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The Sensitivity of a Laplacian Family of Ranking Methods

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Abstract

Ranking from pairwise comparisons is a particularly rich subset of ranking problems. In this work, we focus on a family of ranking methods for pairwise comparisons which encompasses the well-known Massey, Colley, and Markov methods. We will accomplish two objectives to deepen our understanding of this family. First, we will consider its network diffusion interpretation. Second, we will analyze its sensitivity by studying the "maximal upset" where the direction of an arc between the highest and lowest ranked alternatives is flipped. Through these analyses, we will build intuition to answer the question "What are the characteristics of robust ranking methods?" to ensure fair rankings in a variety of applications, ranging from choosing political candidates to ranking web pages to comparing sports teams.

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Preface

How should you read through this work? I recommend starting with Section 1.3 to understand the purpose of this thesis, and skimming the rest of Chapter 1 to understand the landscape in which this work lies. The next step depends on the reader's interest.

1. For the undergraduate reader, you should be familiar with basic ideas of linear algebra, but not much else is required! If you're curious about some fun intuition for a ranking method, you can start by reading Section 2.5 and Section 3.5. If you're still interested, you may wish to return to the rest of Chapter 2 and Chapter 3 to skim for background information you are unfamiliar with. Along the way, I will also leave asides to provide background information, which will look like this green box:

   > **Background Box**
   >
   > Here is some background information relevant to my thesis.

2. For any reader already familiar with background about the Laplacian and ranking methods, I would recommend starting at Section 4.2 and Section 5.2 for highlights of my main contributions, then reading deeper into the rest of Chapters 4 and 5 if you are still interested in learning more.

I will also continue leaving general asides as well in red boxes, which look like this:

> **Author's Note**
>
> One helpful tip is to print out Appendix A if you are planning to delve into Chapters 4 and 5 to reference side-by-side.

Chapter 1

Questions in Ranking

After seeing applications ranging from choosing political candidates to ranking web pages to comparing sports teams, you have likely grown to appreciate the power of ranking algorithms. Furthermore, these applications can all be handled using similar ranking methods, as they use pairwise comparison data. In general ranking problems, a set of n alternatives is ordered into a single list by importance. Using pairwise orderings adds complexity to this problem, because there may be contradictory or missing ordered pairs of alternatives among the $\binom{n}{2}$ total pairs.

For this thesis, we will focus on the sensitivity of a family of ranking methods through the lens of ranking sports teams. However, the results and structure of our analysis will still provide insight into other critical applications of ranking. The rest of this thesis is dedicated to answering two main questions about this ranking family: which ranking methods in the family outputs are more stable when ranking high variability data, and which methods are more suitable for partial ranking problems? The remainder of Chapter 1 will motivate and contextualize these questions, Chapters 2 and 3 introduce this family of ranking methods under a network diffusion interpretation, and Chapter 4 and 5 will answer the questions that we pose. Finally, we summarize and present future directions in Chapter 6. There are also several appendices for more tangential explanations along the way, as well as Appendix A, which will provide a useful summary of the key definitions and examples for easy reference.

1.1 Motivation

Have you ever wondered things like "If Tomoa Narasaki's foot hadn't slipped during his speed climbing finals in the 2020 Tokyo Summer Olympics, would he have won the gold medal?[1]"? If we are not confident in pairwise comparison data being inputted, we should apply ranking methods that are more robust and will more fairly determine an overall ranking. Furthermore, we primarily care about the top three placements in the Olympics, which means we might want to prioritize a ranking method that will clearly distinguish the top few athletes from the rest, as opposed to a method that maximizes the overall spread between ratings. What constitutes a "good" or a "fair" ranking method, and how can we tell? In this thesis, we will explain why the Laplacian family of ranking methods introduced by Devlin and Treloar (2018) yields an answer to these questions, with a parameter that allows us to maximize the effectiveness of the ranking method by accounting for questionable pairwise ordering inputs and focusing on higher quality partial rankings.

In the interest of having concrete examples, we will frame our work using the analogy of ranking sports teams, but ranking also has a plethora of other application areas. We can rank anything from sports teams to web pages, but how do some of these terms line up with a graph, or with more mathematical definitions for ranking? To answer some of these questions, we have constructed a "translation table" between applications beyond sports, alongside how they can be visualized as a graph, and how more mathematical ranking methods might abstract these terms. We have given sports, voting, and website recommendations as examples of ranking applications, but there are still many more, ranging from college applications to food chains. While reading the remainder of this thesis, you may wish to substitute analogous terminology from other applications or mathematical ranking terminology depending on your interests.

In this table, the first two rows focus on setting up a common language for different settings of ranking problems in graphs. The next two rows examine two possible perturbations if the data used to rank the alternatives is changed. The third row, with an edge value change, builds off Chartier et al. (2011) in examining the sensitivity of rating vectors, and is a primary focus for this thesis. In the last row, the idea of removing a node in an election ("candidate elimination") has been studied in voting theory as the

[1]Yes, he would have.

Ranking Translations

Mathematics of Ranking	Graph	Sports	Voting	Web
Alternatives	Nodes	Teams	Candidates	Web pages
Data used for ranking	Edge values	Game scores	Voter choices	External links
Perturbation	Edge value change	Upset	Voter fraud	Link spamming
Perturbation	Node removal	Team disqualified	Candidate eliminated	Page deleted

Table 1.1 A translation table between ranking theory and applications. We consider the graph elements like nodes and edges, and also a few perturbations that are possible and how they translate into ranking for sports teams, elections, and Web pages. For the data and both types of perturbations, there are other possible interpretations, but we have listed some examples to give you an idea.

idea of a spoiler. Even beyond voting theory, removing a node has interesting connections to other applications. In voting, researchers wonder whether removing a lower-ranked candidate would cause the winner, or the top k candidates, to change. In networks, we might wonder which Internet routers would most damage the network connectivity, which is connected to the idea of centrality (Franceschet, 2019).

Furthermore, methods that are originally designed for specific applications are still highly adaptable. For example, the Massey and Colley methods can be used for ranking movies and web pages (Chartier et al., 2011). The Markov method has been used for ranking political candidates and web pages, but it has also been tailored for ranking species, genes, and social networks (Chartier et al., 2011). Other ranking methods like the Elo method for chess tournaments or Analytic Hierarchy Process (AHP) for high-stakes decision-making are also used to hire job candidates, analyze food chains, compare colleges, and identify hubs of social networks (Langville and Meyer, 2012).

Besides this table, other terminology used by literature studying the mathematics of ranking includes the variables n and k. Typically, n describes the number of alternatives (or teams or candidates), and k relates to the partial ranking problem. There, we are primarily concerned with the ranking of the top k alternatives. Perhaps the top k alternatives are the teams that will make the playoffs or the Web pages that will show up on the first page of a Google search (because I doubt you always click through until the tenth page).

1.2 Methodology

How have mathematicians historically addressed and attempted to solve these problems? Literature about ranking spans a variety of fields, including social choice theory, voting theory, operations research, applied mathematics, and machine learning.

We center our research around four categories of research in ranking. The problems that we introduced in Section 1.1 are related to the first two categories: the sensitivity of the ranking method (**A1**) and partial ranking (**A2**). The third category (**A3**) concentrates on desirable properties of ranking methods that are a promising extension of this thesis. Finally, the fourth category (**A4**) is based on the underlying geometry and graph theoretic interpretations for ranking methods. Our work contributes to (**A1**) and (**A2**), while drawing from fundamental ideas in (**A3**) and (**A4**).

First, (**A1**) is concerned with the sensitivity of ranking methods. In this context, sensitivity refers to the degree to which the output ranking is altered when the inputted pairwise comparisons are changed. Especially when we suspect the data we have might be inaccurate, it is critical to use robust ranking methods that do not drastically change their output after a minor adjustment to the input is made. One of the primary works we build off in this thesis is Chartier et al. (2011), which tests the sensitivity of the Massey method, the Colley method, and the Markov method. They conduct perturbation analysis on "the perfect season," an ideal tournament structure, and find that the Markov method is less stable than the Massey and Colley methods. To address the issue of sensitivity, there has also been research attempting to modify existing ranking methods to be more robust. For example, Burer (2012) modifies the Colley method and turns it into a mixed integer nonlinear programming (MINLP) model that is less sensitive to the win-loss outcomes of inconsequential games.

Next, in (**A2**), we hone in on the ranking of the top k teams, or on the partial ranking. What exactly is a partial ranking?

Definition 1.1 (Partial Ranking)**.** *A partial ranking is a partially known total ordering of a set of alternatives. A common example is a top-k list, where the first* *k alternatives are ranked and the remaining elements are all tied at rank* $k + 1$*. For that reason, it can also be called ranking with ties.*

When voter preference data is not complete, it is often not possible to return a strict ordering of the alternatives. Can you imagine if Google asked everyone to rank every single website on the Internet so they could provide more reliable search results? Papers like Ailon (2010) work on algorithms for improving partial rank aggregation. In this thesis, we will be able to highlight the sensitivity in rankings for the top k teams as a consequence of learning about the sensitivity for the complete n team ranking, which allows us to recommend better ranking methods for partial ranking problems.

In (**A3**), researchers explore the quality of ranking methods, detailing properties of fair ranking methods that may be sensible in a variety of applications. In particular, González-Díaz et al. (2014) lists a plethora of properties that are beneficial for ranking methods. For example, the idea of inversion: if the results of a tournament are reversed, then the ranking should also be reversed. Vaziri et al. (2018) hone in on three specific axioms that should be critical for fair and comprehensive sports rankings. They study the Win-Loss method, the Massey method, the Colley method, the Markov method, and the Elo method. Then, they evaluate these five methods based on three criteria: opponent strength, incentive to win, and sequence of matches. Another way to measure the quality of a ranking is by counting the number of violations, or the number of times a team is ranked lower than a team it has beaten. Minimum violations ranking methods search for rankings with the minimum number of violations, such as Chartier et al. (2010), which applies evolutionary optimization and binary integer linear program approaches in order to solve for the minimum violation ranking.

One important problem within the topic of ranking quality concerns fairness. Traditionally, research in ranking focuses on a single optimal ranking. Anderson et al. (2022) argue that it is important to find multiple optimal rankings, and that if there are multiple rankings, the most fair one should be used. In Pitoura et al. (2022) and Kuhlman and Rundensteiner (2020), a fair ranking is one in which protected groups are ranked similarly to the group as a whole. However, within the context of sports and this thesis, we will consider a fair ranking as approximately equivalent to a "good" ranking.

Several works illustrate the failings of ranking methods. Boudreau et al. (2018) explains that although not common, the scoring of cross country running contains social choice violations, and Truchon (2004) similarly studies manipulability in figure skating. Stinson and Stinson (2021) and Nguyen et al. (2022) criticize the rank product aggregate scoring of competitive climbing in the Tokyo 2020 Olympics. From these works, it is clear that research on more robust ranking methods is vital.

