, never happen". In *Before the Law*, a man is confronted with what we

are told is an infinity of gatekeepers, each more terrible than the last, guarding a series of doors that lead to the law. The man never proceeds beyond the first door and dies outside the gates. These stories have been analyzed at length and compared to various versions of Zeno's Paradox (see, e.g., [8]). Kafka also employs paradoxes of time and space to induce uneasiness in his readers; see *A Common Confusion* [4], for example.

If any branch of mathematics could be called Kafkaesque, it would have to be Georg Cantor's anxiety-inducing, intuition-challenging study of the infinite. Before Cantor, everyone more or less assumed that infinity is infinity; that is, there is only one level of infinity, or more accurately that all infinite sets have the same cardinality. Cantor demonstrated rather dramatically that this is false. In fact, a consequence of his work is that there is an infinity of infinities, each larger than the last.

I imagine Kafka, who claimed to have great difficulties with all things scientific, would have appreciated the mathematical abyss Cantor opened up for us. I do not use the term abyss lightly—Cantor was attacked and mocked by his contemporaries, often viciously, and this fueled his depression and ultimately led to multiple hospitalizations for treatment; he died poor and malnourished in a sanatorium in 1918. Decried by Poincaré as a "grave disease" afflicting mathematics [2, page 266] and condemned by Wittenberg as "utter nonsense" [7], Cantor's work survived this initial rejection and is considered fundamental to modern mathematics.

Mathematics is often used as metaphor in literature and film. Aside from the instances of Zeno's Paradox in the work of Kafka and others (Borges, Calvino, Eco, etc.), writers and filmmakers have employed ideas from dynamical systems (e.g., Stoppard's *Arcadia*), probability (e.g., *Hamlet, Candide*), number theory (e.g., Aronofsky's π: *Faith in Chaos*), uncertainty (e.g., Frayn's *Copenhagen*), and other fields to tell a story. These metaphors help the reader/viewer understand the human condition through another lens, revealing unexpected connections that might elude a traditional literary analysis.

In this paper I will illustrate an instance of the Cantor set (defined below) in the narrative of Kafka's story *The Great Wall of China*. It is almost certainly unintentional on Kafka's part, but his description of the construction of the Wall mirrors a process from the theory of iterated systems of functions. It is this parallel we discuss in the final section, yielding an enhanced understanding of Kafka's description of the Wall.

The Cantor Set

We start with an overview of the construction of the middle-thirds Cantor set C. Begin with the interval $[0, 1]$ and remove the middle open interval $(1/3, 2/3)$. Then remove the middle third of each of the remaining subintervals: $(1/9, 2/9)$ and $(7/9, 8/9)$. Continue this process; at the nth stage removing 2^{n-1} intervals of length 3^n. The set C is what remains at the end of this process. See Figure 1 below for an illustration.

Figure 1: Cantor set, in seven iterations. Public domain image from https://commons.wikimedia.org/wiki/File:Cantor_set_in_seven_iterations.svg, accessed on January 27, 2017.

The Cantor set is one of the standard examples mathematicians use to challenge the intuition of undergraduates. Indeed, it is a bizarre set. Note that the total length of the intervals we remove is

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1/3}{1 - 2/3} = 1;$$

it would appear, then, that we have removed "everything" from the interval. Yet we know the set C is nonempty since it contains the endpoints of the intervals we removed ($1/3$ and $7/9$, for example). The situation is even stranger than that, however: the set C is *uncountable*. The simplest way to see this is to use the ternary expansion of numbers in the interval $[0, 1]$. Given any $x \in [0, 1]$ we may write

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

where each a_i lies in the set $\{0, 1, 2\}$. Elements of C correspond to those real numbers having ternary expansions with each a_i in $\{0, 2\}$; that is, there are no 1's in the ternary expansion. Note that, for example,

$$\frac{1}{3} = 0.10000\ldots_3 = 0.022222\ldots_3,$$

so that $1/3$ is indeed a number of this form. We may then show that C is uncountable via Cantor's diagonalization argument: If C were countable we would have a bijection $f : \mathbb{N} \to C$. Consider the following real number x: the ith ternary digit of x is 2 if the ith digit of $f(i)$ is 0, and is 0 if the ith digit of $f(i)$ is 2. Then x is not in the image of f, a contradiction.