Finally, we return to (**A4**), which ties into more theoretical mathematics and presents new interpretations for ranking methods. This category seeks to provide tools and frameworks to better explain the results of the other three fields. For example, Saari (2011) describes the geometry underlying ranking, and introduces the Saari triangle for visualizing results of elections with three candidates. Outside of voting theory, Jiang et al. (2010) examines how HodgeRank, a ranking technique, can be used for incomplete and imbalanced data and integrates ideas from graph theory. This work is extended and applied to sports by Sizemore (2013). Moreover, many analyses have been made of graph theoretic interpretations of ranking methods. Besides Chartier et al. (2011), the other paper that is a cornerstone of this thesis is Devlin and Treloar (2018). In this thesis, a network diffusion interpretation of ranking is presented that allows us to unify the Massey, Colley, and Markov methods under a single family from Devlin and Treloar (2018), and much of our work leans on graph and network theory.

In summary, we have organized the literature about ranking problems in Section 1.1 into four categories. The two pillars to which we contribute are (**A1**) on the sensitivity of ranking methods and (**A2**) on partial ranking. To answer questions from these two categories, we will rely on work from (**A3**) on properties of fair ranking methods and (**A4**) on graph theoretic interpretations of ranking methods.

1.3 Research Objectives

In this thesis, we are curious about the sensitivity of a family of ranking methods that include the Massey method, the Colley method, and the Markov method. There are two main goals:

(**G1**) analyze the sensitivity of this family of ranking methods and their implications on (**A1**) and (**A2**) in light of (**A3**), and

(**G2**) study the connections to (**A4**) in order to gain intuition about this family and generalize our results to any application area.

These objectives extend the work of Chartier et al. (2011), which examines how rank-one updates ("upsets," in the context of sports) to the perfect season affect the Massey, Colley, and Markov rankings, and Devlin and Treloar (2018), which presents a generalized family of methods that includes the Massey, Colley, and Markov methods. From those results, we wish to create a framework to analyze the sensitivity of any method in this parametrized family, initially just for rank-one updates to the perfect season (involving two teams). Furthermore, we will connect this analysis on sensitivity to the network diffusion interpretation of this family from Devlin and Treloar (2018).

Due to interdisciplinary interest from researchers within computational social choice, voting theory, operations research, applied mathematics, and machine learning, the sensitivity of ranking methods is an important question and has been studied for many commonly used methods (Chartier et al., 2011; Burer, 2012; Morin et al., 2018). Through the results of this work, researchers will be better able to characterize the robustness of ranking methods. Moreover, they will be able to select a parameter corresponding to a method within the family based on sensitivity properties that they desire in their work. Finally, while this thesis describes the ranking problem for sports ranking applications, this is only for the sake of concreteness; in fact, all this work applies equally to any other application aggregating pairwise comparisons as we saw in Table 1.1.

Chapter 2

A Laplacian Family of Ranking Methods

In this chapter, we introduce the Laplacian family of ranking methods from Devlin and Treloar (2018). Before doing so, we highlight the three well-studied methods that lie within the family: the Massey, Colley, and Markov methods. These three linear algebraic methods perform well: the Massey and Colley methods were used by the NCAA Football Bowl Subdivision to calculate the Bowl Championship Series rankings, and the Markov method is the foundation for PageRank, a popular algorithm for ranking websites.

2.1 Rankings and Ratings

Before beginning, we will define a few terms and clarify some notation.

Definition 2.1. *A ranking is an ordered list of alternatives or choices.*

We can write the ranking from Example 2.1 of

1. AA
2. BB
3. CC

as $AA \succ BB \succ CC$ using the *succeeds* symbol, representing a preference relation.

Definition 2.2. *A rating is a list of numerical scores for each alternative.*

Ratings can be sorted in increasing or decreasing order to yield rankings.

One challenge that ranking methods face is how to resolve paradoxes that may arise. We can understand this idea through a few examples.

Example 2.1. *Imagine we have three teams that play each other exactly once. The Academic Alpinists* (*AA*) *beat the Boba Bears* (*BB*)*, and the Boba Bears beat the Coffee Crew* (*CC*)*. We might expect the teams to be ranked as follows*:

1. *AA*
2. *BB*
3. *CC*

But in the next game, the Coffee Crew beats the Academic Alpinists!

How should a ranking of the three teams reflect this upset? Furthermore, we can generalize our notion of an upset to a paradox. To see how we could translate Example 2.1 to another context, consider Example 2.2.

Example 2.2. *Now, imagine that we are trying to elect the next leader for the Upbeat Storks of Adventure. Three voters give the following votes*:

1. *AA*	1. *BB*	1. *CC*
2. *BB*	2. *CC*	2. *AA*
3. *CC*	3. *AA*	3. *BB*

There is a voting paradox here with no clear winner, which parallels the idea of upsets within sports seasons. Furthermore, these questions also emerge in other applications of ranking problems.

It turns out that at least for voting, finding a perfect ranking is impossible! A famous result in social choice theory from Arrow (1950) is that no ranked voting electoral system with at least three alternatives can meet all of his criteria. These are unrestricted domain, non-dictatorship, Pareto efficiency, and independence of irrelevant alternatives.

Theorem 2.1 (Arrow's Impossibility Theorem)**.** *When aggregating individual voter's preferences into the group's preference, it is not possible to satisfy all of the four criteria*:

1. (*Universality/Unrestricted Domain*) *The group's preference must be complete and unique* (*deterministic*)*.*
2. (*Non-dictatorship*) *No single voter can determine the group's preference.*
3. (*Pareto efficiency*) *If every voter prefers A over B, then the group prefers A over B.*

4. (*Independence of Irrelevant Alternatives*) *If every voter's preference between A and B remains unchanged, then the group's preference between A and B will also remain unchanged (even if voters' preferences between other pairs like X and Z, Y and Z, or Z and W change).*

It is worthwhile to note that not all of Arrow's criteria are relevant to every application. With sports, it is easy to see that the various strategies, strengths, and weaknesses of each team that could explain Example 2.1 also mean that overall rankings would change if one team were taken out of the league.

2.2 Massey Method

The Massey method is often used for ranking sports teams, and it finds the least squares solution based on point differentials for games. It was developed by Kenneth Massey for his undergraduate thesis at Bluefield College in 1997 (Massey, 1997). The underlying idea is that the ratings r_i and r_j for teams i and j respectively can be determined by $r_i - r_j = y$, where y is the point difference in games between the two teams. To generalize to rating n teams and m games, we construct a system of equations using the vectors of rankings $\mathbf{r}$ for all n teams and $\mathbf{y}$ for all m games. Then, Massey's method can be represented by the equation

$$\underset{m\times n}{X}\ \underset{n\times 1}{\mathbf{r}} = \underset{m\times 1}{\mathbf{y}},$$

where X is a matrix with entries

$$X_{ki} = \begin{cases} 1 & \text{if team } i \text{ won in game } k \\ -1 & \text{if team } i \text{ lost in game } k \\ 0 & \text{otherwise} \end{cases}$$

However, this system likely has no solution. Intuitively, we can imagine that it is hard to find ratings that exactly satisfy $r_i - r_j = y$ for all of the games. From a linear algebra standpoint, there will be many more games played than teams ($m >> n$), so the system will be overdetermined. To remedy this issue, we use the method of least squares to find the ratings that most closely match the point differentials of each game. Hence we multiply both sides by $X^\top$ to obtain the normal equations,

$$X^\top X\mathbf{r} = X^\top\mathbf{y}.$$

Renaming this equation, we have $M\mathbf{r} = \mathbf{p}$, where $M = X^\top X$ and $\mathbf{p} = X^\top \mathbf{y}$. There is still another issue, though. The matrix M in this equation is not full rank, so there are infinitely many solutions. To fix this issue, Massey replaces the last row of M to all ones and the last entry of p to be zero, so that the ratings must sum to zero. With this modification, we finally arrive at the equation $\overline{M}\mathbf{r} = \overline{\mathbf{p}}$, corresponding to the modified M matrix and $\mathbf{p}$ vector.

Furthermore, to determine M and $\mathbf{p}$, we do not need to calculate X and $\mathbf{y}$. Instead, we can define M based on the number of games played between each team, and $\mathbf{p}$ based on cumulative point differentials for each team. Let t_i be the total number of games played by team i, and let n_{ij} be the total number of times team i and team j face each other. Let f_i be the total points scored by (*for*) team i during the season, and let a_i be the total points scored against team i during the season. In summary, we can write Massey's method as

$$\begin{aligned} \underset{n\times n}{\overline{M}}\ \underset{n\times 1}{\mathbf{r}} &= \underset{n\times 1}{\overline{\mathbf{p}}}, \\ \text{where matrix } \overline{M} \text{ has entries } \overline{M}_{ij} &= \begin{cases} t_i & i < n, i = j \\ -n_{ij} & i < n, i \neq j \\ 1 & i = n, \end{cases} \\ \text{and vector } \overline{\mathbf{p}} \text{ has entries } \overline{\mathbf{p}}_i &= \begin{cases} f_i - a_i & i < n \\ 0 & i = n. \end{cases} \end{aligned} \tag{2.1}$$

Then the goal of Massey's method is to solve for $\mathbf{r}$ in Equation 2.1.

2.3 Colley Method

The Colley method is very similar to the Massey method. It is another least squares problem and was developed by Wesley Colley to rank football teams (Colley, 2002). At its core, the Colley method is motivated by a modification of the winning percentage for each team called Laplace's rule of succession. With the modification, the rating for team i is approximately $r_i = \frac{1+w_i}{2+t_i}$, where w_i is the number of wins and t_i is the number of total games played by team i. We say that the rating is approximately based on the modification of the winning percentage because Colley uses an estimation for w_i.

Laplace's Rule of Succession

Theorem 2.2 (Laplace's Rule of Succession)**.** *Let s be the number of successes out of n trials. Traditionally, the probability of success is determined by*

$$p = \frac{s}{n}. \tag{2.2}$$

Laplace's Rule of Succession suggests that sometimes,

$$p = \frac{s+1}{n+2} \tag{2.3}$$

is a better estimate for the probability of success.

For a small number of trials, using Laplace's rule can be more accurate. If there are no successes, Equation 2.2 would suggest there is no chance of success. Similarly, if there are no failures, Equation 2.2 would suggest there is no chance of failure. On the other hand, Equation 2.3 maintains a possibility of success or failure in either case. Similarly, with sports, the basic win-loss method gives a rating by

$$r_i = \frac{w_i}{t_i}. \tag{2.4}$$

Equation 2.4 parallels Equation 2.2. To apply the idea of Equation 2.3, Colley's method defines

$$r_i = \frac{w_i + 1}{t_i + 2}, \tag{2.5}$$

so that the rating for teams with only a few games played is more accurate than in the basic win-loss method. Notice that initially (when $w_i = t_i = 0$), the rating for each team is $\frac{1}{2}$, which only changes after teams have won or lost games.

From Equation 2.5, we want to rewrite w_i. We can use a clever trick to split w_i up. We have

$$
\begin{aligned}
w_i &= \frac{w_i - l_i}{2} + \frac{w_i + l_i}{2} && (2.6)\\
&= \frac{w_i - l_i}{2} + \frac{t_i}{2} && (2.7)\\
&= \frac{w_i - l_i}{2} + \sum_{j=1}^{t_i} \frac{1}{2} && (2.8)\\
&\approx \frac{w_i - l_i}{2} + \sum_{j=1}^{t_i} r_j^i. && (2.9)
\end{aligned}
$$

Why is the approximation in Equation 2.9 valid? Notice that initially, all teams are ranked at $\frac{1}{2}$, since $w_i = t_i = 0$. Since the rating is "conserved" (if one team wins, the other team loses), it can be shown that the average rating for an arbitrary team is $\frac{1}{2}$. Thus, we can say that $\sum_{i \in TEAMS} \frac{1}{2} = \sum_{i \in TEAMS} r_i$. If we assume that the set of all teams is close enough to the set of all the opponents one team has played, we can say that $\sum_{j=1}^{t_i} \frac{1}{2} \approx \sum_{j=1}^{t_i} r_j^i$ in Equation 2.9, where r_j^i is the rating for the jth opponent of team i.

We can now rearrange Equation 2.9 to

$$(2 + n_i)r_i - \sum_{j=1}^{t_i} r_j^i = 1 + \frac{w_i - l_i}{2},$$

and switch to matrix form by defining C, the Colley matrix. Hence, the Colley method can be summarized using the following equations:

$$
\begin{aligned}
\underset{n\times n}{C}\ \underset{n\times 1}{\mathbf{r}} &= \underset{n\times 1}{\mathbf{b}},\\
\text{where } C_{ij} &= \begin{cases} 2 + n_i & i = j \\ -n_{ij} & i \neq j \end{cases} && (2.10)\\
\text{and } b_i &= 1 + \frac{1}{2}(w_i - l_i).
\end{aligned}
$$

To solve for the ratings, we solve for $\mathbf{r}$ in this system of equations. Also notice that Equation 2.10 is very similar to Equation 2.1! We will discuss the similarities more in Chapter 3.

2.4 Markov Method

The Markov method, which is a generalization of PageRank, uses Markov chains to rank and rate teams. The main idea is that each team will vote for teams they think are better. The votes can be based on wins as in the Colley method, point differentials as in the Massey method, or any other statistics such as yardage information. For simplicity, we will describe the version of the Markov method that only leverages win-loss data. This data is encapsulated in the matrix V, where $V_{ij} = 1$ if team i lost to team j. Next, we create a stochastic matrix S by normalizing the rows of V and replacing any rows of $\mathbf{0}^\top$ with $\frac{1}{n}\mathbf{e}^\top$. (This solution comes from the dangling node problem in webpage ranking.) Conceptually, these are equivalent because a row of zeros means a team is undefeated, so that team votes for all teams equally.

Markov Chains

Definition 2.3. *A Markov chain is a stochastic model describing a sequence of events whose transition probabilities depend only on the current state.*

Definition 2.4. *A stochastic matrix, or Markov matrix, describes the transitions of a Markov chain. Its rows or columns must be probability vectors, with entries between 0 and 1 that sum to 1.*

To determine the rating vector, we wish to solve the system

$$\underset{n\times n}{S}\;\underset{n\times 1}{\mathbf{r}} = \underset{n\times 1}{\mathbf{r}}. \tag{2.11}$$

The solution $\mathbf{r}$ is the stationary vector of the stochastic matrix S, which we find by calculating the dominant eigenvector of S.

Markov Chains

Definition 2.5. *The stationary distribution of a Markov chain is the long-run probability distribution. In other words, enough time has passed that the distribution does not change any longer.*

Definition 2.6. *The dominant eigenpair consists of the dominant eigenvalue and its corresponding eigenvector, the dominant eigenvector. The dominant eigenvalue is real and has the greatest magnitude of all the eigenvalues. The dominant eigenpair provides useful information about the steady-state behavior of linear systems.*

Intuitively, the stationary vector represents the stationary distribution and tells us about the proportion of time spent in each state. In this case, those fractions indicate the dominance of each team, so they are used as ratings. We will further discuss the interpretation of the Markov method in Chapter 3 as well. At first glance, the Markov method seems to be completely unrelated to the Massey and Colley methods. However, we will see in the next section, and in Chapter 3 that we can connect all these methods.

2.5 The Family of Ranking Methods

Devlin and Treloar (2018) define a family of ranking methods: the Massey, Colley, and Markov methods are all members of this one-parameter family. The parameter p separates these methods, where $p = 0$ for the Markov method and $p = 1$ for the Massey and Colley methods.

Suppose there are n teams in a league. Let w_{ij} and l_{ij} be the number of wins and losses for team i against team j respectively, and W_i and L_i be the total number of wins and losses for team i. Then, define the matrices W and L entrywise as

$$W_{ij} = \begin{cases} -w_{ij} & i \neq j \\ L_i & i = j \end{cases}, \tag{2.12}$$

and

$$L_{ij} = \begin{cases} -l_{ij} & i \neq j \\ W_i & i = j. \end{cases} \tag{2.13}$$

Similarly to the Colley method, the right hand side vector will be defined by the difference in wins and losses, and here we have

$$\overline{\mathbf{s}_p} = \left[W_i - L_i\right]_{n\times 1}.$$

Now, we will then define a weighted combination of W and L,

$$\overline{\mathcal{L}_p} = W + pL.$$

Since $\overline{\mathcal{L}_p}$ is not full rank, Devlin and Treloar (2018) follows the convention of the Massey method and specifies that the ratings must all sum to zero. To do so, they add a row of ones in $\overline{\mathcal{L}_p}$ and a zero in the last entry of the right hand side vector $\mathbf{s}_p^+$. For this modified system, define the matrix $\mathscr{L}_p^+$ as the $(n+1)\times n$ matrix

$$\mathscr{L}_p^+ = \left[\frac{W + pL}{1 \dots 1}\right]$$

and right hand side vector $\mathbf{s}_p^+$ as the $(n+1)\times 1$ vector

$$\mathbf{s}_p^+ = \begin{cases} \left[\left[W_i - L_i\right]_{n\times 1} \quad \middle| \quad 0\right]^\top & p > 0 \\ \mathbf{0} & p = 0. \end{cases} \tag{2.14}$$

Author's Note

For clarity of notation, we will use $\overline{\mathscr{L}_p}$ rather than $\mathscr{L}_p$ as Devlin and Treloar (2018) originally have. We will see later, in Chapter 4, why this adjustment is necessary (hint: we want to invert $\mathscr{L}_p$!). In general, we will use the bar symbol over the original symbols in Devlin and Treloar (2018) that we will later redefine ourselves, and we will use the + symbol for the system with an extra row added. Additionally, it turns out that we use the symbol $\mathscr{L}$ because this matrix is very similar to the graph Laplacian: the graph Laplacian is $M = \mathscr{L}_1$, so $\mathscr{L}_p$ can almost be thought of as a weighted graph Laplacian.

Then, the rating vector $\mathbf{v}_p$ from Devlin and Treloar (2018) is defined by

$$\mathscr{L}_p^+\mathbf{v}_p = \mathbf{s}_p^+. \tag{2.15}$$

Equation 2.15 encompasses Equation 2.1, Equation 2.10, and Equation 2.11. It has a matrix $\mathscr{L}_p^+$ and vector $\mathbf{s}_p^+$ that characterize the strength of a team based on the number of wins and losses, and the total number of games played by each team. Notice that in this framework from Devlin and Treloar (2018), there is no point differential data used to determine the rating vector.

2.5.1 The Case of $p = 1$

For $p = 1$, we have a combination of the Massey and Colley methods, the modified Colley method or the m-Colley method, as we will refer to it in the remainder of this thesis. The matrix $\mathscr{L}_1^+$ is the same as the Massey matrix M, but the right hand side vector $\mathbf{s}_1^+$ is more similar to the right hand side of the Colley method from Equation 2.10. How does this case of $p = 1$ in the ranking family directly compare to the Massey and Colley methods?

In the Massey method from Equation 2.1, point differentials are used as the right hand side vector. For $\mathbf{s}_1^+$, we use wins and losses instead. However, $\mathcal{L}_1^+$ is exactly the Massey matrix! So, all that changes is the right hand side vector.

In the Colley method from Equation 2.10, recall that the right hand side vector $\mathbf{b}$ has entries $b_i = 1 + \frac{1}{2}(w_i - l_i)$. In this family of ranking methods, $\mathbf{s}_1$ (not augmented in the Colley right hand side vector) has entries $w_i - l_i$. Hence, $\mathbf{b}$ and $\mathbf{s}_1$ are closely related, and we can characterize their relationship with the equation

$$\mathbf{b} = 1 + \frac{1}{2}\mathbf{s}_1.$$

Now, let us consider the matrix $\mathcal{L}_1^+$. We know that this matrix is almost exactly the Massey matrix M, which connects to the Colley matrix C through the equation $C = M + 2I$. Devlin and Treloar (2018) introduce the notion of "virtual games," artificial games that increase the total number of games played. Hence we can generalize the Colley matrix to a related family determined by the parameter k, the number of virtual games played:

$$C = M + kI.$$

Recall that the number of total games played by each team is on the diagonal of the $\mathcal{L}_p^+$ matrix. Therefore, when k increases, that number of games is artificially increased. Using a larger value of k essentially dilutes the effect of wins and losses: intuitively, it is less significant to have two losses or two wins if there are fifty total games, as opposed to if there are only five total games (Devlin and Treloar, 2018). In our family of ranking methods, we set $k = 0$, so that $\mathcal{L}_1^+$ is merely the Massey matrix M. Since $k = 0$, the win-loss record becomes more important. Although we will assume that $k = 0$ for the rest of this thesis, one avenue of future research could explore the family of ranking methods with both p and k as parameters.

Chapter 3

Network Diffusion and the Laplacian

In this chapter, we will introduce the network diffusion interpretation for this Laplacian family of ranking methods. We will first explain how the Markov and m-Colley methods can be described with dominance graphs, then build upon this foundation by leveraging the graph Laplacian to incorporate these methods into a parametrized family.