The Cantor set then forms an uncountable dust sprinkled along the unit interval. It is furthermore a fractal. Indeed, it may be divided into 2^n subsets, each of which may be scaled by 3^n to get the whole set C; its fractal dimension is then

$$\frac{\ln 2^n}{\ln 3^n} = \frac{n \ln 2}{n \ln 3} \approx 0.631.$$

As a topological space, the Cantor set is totally disconnected of topological dimension 0. In other words, C has no nontrivial connected subsets and any open cover of C has a refinement consisting of disjoint open sets such that every point of C lies in exactly one element of the refinement.

We may also construct C as the attracting set of a system of iterated functions. Consider the functions $F_0, F_1 : [0, 1] \to [0, 1]$ defined by

$$F_0(x) = \frac{1}{3}x \qquad \text{and} \qquad F_1(x) = \frac{1}{3}(x - 1) + 1 = \frac{1}{3}x + \frac{2}{3}.$$

Note that F_0 moves a point x two-thirds of the way toward 0 and F_1 moves x two-thirds of the way toward 1. Now, suppose we begin with an arbitrary $x_0 \in [0, 1]$ and iteratively apply the functions F_0 and F_1 with equal probability at each step to generate a sequence

$$x_0, x_1, \ldots, x_n, \ldots$$

where for $j \geq 1$, $x_j = F_i(x_{j-1})$ for some $i = 0, 1$. If $x_0 \in (1/3, 2/3)$ then both F_0 and F_1 take x_0 outside $(1/3, 2/3)$. Also, F_0 takes $[0, 1/3]$ to $[0, 1/9]$ and $[2/3, 1]$ to $[2/9, 1/3]$. Similarly, F_1 takes $[0, 1/3]$ to $[2/3, 7/9]$ and $[2/3, 1]$ to $[8/9, 1]$. It follows that the first iterate x_1 cannot be in $(1/3, 2/3)$ and therefore no further iterates can be in this interval either. Note then that x_2 cannot lie in either $(1/9, 2/9)$ or $(7/9, 8/9)$; x_3 cannot lie in the next collection of intervals removed from $[0, 1]$ to form C; and so on. It follows that for any $x_0 \in [0, 1]$, the limit

$$x^* = \lim_{n \to \infty} x_n$$

lies in C.

If we want to be precise, we should provide the following calculation. Represent the sequence of iterations applied to x_0 by $(s_1, s_2, s_3, \dots)$, where $s_i = 0$ if we apply F_0 at the ith step and $s_i = 1$ if we apply F_1. Then the orbit of x_0 is

$$
\begin{aligned}
x_1 &= \frac{x_0}{3} + \frac{2s_1}{3} \\
x_2 &= \frac{1}{3}\left(\frac{x_0}{3} + \frac{2s_1}{3}\right) = \frac{x_0}{3^2} + \frac{2s_1}{3^2} + \frac{2s_2}{3} \\
&\vdots \\
x_n &= \frac{x_0}{3^n} + \frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + \frac{2s_3}{3^{n-2}} + \cdots .
\end{aligned}
$$

Now, as $n \to \infty$ the first term in this expression goes to zero and we find

$$
x^* = \lim_{n \to \infty} x_n = \sum_{i=1}^{\infty} \frac{t_i}{3^i},
$$

where each t_i is either 0 or 2; that is, $x^* \in C$.