3.1 Dominance Graphs

We will begin by introducing some basic concepts at the intersection of ranking and graph theory. The simplest example we can provide is the *perfect season*.

Definition 3.1 (Perfect Season)**.** *A perfect season is one in which all n teams play each other exactly once, and there are no upsets. If Team 1 is the best and Team n is the worst, then the first team wins $n-1$ games, the second team wins $n-2$ games (only losing to the first team), and so on, with the nth team losing all $n-1$ games.*

Example 3.1. *Let's consider the perfect season for a league comprised of the five undergraduate Claremont Colleges (5Cs). They are ranked in the following order:*

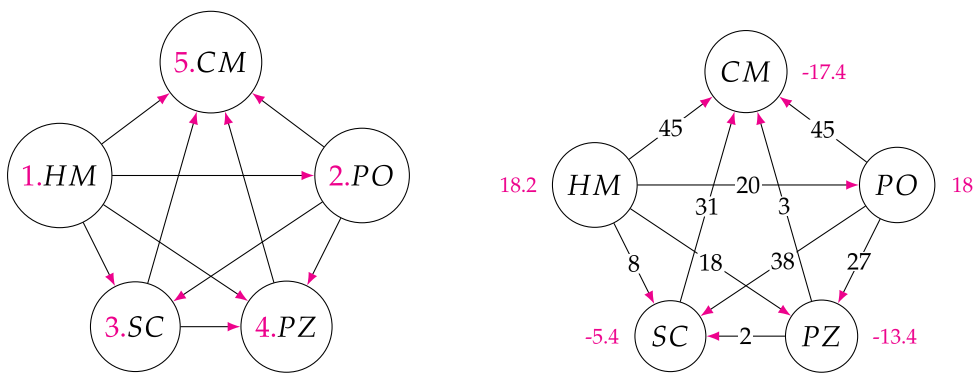
1. *Harvey Mudd College (HM)*
2. *Pomona College (PO)*
3. *Scripps College (SC)*
4. *Pitzer College (PZ)*
5. *Claremont McKenna College (CM)*

Thus, each team plays exactly four games, and the number of wins for HM, PO, SC, PZ, and CM respectively are 4, 3, 2, 1, and 0.

How can we represent Example 3.1 with a graph? We can use a *dominance graph*.

Definition 3.2. *A dominance graph is a directed graph such that for every pair of vertices P_i and P_j, either $P_i \longrightarrow P_j$ or $P_j \longrightarrow P_i$, but not both. We can consider i the winner and j the loser of a game within a tournament.*

For Example 3.1, the dominance graph is illustrated by Figure 3.1a. Each node represents one of the college teams, and an edge directed from node A to node B represents that team A beat team B in a game. For the perfect season, we can interpret the direction of the edges to discern the winning and losing teams for each game.



a. The dominance graph for Example 3.1, with the rankings in magenta inside each node.

b. The weighted dominance graph for Example 3.2, with the Massey ratings illustrated in magenta outside each node.

Figure 3.1 Examples of dominance and weighted dominance graphs.

To generalize the concept of a dominance graph, we can give the weights of the edges representing point differentials for each game. Then we have a weighted dominance graph, as in Figure 3.1b. To further illustrate this concept, we will add this measurement to our example.

Example 3.2. *Suppose that, as in Example 3.1, we have a league consisting of the 5Cs with the same overall ranking ($HM > PO > SC > PZ > CM$). There is one upset, where Pitzer wins their game against Scripps. The point differentials for each game are listed in the table below.*

Winning Team	*Losing Team*	*Point Differential*
HM	*PO*	20
HM	*SC*	8
HM	*PZ*	2
HM	*CM*	45
PO	*SC*	38
PO	*PZ*	27
PO	*CM*	45
SC	*CM*	31
PZ	*SC*	2
PZ	*CM*	3

With the point differentials from Example 3.2, we end up with the weighted dominance graph in 3.1b. Notice that the edge between Scripps and Pitzer flips between the perfect season in 3.1a and the example with an upset in Figure 3.1b, so the two graphs depict different scenarios. However, since the upset between Scripps and Pitzer is very minor in Example 3.2 with a point differential of only two points, the rankings for that example still match the ranking in the perfect season:

1. HM
2. PO
3. SC
4. PZ
5. CM.

By introducing the notion of a dominance graph, we can also note that the Massey matrix, or $\mathscr{L}_1^+$, is the same as the graph Laplacian of the game graph.

Definition 3.3. *The graph Laplacian is a matrix representation of a graph. For a graph with vertices v_i, the elements of the graph Laplacian L are defined by*

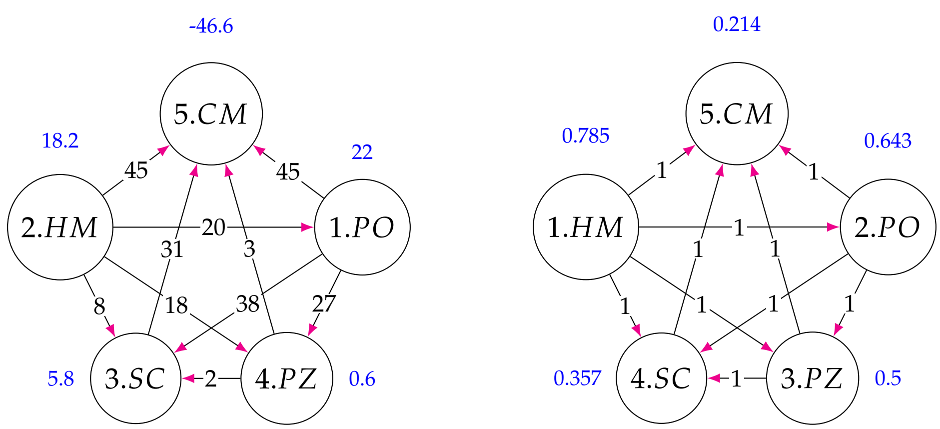
$$L_{ij} = \begin{cases} \deg(v_i) & i = j \\ -1 & i \neq j, \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

It can also be defined by the degree matrix D and adjacency matrix A so that $L = D - A$.

3.2 Least Squares Methods

Both the Massey and Colley method are least squares methods, so they have similar interpretations. As before, we will allow the nodes to represent teams and the edges to represent games between teams. In the Massey method, we try to find ratings such that the difference between the ratings of team i and team j is equal to the point differential of a game played between them. That is, $r_i - r_j = y$ where y is the point differential. These ratings correspond to the node weights, and the point differentials correspond to the edge weights. However, we solve the normal equations to ensure that this problem has a solution. In other words, we find the least squares solution and try to find the node weights that minimize the cumulative difference between the point differentials and the ratings of the teams involved. This idea of minimizing the offsets is analogous to finding the tightest fit of the node weights on the graph (Langville and Meyer, 2012).

Similarly, in the Colley method, we can consider a similar setup, but with differences in wins and losses instead of point differentials.



a. The graph for Example 3.2 corresponding to the Massey method, with the ratings from Equation 2.1 illustrated in blue outside each node. The rankings associated with each team are inside the nodes.

b. The graph for Example 3.2 corresponding to the Colley method, with the ratings from Equation 2.10 illustrated in blue outside each node. The rankings associated with each team are inside the nodes.

Figure 3.2 The graphs from Example 3.2 for the Massey and Colley methods.

We know that the Colley method does not take into account point differentials for games, whereas the Massey method does consider point differentials. By ignoring the margin of victory, Colley creates a "bias free" method that does not depend on the "conference, tradition, or region" (Colley, 2002). Subsequently, some of the rankings in Figure 3.2b and Figure 3.2a change between the Colley and Massey methods. In particular, notice that there is a minor upset between Scripps and Pitzer in Example 3.2. In the Massey method, Scripps is still ranked above Pitzer because it has beaten other teams with a larger margin, and lost to other teams with a smaller margin when compared to Pitzer. On the other hand, Scripps is ranked below Pitzer in the Colley method because it has one win and three losses, whereas Pitzer has two wins and two losses. Accordingly, we can see that although the graph and least squares methodology for both the Massey and Colley methods are similar, considering point differentials as opposed to just the win-loss record affects the final rating and subsequently, the final ranking.

3.3 Markov Method

Before discussing the Markov method, we will review some key definitions and reminders about Markov processes. In a Markov process, the future behavior is independent of the past behavior. So, the probability of transitioning between states at each time step is only based on the current state, and not on the current time.

Markov Processes

Given a transition matrix of a Markov chain P, the transition probability of moving from state i to state j is given by the (i, j) entry of P.

Theorem 3.1. *If the system starts in state i, then the probability of the system being in state j after t time periods is the (i, j) entry of P^t.*

A common question to ask about Markov processes is about the long-term behavior of the system, which is described by the steady state vector. In Section 2.4, we previously introduced the stationary distribution, but we will now specify how this vector relates to the probability matrix P.

Definition 3.4 (State Vector)**.** *The state vector for a Markov chain with n distinct states is an $n \times 1$ vector $\mathbf{x_t}$, with entry i of $\mathbf{x_t}$ describing the probability that the system is in state k at time t.*

Definition 3.5 (Steady State Vector)**.** *The steady state vector does not change from one time step to the next. So, it satisfies the equation $Px_t = x_{t+1}$ for any time t.*

Also, the steady state vector is an eigenvector for P corresponding to the eigenvalue of 1! Furthermore, for entries in P, it is natural that we might want entries to be nonnegative with columns that sum to 1 to represent probabilities. In other words, we wish for P to be a stochastic matrix (defined in Section 2.4).

The graph interpretation of the Markov method is a random walk on the graph.

Definition 3.6 (Random Walk)**.** *A random walk is a process for determining the likely location of a point subject to random motions, which are determined by the probabilities of moving some distance in some direction. The probabilities are the same at each step, which makes random walks an example of a Markov process.*

Pretend that you are Fibonacci, a prospective student of the Claremont Colleges, walking around to determine which college has the best sports team. Fibonacci starts at a random school and moves to another school based on the outcomes of the game in Example 3.2. Suppose Fibonacci starts at Pitzer College and asks a student at Pitzer "Which school has the strongest sports team?" Since Pitzer was defeated by Scripps, Fibonacci then moves to Scripps and asks a student there which school has the strongest sports team. So, Fibonacci might next move to Harvey Mudd College. As Fibonacci continues moving through the colleges, they will have some average time spent at each school. This proportion is the steady state vector, which is equivalent to the Markov rating vector. This vector is also equivalent to the dominant eigenvector of S from Equation 2.11.

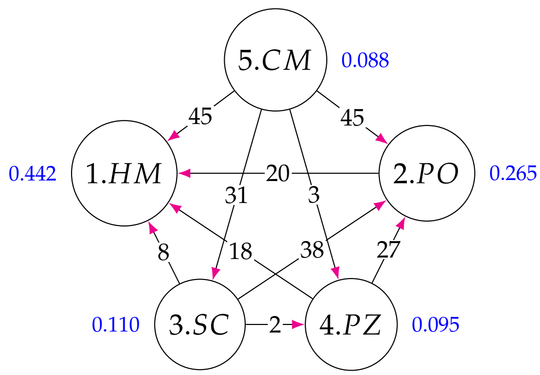


Figure 3.3 The graph for Example 3.2 corresponding to the Markov method, with the ratings from Equation 2.11 illustrated in blue outside each node. The rankings associated with each team are inside the nodes.

In Figure 3.3, we can see that the ranking for the Markov method turns out to be the same as the ranking for the Massey method! Since we consider point differentials when calculating these ratings, these rankings make sense. However, the ratings for the Markov method are completely different than the ratings for the Massey method. Recall from Equation 2.11 that the Markov rating vector should sum to 1 since we can think of each entry as the fraction of time spent in a specific state. On the other hand, in Equation 2.1, we can see that the Massey ratings should sum to zero (a convention set by altering the last entry of the right hand side vector). While this example results in the same ranking for the Markov and Massey methods, it is not always the case that these rankings agree. The Markov method places more weight on the strength of the opponent, so for a larger upset (say, if Claremont McKenna beat Pomona), the rankings would likely differ. We will be able to see this difference more in Chapter 5.

3.4 The Laplacian

Now that we have a better understanding of how directed weighted graphs can be used to represent data for each of the ranking methods in our family, let us delve deeper into the linear algebraic connections. In particular, recall that we defined the ranking family equation as $\mathscr{L}_p^+ \mathbf{v}_p = \mathbf{s}_p^+$. How does this equation translate to a network where rank diffuses over a graph? To answer this question, we must start by investigating the graph Laplacian, and how it relates to the typical Laplacian for continuous multivariate functions. If you were wondering, the connection to the Laplacian is why we use the symbol $\mathscr{L}$!

Intuitively, the Laplacian describes the shape or structure of a function. For continuous functions, the Laplacian is defined as the divergence of a function's gradient:

$$\Delta f(\mathbf{x}) := \nabla \cdot \nabla f(\mathbf{x}).$$

Recall that the gradient of a function returns a vector field describing the direction of steepest ascent, and the divergence returns a multivariate function describing the "flow" in and out of $\mathbf{x}$. The gradient informs us of how much the function is changing at each point, and the divergence tells us about the magnitude and sign of those changes. The flow determines the sign of the divergence: if there is more flow moving inward, the divergence is negative. If there is more flow moving outward, the divergence is positive, and if there is an equal amount of flow in and out, the divergence is zero. We can see an example of how the Laplacian of a continuous function is formed in Figure 3.4, transforming our original multivariate function into a vector field by taking the gradient, then graphing the Laplacian of the function by taking the divergence of the gradient. Pay attention to the center point, where we can see that in the vector field formed by taking the gradient of $f(x, y)$, arrows follow the direction of "steepest ascent," and so all the arrows in $\Delta f(x, y)$ point outwards at the center point because we are at a "hole" in the graph where every direction is sloped sharply upwards, and all equally. Then, in $\nabla f(x, y)$, this point has a high "flow" outwards because all the arrows are pointing outwards from the center point in $\Delta f(x, y)$, which means that the divergence yields a large positive number.

For graphs, the analog of the Laplacian is the graph Laplacian, which can be calculated as $L = D - A$ for an unweighted graph, where A is the

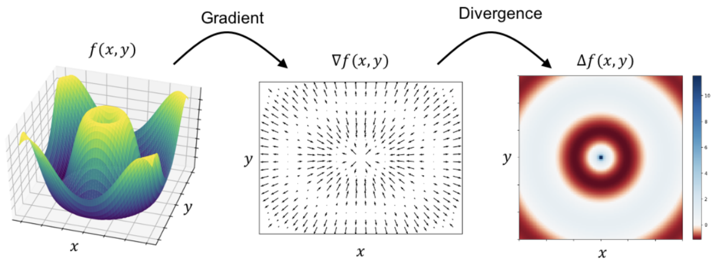


Figure 3.4 Visualization of the continuous Laplacian as the divergence of a function's gradient. Reproduced from "The graph Laplacian," by Bernstein (2020). Reprinted with permission.

adjacency matrix with elements

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases},$$

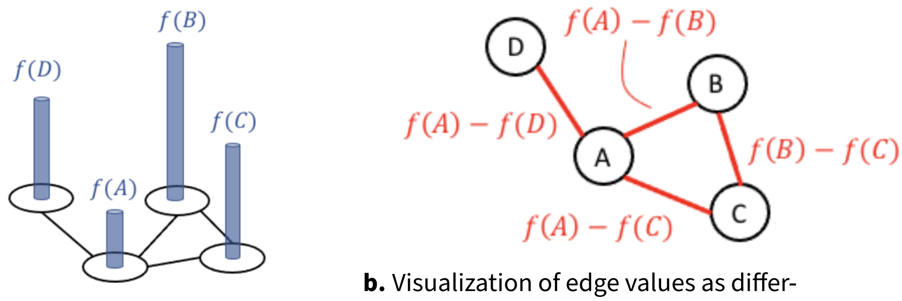
and D is the diagonal degree matrix for a graph with elements

$$D_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

However, we can also interpret the graph Laplacian using the definition of the continuous Laplacian Bernstein (2020). Imagine a graph as being a discrete function, where the values of the nodes are function values (see Figure 3.5a for an example). Again, the gradient embeds information about the function's change. So, the "gradient" on an unweighted graph can be thought of as the difference in function values between two vertices, which is represented by the edges (see Figure 3.5b)! What about the divergence? The "divergence" for a graph measures the flow coming in and out of each point (from each edge). To determine the divergence at a point for an unweighted graph, we can simply subtract all the edge values flowing "into" that point from all edge values flowing "out" of that point.

3.5 Rank Diffusion and Infusion

After reviewing the Laplacian, we can now study our equation $\mathcal{L}_p^+ \mathbf{v}_p = \mathbf{s}_p^+$ under a new light. First, we will motivate why the graph Laplacian shows



a. Visualization of node values as discrete values of functions at points.

b. Visualization of edge values as differences of the node values, representing the gradient of the graph.

Figure 3.5 Visualizations of the discrete version of the Laplacian, interpreting the node and edge values as function values analogous to points in the continuous Laplacian. Reproduced from "The graph Laplacian," by Bernstein (2020). Reprinted with permission.

up in our equation. Then, we will highlight the connections that emerge between the idea of rank diffusion and the solution for Markov and least squares methods. Finally, we will briefly mention some other interpretations and implications of the Laplacian emerging besides rank diffusion.

3.5.1 A Stable Solution

Suppose we wish to formulate an equation describing this idea of rank diffusion. We will assume that in one time step, the quantity of rank flowing from node j to node i is proportional to the difference in the rank at those nodes. So, we can use the number of wins of team i against team j as the flow from team j to team i and the opposite for the number of losses. Let A be the adjacency matrix, and k a parameter determining the rate of diffusion. Then if v_i^t is the quantity of rank at node i at time t, from Devlin and Treloar

(2018), the change in rank (flow) for node i is

$$\begin{aligned}\Delta v_i^t &= \sum_j kA_{ij}(v_j^t - v_i^t)\\ &= k\sum_j A_{ij}v_j^t - kv_i^t\sum_j A_{ij}\\ &= k\sum_j (A_{ij}v_j^t) - kd_iv_i^t\\ &= k\sum_j (A_{ij}v_j^t - \delta_{ij}d_jv_j^t),\end{aligned}$$

where we sum the rank that flows to node i, expand that sum, simplify $\sum_j A_{ij}$ as d_i, the degree of node i, and finally introduce the Kronecker delta δ_{ij} to combine the sum. The Kronecker delta δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

Then, to look for a stable solution (Markov method) where there is no flow in the system long term, set $\Delta v_i^t = 0$ for all nodes i, so we obtain the matrix equation

$$\mathbf{0} = (D - A)\mathbf{v}$$

Since the left hand side is 0, we can divide by $-k$. The summation works due to matrix multiplication of A and $\mathbf{v}$. And, notice that $D - A = L$, the graph Laplacian!

We can also "infuse" rank into the system by *not* looking for a stable solution. That is, letting the long-term change in rank for individual nodes (and hence the system) be positive or negative. We will discuss rank infusion in Subsection 3.5.2, but first we will think about the rank flow a bit more.

Returning to our family $\mathcal{L}_p^+\mathbf{v}_p = \mathbf{s}_p^+$, we now have addressed the case of $p = 0$ where $\mathbf{s}_0 = 0$. Infusing rank into our system is equivalent to having $p > 0$ such that $\mathbf{s}_p^+ \neq \mathbf{0}$. As p changes, the "Laplacian" in our equation also changes. Recall that in our family, the "Laplacian" matrix is defined as $\mathcal{L}_p = W + pL$. Notice that W and L have the same sign, so they are treated equivalently. As the elements of either W or L increase, the flow of the system between the corresponding nodes also increases.

What happens as p increases? If $p = 0$, then we have the Markov method discussed in the previous section, where transitions in a Markov process are determined only by the win percentage. As p increases, the number of losses contributes more and more to the flow of the system. For higher values of p, losing against a strong team increases the rank flow out of the node of the stronger team. Thus, credit is given for merely playing other teams, especially strong ones. Intuitively, a stronger team has less flow (there are fewer losses, and only wins), since more games mean more flow. So if a weaker team plays a stronger team, the flow of the stronger team is increased, almost like it is "leaked" to all the other teams. In Chapter 5, we will see how increasing p means that less weight is placed on the strength of the schedule since the rank of stronger teams is diffused through losses as well. At $p = 1$, the flow from wins and losses is the same—how can we differentiate between teams? To answer this question, we will return to our question of rank infusion, which will allow us to take into account the overall win-loss record of teams.

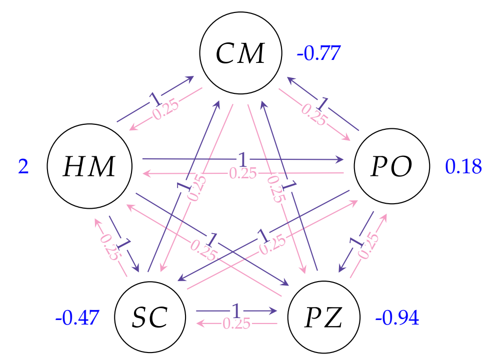
Author's Note

These realizations are central to this thesis! To summarize, the flow through $\mathscr{L}_p^+$ accounts for the strength of the schedule, and $\mathbf{s}_p$ accounts for the record. By varying p, we can adjust the weight we want to put on each of these factors. As p increases, more weight is put onto a team's record and less weight is placed on the strength of the schedule.