The Cantor Set and the Great Wall of China

In the opening paragraph of the story, Kafka tells us how the Great Wall was constructed:

> *The Great Wall of China was finished at its northernmost location. The construction work moved up from the south-east and south-west and joined at this point. The system of building in sections was also followed on a small scale within the two great armies of workers, the eastern and western. It was carried out in the following manner: groups of about twenty workers were formed, each of which had to take on a section of the wall, about five hundred meters. A neighboring group then built a wall of similar length to meet it. But afterwards, when the sections were fully joined, construction was not continued on any further at the end of this thousand-meter section. Instead the groups of workers were shipped off again to build the wall in completely different regions. Naturally, with this method many large gaps arose, which*

> *were filled in only gradually and slowly, many of them not until after it had already been reported that the building of the wall was complete. In fact, there are said to be gaps which have never been built in at all, although that's merely an assertion which probably belongs among the many legends which have arisen about the structure and which, for individual people at least, are impossible to prove with their own eyes and according to their own standards, because the structure is so immense.* [5]

The Great Wall is more than 20,000 kilometers in length. Imagine viewing the construction from high above. Initially, as the workers build their 500 meter sections, the Wall is disconnected and barely visible, yet the sections are scattered throughout the expanse, much like the points of the Cantor set. We do not know the number of teams, but they are described as "armies" and so we may assume the number is large. Still, no matter how many there are, the first iteration would result in a "wall" that resembles C more than an actual wall. The segments are totally disconnected from each other, and relative to the vast scale of the total Wall, they may as well be points (they would certainly appear so from high above). Indeed, if one were to choose a point at random along the path of the finished Wall, the probability of hitting a completed section in this first iteration is effectively zero.

Once a section is complete, the workers move to a completely different area to begin construction on another segment. The set C is the attracting set of the iterated function system described in the previous section. We may think of the construction in these terms. View the map F_0 as moving a team to the western section and F_1 as moving a team to the eastern section; the Wall is built by beginning at a random point and then iteratively moving workers west and east with equal probability. Of course, the workers are filling in gaps via this method rather than closing in on points of C, but the mathematical process provides a useful metaphor.

Strictly speaking, we do not know if the eastern and western armies of workers mix with each other, but even if it they do not, we may still view the construction of each of the two halves in this manner by taking F_0 to move workers south and F_1 to move workers north within each half. We then have two separate copies of C with a gap near the completion point, and this process reveals the self-similarity feature inherent in fractal sets. Each half is a copy of the whole on a smaller scale.

Kafka also tells us that there are reports of gaps which have never been filled, even though the Wall is complete. That is, there is a Cantor set of gaps remaining whose existence we cannot verify. These gaps are points along the wall that have yet to be reached by teams of workers; presumably, they will be filled in eventually if we iterate the functions F_0 and F_1 long enough. In typical Kafka fashion, however, we get the sense that these gaps will never be reached, or in any case, even if they are we will never know.

Mathematics as Metaphor

The mapping between the Cantor set and the construction of the Wall is not perfect, of course, but the image of the teams of workers bouncing around the countryside according to the iterated system of functions is evocative. Thinking about the story in these terms gives some precision to the loose description of the workers' movements, and the link between the sections of the Wall and points in the Cantor set reveals the immensity of the project, hinting at its impossibility. Indeed, no matter how long we let the workers move from section to section, there will always be gaps. They will grow smaller and smaller, but given the randomness of the teams' movements, sections will always remain uncompleted.

Paradoxical constructions such as these—the Wall is reported complete, but there are observable gaps; the Cantor set has measure zero, but it is uncountable—invoke feelings of hopelessness at first, but I argue that they contain beauty and truth essential to a thorough understanding of the world around us. Does it really matter that the Wall will never be complete? The workers and their families feel a strong sense of pride in the labor; dwelling on the futility of the project does no good. Similarly, the mathematician never really reaches the Cantor set. It is obtained via a limiting process and there are elements in the set whose ternary expansion we could never write down since it would require more space and time than the universe can hold. It is through these disruptions to our senses that we evolve. Using this mathematical idea to examine literature adds another layer to our understanding of not only the text, but of the human condition.

in the Humanities Initiative. The course was titled $2 + 2 = 5$: *Reframing Literature through Mathematics*, and we spent the semester examining pieces of literature through the lens of mathematics. We chose works by Plato, Stoppard, Woolf, Borges, Kafka, and Frayn, along with films by Aronofsky and the Coen Brothers. Eric and I agree that it was the most satisfying teaching experience we have ever had and I thank him for his collaboration and insight. Thanks are also due to an anonymous referee for valuable suggestions.

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