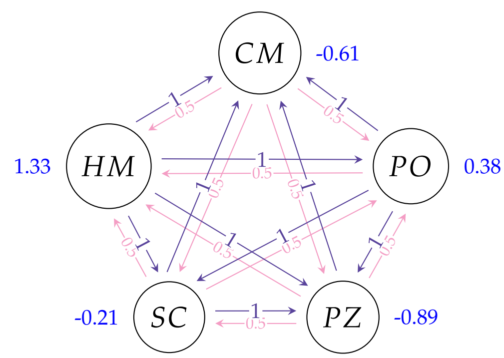
3.5.2 Infusion of Rank

As defined, this method of flow with $\mathscr{L}_p^+$ might feel unintuitive, since wins and losses are treated the same. To compensate, there is an external infusion of rank through $\mathbf{s}_p^+$. Recall that $\mathbf{s}_p = W_i - L_i$ for each team i, so $\mathbf{s}_p$ doesn't depend on the strength of teams, just their record. To better grasp the effect of $\mathbf{s}_p$ on the system, we can study graphs depicting the rank flow system for $p = 0, p = 0.25, p = 0.5, p = 0.75$, and $p = 1$. First, in Figure 3.6, we can see that the magnitude of the ratings decreases as p increases, and also that the gaps in ratings between each team decrease.

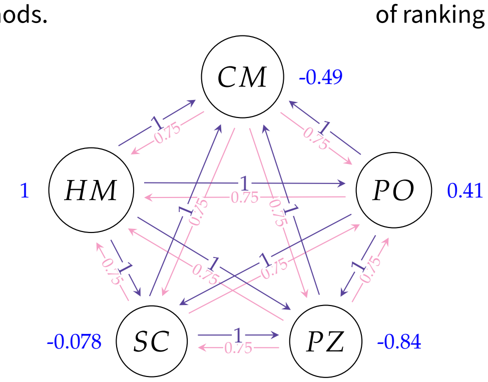
Having studied the general patterns in the rank flow for our family, we can return to the extreme cases of $p = 0$ and $p = 1$. $p = 0$ corresponds to the Markov method. However, in our interpretation, the flows have the reversed direction as the original Markov method, so we interpret it as a "reverse random walk." We can see the difference in these two graphs in Figure 3.7.



a. The flows for $p = 0.25$ in the family of ranking methods.

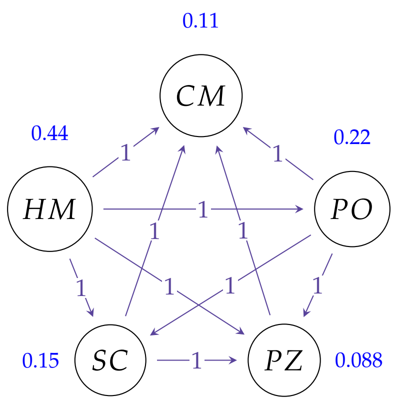


b. The flows for $p = 0.5$ in the family of ranking methods.

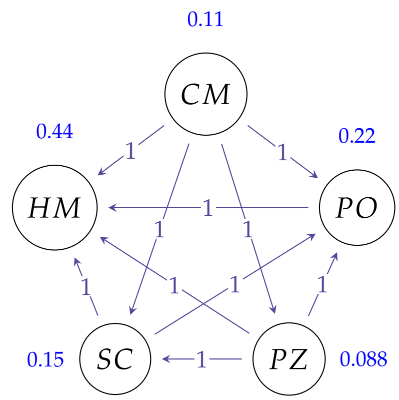


c. The flows for $p = 0.75$ in the family of ranking methods.

Figure 3.6 Flows for $p = 0.25$, $p = 0.5$, and $p = 0.75$ in the network diffusion family of ranking. The blue arrows are the flows from wins, and the pink arrows are flows from losses. The ratings are in blue outside each node.



a. The flows for $p = 0$ in the family of ranking methods, whose direction is flipped from the graph of the random walk.

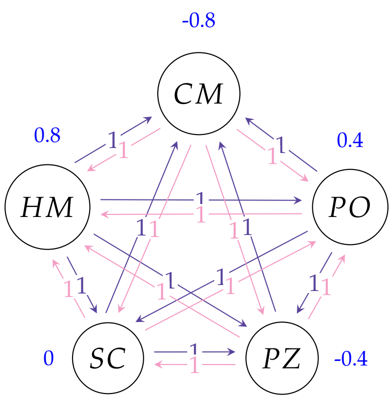


b. The flows in the original Markov ranking method correspond to a random walk whose direction is flipped from the network diffusion flow graph.

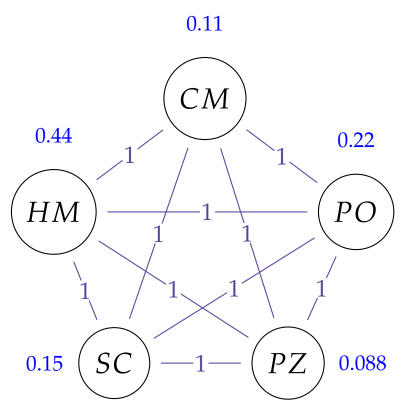
Figure 3.7 Flows corresponding to $p = 0$, which is associated with the Markov method.

Next, for $p = 1$, we have a highly symmetric graph for the perfect season, where each pair of nodes has two directed edges between them in opposite directions with equal weight. In light of the m-Colley method that this graph corresponds to, we can interpret the pair of arcs as an undirected edge. In this least squares method, we are trying to best find the node values such that their differences are as close as possible to their corresponding edges. Why is this idea related to least squares? In statistics, least squares methods fit a curve to a collection of data, which we can also view as projecting a vector. Here, we fit node values to the edge values. The residual represents inconsistencies in the pairwise data. The details of this idea are explored in the HodgeRank algorithm, a generalization of PageRank (Jiang et al., 2010).

In this chapter, we have seen how the Laplacian connects to the game graph, as well as the equation for our parametrized family. This framework sets the stage for the next chapter, where we will return to determine ratings for teams in a perfect season.



a. The flows for $p = 1$ in the family of ranking methods.



b. An alternative interpretation of flows for $p = 1$, more related to the concept of least squares.

Figure 3.8 Flows corresponding to $p = 1$, which is associated with the Massey and Colley methods.

Chapter 4

Laplacian Family Ratings

In this chapter, we find and prove an analytical solution for the ratings $\mathbf{v}_p$ in the perfect season that holds for any number of teams n and any parameter p. Then, we will start to hypothesize about the sensitivity of the family based on the numerical distribution of these ratings.

4.1 The Canonical Example

To begin with, we will run through an example with $n = 5$ teams to build intuition. How can we find $(\mathscr{L}_p^+)^{-1}$ and $\mathbf{s}_p^+$?

4.1.1 Redefining $\mathscr{L}_p$ and $\mathbf{s}_p$

Before we can find $(\mathscr{L}_p^+)^{-1}$, we need to address the issue that $\mathscr{L}_p^+$ is not square and hence is not invertible! Recall that as defined in Devlin and Treloar (2018), we have the general augmented system

$$\mathscr{L}_p^+ \mathbf{v}_p = \mathbf{s}_p^+.$$

For an $n = 5$ team system, we have

$$\mathscr{L}_p^+ = \begin{bmatrix} 4p & -1 & -1 & -1 & -1 \\ -p & 3p+1 & -1 & -1 & -1 \\ -p & -p & 2p+2 & -1 & -1 \\ -p & -p & -p & p+3 & -1 \\ -p & -p & -p & -p & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \tag{4.1}$$

which is a 6×5 matrix. Since $\mathcal{L}_p^+$ is not square, it is not invertible, and hence our method of finding $\mathbf{v}_p$ as $(\mathcal{L}_p^+)^{-1}\mathbf{s}_p^+$ would not work. In general, the augmented matrix $\mathcal{L}_p^+$ has dimensions $(n+1) \times n$, so it is also not square or invertible. To remedy this issue, we will redefine $\mathcal{L}_p$ as the matrix that removes the second to last row of $\mathcal{L}_p^+$, as in the Massey method. Since the nth row is linearly dependent on the first $n-1$ rows, replacing it with a row of all ones will still yield a valid answer in the plane of the original solution space.

We now have a revised family of ranking methods given by the equation

$$\mathcal{L}_p \mathbf{v}_p = \mathbf{s}_p, \tag{4.2}$$

where $\mathcal{L}_p$ is an $n \times n$ matrix with entry (i, j) described by

$$(\mathcal{L}_p)_{ij} = \begin{cases} W_{ij} + pL_{ij} & i \leq n-1 \\ 1 & i = n \end{cases}. \tag{4.3}$$

So, $\mathcal{L}_p$ for 5 teams in the perfect season is

$$\mathcal{L}_p = \begin{bmatrix} 4p & -1 & -1 & -1 & -1 \\ -p & 3p+1 & -1 & -1 & -1 \\ -p & -p & 2p+2 & -1 & -1 \\ -p & -p & -p & p+3 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

After redefining $\mathcal{L}_p$, we also need to modify $\mathbf{s}_p$ from Equation 2.14 so our new system of equations has a solution. In general, $\mathbf{s}_p$ is a $n \times 1$ vector defined by

$$\mathbf{s}_p = \begin{cases} \left[\left[W_i - L_i\right]_{(n-1)\times 1} \quad \middle| \quad 0\right]^\top & p > 0 \\ \mathbf{0} & p = 0. \end{cases} \tag{4.4}$$

For the perfect season, we know the precise number of wins and losses for each team, as team i plays $n-1$ games: $n-i$ wins and $i-1$ losses. Thus, in this case $\mathbf{s}_p$ can be explicitly written in terms of n and i:

$$\mathbf{s}_p = \begin{cases} \left[\left[n - 2i + 1\right]_{n\times 1} \quad \middle| \quad 0\right]^\top_{(n+1)\times 1} & p > 0 \\ \mathbf{0} & p = 0. \end{cases}$$

For $p > 0$, instead of augmenting $\mathbf{s}_p$ by adding a 0, we will replace the last entry with 0, so that in the perfect season we have

$$\begin{aligned}\mathbf{s}_p &= \begin{bmatrix} n-2(1)+1 & \cdots & n-2i+1 & \cdots & n-2(n-1)+1 & 0\end{bmatrix}^\top \\ &= \begin{bmatrix} n-1 & \cdots & n-2i+1 & \cdots & 3-n & 0\end{bmatrix}^\top_{n\times 1},\end{aligned}$$

or for $0 \leq p \leq 1$,

$$(\mathbf{s}_p)_i = \begin{cases} n-2i+1 & \text{if } p > 0, 1 \leq i \leq n-1 \\ 0 & \text{if } p > 0, i = n \text{ or } p = 0 \end{cases}, \tag{4.5}$$

mimicking the alteration in the Massey method to have all ratings sum to 0 in order to resolve the issue of linear dependence (and non-invertibility) in the original $\mathscr{L}_p$ matrix.

4.1.2 Calculating $\mathbf{v}_p$

Now that $\mathscr{L}_p$ and $\mathbf{s}_p$ have been defined more clearly, we can finally determine $\mathbf{v}_p$ for our five-team system. Since $\mathscr{L}_p$ with $n = 5$ is a 5×5 matrix with known entries that are functions of only p, it is fairly trivial to compute $\mathscr{L}_p^{-1}$ for $n = 5$ with mathematical software. It turns out that

$$\mathscr{L}_p^{-1} = \begin{bmatrix} \frac{1}{4p+1} & 0 & 0 & 0 & \frac{1}{4p+1} \\ \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \frac{5p}{(4p+1)(3p+2)} \\ \frac{p-1}{(3p+2)(2p+3)} & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3} & 0 & \frac{5p}{(3p+2)(2p+3)} \\ \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{1}{p+4} & \frac{5p}{(2p+3)(p+4)} \\ -\frac{1}{p+4} & -\frac{1}{p+4} & -\frac{1}{p+4} & -\frac{1}{p+4} & \frac{p}{p+4} \end{bmatrix}. \tag{4.6}$$

We also know from Equation 4.5 that

$$\mathbf{s}_p = \begin{bmatrix} 4 & 2 & 0 & -2 & 0 \end{bmatrix}^\top, \tag{4.7}$$

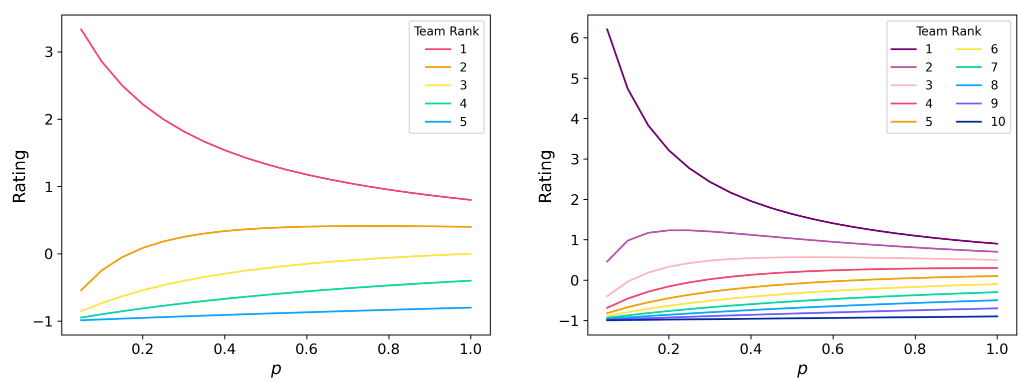
which means we are now ready to find $\mathbf{v}_p$!

Multiplying, we can now find

$$\mathbf{v}_p = \begin{bmatrix} \frac{4}{4p+1} & \frac{12p-2}{(4p+1)(3p+2)} & \frac{6(p-1)}{(3p+2)(2p+3)} & \frac{2(p-6)}{(2p+3)(p+4)} & \frac{-4}{p+4} \end{bmatrix}^\top . \tag{4.8}$$

Hooray! We now know what the rating vectors look like for $n = 5$. But what meaning can we derive from this computation, and how can we generalize to n teams? In the calculation above, notice that since a lot of terms in $\mathscr{L}_p$ repeat, there are sums of $(4+2)$ and $(4+2+0)$. As n increases, this summation can be simplified with a finite sum formula, and we will see this in the next section. In fact, all the middle terms in the vector from the second to second to last position follow this pattern, and only the first and last term may need to be treated separately.

Now, let us return to the question of $n = 5$. We can graph this rating vector across the range where $p \in (0, 1]$ (recall that the Markov method must be handled separately) in Figure 4.1. Furthermore, we can repeat this exercise for $n = 10$ to see that our findings should still hold merit for any general n.



a. $n = 5$ Teams **b.** $n = 10$ Teams

Figure 4.1 The spread of ratings for the perfect season for five and ten teams for $p \in (0, 1]$.

In Figure 4.1, we notice several key takeaways.

Remark 1: The ratio of ratings between teams for the first two teams and the last two teams is very different as we slide the scale from near $p = 0$ to $p = 1$. Near $p = 0$, the highest two ratings are highly distinct and separated, whereas the last two teams are almost indistinguishable. At $p = 1$, on the other hand, there is even spacing between the ratings of all the teams. This difference has implications for the sensitivity of this family of methods, since it seems that using a lower p will lead to much larger upsets than with a higher p value.

Remark 2: The magnitude of the ratings decreases as p increases. Near $p = 0$, the rating of the first team has a rating of over 3 for $n = 5$ and over 6 for $n = 10$. However, the magnitude of the ratings decreases as p approaches 1. It turns out that at $p = 1$, the ratings are evenly spaced intervals centered at 0, but this decrease in magnitude motivates us to examine the family's behavior for $p > 1$ (out of bounds). In Chapter 5, we will examine exactly what the ratio between the ratings of these teams looks like.

4.2 Finding $\mathbf{v}_p$

To find $\mathbf{v}_p$, we once again consider the ranking family equation $\mathscr{L}_p \mathbf{v}_p = \mathbf{s}_p$.

> **Author's Note**
>
> Take out your cheat sheet (Appendix A)!

Recall that the ranking family equation is now fully defined, with $\mathscr{L}_p$ in Equation 4.3 and $\mathbf{s}_p$ in Equation 4.5. Then, if $\mathscr{L}_p$ is invertible, we can find $\mathbf{v}_p = \mathscr{L}_p^{-1}\mathbf{s}_p$. But, since we foresaw this problem, we modified $\mathscr{L}_p$ to be square, so it is invertible for $p \neq 0$!

Using the definition in Equation 4.3, we can say that $(\mathscr{L}_p)_{n\times n}$ has entries

$$(\mathscr{L}_p)_{ij} = \begin{cases} -1 & i \neq n, i < j \\ (n-i)p + (i-1) & i \neq n, i = j \\ -p & i \neq n, i > j \\ 1 & i = n \end{cases}, \tag{4.9}$$

or equivalently

$$\mathscr{L}_p = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & & i & n-1 & n \end{array} \\ \begin{bmatrix} (n-1)p & -1 & \cdots & \cdots & \cdots & -1 \\ -p & (n-2)p+1 & -1 & \cdots & \cdots & -1 \\ -p & -p & (n-3)p+2 & -1 & \cdots & -1 \\ \vdots & \ddots & & \ddots & & \vdots \\ -p & \cdots & -p & (n-i)p+(i-1) & -1 \cdots & -1 \\ \vdots & & & & \ddots & \vdots \\ -p & \cdots & \cdots & -p & p+(n-2) & -1 \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \end{array}. \tag{4.10}$$

We have a highly-structured matrix, with $-p$ on the lower triangular and -1 on the upper triangular parts, and 1 in the final (nth) row. On the diagonal, there are entries of $(n-i)p + (i-1)$.

Leveraging our intuition from our canonical example (which we could further build by testing with other n values), we claim that we have found $\mathscr{L}_p^{-1}$. We propose that the following theorem holds, which we will prove in the following section.

Theorem 4.1. *$\mathscr{L}_p^{-1}$ has (i, j)th entry defined by*

$$(\mathscr{L}_p)_{ij}^{-1} = \begin{cases} \dfrac{-1}{p+n-1} & i = n, i \neq j \\ \dfrac{p}{p+n-1} & i = n = j \\ 0 & i \neq n, i < j \\ \dfrac{1}{(n-i)p+i} & i \neq n, i = j \\ \dfrac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & i \neq n, i > j. \end{cases} \tag{4.11}$$

We can also see this matrix written out in Appendix B.

With this theorem for $\mathscr{L}_p^{-1}$, we can now continue to find $\mathbf{v}_p$.

Now, using Equation 4.5 and Equation 4.11, we can find $\mathbf{v}_p = \mathscr{L}_p^{-1}\mathbf{s}_p$ as

$$\mathbf{v}_p = \begin{bmatrix} \dfrac{n-1}{(n-1)p+1} \\ \dfrac{p-1}{((n-1)p+1)((n-2)p-2)}(n+1-2) + \dfrac{1}{(n-2)p+2}(n+1-4) \\ \dfrac{p-1}{((n-2)p+2)((n-3)p+3)}\left[(n+1-2)+(n+1-4)\right] + \dfrac{n+1-6}{(n-3)p+3}(n+1-6) \\ \dfrac{p-1}{((n-3)p+3)((n-4)p+4)}\left[(n+1-2)+(n+1-4)+(n+1-6)\right] + \dfrac{n+1-8}{(n-4)p+4}(n+1-8) \\ \vdots \\ \dfrac{p-1}{(n-(n-1)p+(n-1))((n-n)p+n)}\left[(n+1-2)+(n+1-4)+\cdots+(n+1-2(n-1))\right] + \dfrac{n+1-2n}{(n-n)p+n}(n+1-2n) \\ \dfrac{-1}{p+n-1}\left[(n+1-2)+(n+1-4)+(n+1-6)+\cdots+(n+1-2(n-1))\right] \end{bmatrix}.$$

Applying the series formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, we find that in general,

$$\mathbf{v}_p = \begin{bmatrix} \dfrac{n-1}{(n-1)p+1} \\ \vdots \\ \dfrac{(p-1)(i-1)(n-i+1)}{((n-i+1)p+i-1)((n-i)p+i)} + \dfrac{n-2i+1}{(n-i)p+i} \\ \vdots \\ \dfrac{1-n}{p+n-1} \end{bmatrix} \begin{matrix} i=1 \\ \\ i=2...(n-1). \\ \\ i=n \end{matrix}$$

Moreover, it turns out that the formula for a general i encompasses the formulas for $i = 1$ and $i = n$, which means that $\mathbf{v}_p$ is defined by

$$\mathbf{v}_p = \left[\frac{(p-1)(i-1)(n-i+1)}{((n-i+1)p+i-1)((n-i)p+i)} + \frac{n-2i+1}{(n-i)p+i}\right]_{n\times 1}, \qquad (4.12)$$

or we can simplify to obtain

$$\mathbf{v}_p = \left[\frac{(n-i+1)(n-i)p - i(i-1)}{(((n-i+1)p+i-1))((n-i)p+i)}\right]. \qquad (4.13)$$

Now, we need to prove that this equation for $\mathbf{v}_p$ holds.

4.3 Verifying $\mathbf{v}_p$

We want to verify that $\mathbf{v}_p$ in Equation 4.12 is correct. To verify, we use proof by induction. Just to check, we can also confirm the rating and $\mathscr{L}_p$ matrix are consistent with the outputs of the individual ranking methods. Since the Massey and Colley method are combined and the Markov method does not work with our method of inversion, this means that we just need to check that our equation is consistent with the m-Colley method.

4.3.1 Colley Method

First, we will confirm that the rating vector is consistent with the Colley rating vector. Let $\mathbf{e}$ be the vector of all ones, and $\mathbf{e}_i$ be the vector consisting of a one in the ith entry and zero everywhere else. Then, for the Colley method, the rating vector is

$$\mathbf{r}_c = C^{-1}\mathbf{b} = \frac{1}{n+2}\mathbf{b} + \frac{n}{2(n+2)}\mathbf{e} = \frac{1}{2(n+2)}\begin{pmatrix} 2n+1 \\ 2n-1 \\ 2n-3 \\ \vdots \\ 2n-2i+3 \\ \vdots \\ 3 \end{pmatrix}, \tag{4.14}$$

where

$$\begin{aligned} C &= (n+2)I - \mathbf{e}\mathbf{e}^{\intercal} && \text{is the Colley matrix, and} \\ \mathbf{b} &= \left[1 + \tfrac{1}{2}(n-2i+1)\right]_{n\times 1} && \text{is the right hand side vector.} \end{aligned}$$

In the analysis of Devlin and Treloar (2018), the number of virtual games k (artificial games that increase the total number of games played) is lowered from 2 to 0. Also, in place of $\mathbf{b}$ we use $\mathbf{s}_1$, where $\mathbf{s}_1 = 2(b-1)$. To ensure the matrix is nonsingular, we also use the Massey modification of replacing the last row of C with all ones. After setting $k = 0$ and replacing the last row in C, we end up with the modified Massey matrix $\overline{M}$.

With these changes, the ranking would then be

$$\begin{aligned}
\mathbf{v}_c &= \bar{M}^{-1}\mathbf{s}_1 \\
&= \frac{1}{n}(I + \mathbf{e}\mathbf{e}_n^\intercal - \mathbf{e}_n\mathbf{e}^\intercal)\mathbf{s}_1 \\
&= \frac{1}{n}\mathbf{s}_1 + \frac{1}{n}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\begin{pmatrix}0 & 0 & \cdots & 0 & 1\end{pmatrix}\mathbf{s}_1 - \frac{1}{n}\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}\mathbf{s}_1 \\
&= \frac{1}{n}\mathbf{s}_1 + \mathbf{0} - \left[\tfrac{n-1}{n}\right]_{n\times 1} \\
&= \left[\frac{n-2i+1}{n}\right]_{n\times 1},
\end{aligned}$$

since

$$\sum_{i=1}^{n-1}(n-2i+1) = n(n-1) - \frac{2(n-1)n}{2} + n - 1 = n^2 - n - n^2 + n + n - 1 = n - 1.$$

If we let $p = 1$ in Equation 4.12, which we calculated with our formula for $\mathscr{L}_p^{-1}$, we have the $n \times 1$ vector with ith entry

$$(\mathbf{v}_c)_i = \frac{n-2i+1}{n},$$

as we would expect - the two equations match!

4.3.2 Gaussian Elimination (5 Teams)

In Subsection 4.3.3, we will use Gaussian elimination on $\mathscr{L}_p$ to determine $\mathscr{L}_p^{-1}$. In order to gain intuition, we will first write out an example with 5 teams. We are trying to determine $\mathscr{L}_p^{-1}$, so we can set up Gaussian elimination. We start with the following matrix:

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|c}
4p & -1 & -1 & -1 & -1 & \\
-p & 3p+1 & -1 & -1 & -1 & \\
-p & -p & 2p+2 & -1 & -1 & I_{5\times 5} \\
-p & -p & -p & p+3 & -1 & \\
1 & 1 & 1 & 1 & 1 &
\end{array}\right] \end{array}.$$

Next, we add row 5 to each of the other rows to create a lower triangular matrix (except the last row),

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 4p+1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -p+1 & 3p+2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ -p+1 & -p+1 & 2p+3 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -p+1 & -p+1 & -p+1 & p+4 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

Next, subtract row 2 from row 3, and row 3 from row 4. We wish to create a bidiagonal matrix (except the last row), so we add the previous row to the current row, skipping the first two rows. This subtraction results in the matrix

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 4p+1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -p+1 & 3p+2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -4p-1 & 2p+3 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array},$$

and we can normalize the first row to obtain

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \frac{1}{4p+1} \\ -p+1 & 3p+2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -4p-1 & 2p+3 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

We can then use the first row to zero out the first entry in the second row. If we add the first row times $p-1$ to the second row, we have

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \frac{1}{4p+1} \\ 0 & 3p+2 & 0 & 0 & 0 & \frac{p-1}{4p+1} & 1 & 0 & 0 & 1+\frac{p-1}{4p+1} \\ 0 & -4p-1 & 2p+3 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}.$$

At this point, it seems like the fifth column of $\mathcal{L}_p^{-1}$ is starting to get a bit messy, but this is not an issue! Recall that we are trying to find $\mathcal{L}_p^{-1}$ to calculate the rating vector $\mathbf{v}_p = \mathcal{L}_p^{-1}\mathbf{s}_p$. But since the last entry of $\mathbf{s}_p$ is 0 for $p > 0$, and we cannot use this method of inversion for $p = 0$ anyways, we can ignore the final column of $\mathcal{L}_p^{-1}$. This column is multiplied by the zero entry, so while we are calculating $\mathcal{L}_p^{-1}$, we can just abstract these elements whose values we can disregard as $\star$.

Now, we can repeat the previous step, dividing the second row by $3p+2$ (in other words, normalizing) and adding that resulting row times $4p+1$ to the third row. Then, we have this matrix:

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5
\end{array}
\\
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\
0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\
0 & -4p-1 & 2p+3 & 0 & 0 & 0 & -1 & 1 & 0 & \star \\
0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & \star \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star
\end{array}\right]
\end{array}.
$$

We repeat this process, combining the two steps of normalizing the second row and adding $4p+1$ times the normalized second row to the third row. Our matrix will then become

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5
\end{array}
\\
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\
0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\
0 & 0 & 2p+3 & 0 & 0 & \frac{p-1}{3p+2} & -\frac{3p+2}{3p+2} + \frac{4p+1}{3p+2} & 1 & 0 & \star \\
0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & \star \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star
\end{array}\right]
\end{array}.
$$

You can hopefully now see a pattern starting to emerge that could also generalize to n teams, where we normalize the $(i-1)$ row and use that normalized row to zero out the $(i-1)$ entry in the ith row. In this example, we have one final iteration, normalizing the third row and zeroing out the third entry in the fourth row by multiplying the normalized third row by $3p+2$.

After normalizing the third row, we have this matrix:

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5
\end{array}
\\
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\
0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\
0 & 0 & 1 & 0 & 0 & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3}\left(-\frac{3p+2}{3p+2}+\frac{4p+1}{3p+2}\right) & \frac{1}{2p+3} & 0 & \star \\
0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & \star \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star
\end{array}\right],
\end{array}
$$

which simplifies to

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5
\end{array}
\\
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\
0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\
0 & 0 & 1 & 0 & 0 & \frac{p-1}{(3p+2)(2p+3)} & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3} & 0 & \star \\
0 & 0 & -3p-2 & p+4 & 0 & 0 & 0 & -1 & 1 & \star \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star
\end{array}\right].
\end{array}
$$

Multiplying the third row by $3p+2$ and adding it to the fourth row, we obtain

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5
\end{array}
\\
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\
0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\
0 & 0 & 1 & 0 & 0 & \frac{p-1}{(3p+2)(2p+3)} & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3} & 0 & \star \\
0 & 0 & 0 & p+4 & 0 & \frac{p-1}{2p+3} & \frac{p-1}{2p+3} & -\frac{2p+3}{2p+3}+\frac{3p+2}{2p+3} & 1 & \star \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star
\end{array}\right],
\end{array}
$$

which results in this matrix after normalizing the fourth row and simplifying:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\ 0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\ 0 & 0 & 1 & 0 & 0 & \frac{p-1}{(3p+2)(2p+3)} & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3} & 0 & \star \\ 0 & 0 & 0 & 1 & 0 & \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{1}{p+4} & \star \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \star \end{array}\right] \end{array}.$$

Finally, we can subtract each of the other rows (one through four) from the fifth row, ending up with

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{4p+1} & 0 & 0 & 0 & \star \\ 0 & 1 & 0 & 0 & 0 & \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \star \\ 0 & 0 & 1 & 0 & 0 & \frac{p-1}{(3p+2)(2p+3)} & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3} & 0 & \star \\ 0 & 0 & 0 & 1 & 0 & \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{1}{p+4} & \star \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{p+4} & -\frac{1}{p+4} & -\frac{1}{p+4} & -\frac{1}{p+4} & \star \end{array}\right] \end{array}$$

after some simplifying.

From this example of $n = 5$ teams, we can start to think about how the Gaussian elimination would work with a general number of teams. Our overall process is to go through the Gaussian elimination one row at a time, starting by simplifying the first and second rows. Then, we iteratively use the previous row to simplify the next row and normalize it. Finally, the nth row is a bit of an exception, but after subtracting, there is some very elegant cancellation and simplification leading us to Equation 4.11.

4.3.3 Inductive Proof of $\mathbf{v}_p$

With the intuition from the five team example in mind, we will now use Gaussian elimination on $\mathscr{L}_p$ to determine $\mathscr{L}_p^{-1}$ for any number of teams n.

We will use induction for the first $n-1$ rows, as we have seen that there is a bit of a shift in the way we determine $\mathcal{L}_p^{-1}$ for row n from Subsection 4.3.2.

Matrix Manipulation: This section describes the first part of the calculations in Appendix C. To see the matrices written out, reference the equations detailed there. First, we will turn the matrix into a lower triangular and bidiagonal matrix to make it easier to work with. Begin by adding the nth row of all ones to all other rows of the matrix to turn the matrix into a lower triangular matrix. This matrix corresponds to Equation C.1. Next, for rows $i \geq 3$, zero most of the entries in the lower diagonal matrix by subtracting the previous row. That is, for $3 \leq i \leq n-1$, do $R_i \to R_i - R_{i-1}$. This matrix corresponds to Equation C.2.

We wish to prove the form of row i for $1 \leq i \leq n-1$ in our Gaussian elimination of $\mathcal{L}_p^{-1}$ using induction on i. Then, we reduce the rows one at a time (up to row $n-1$) to invert $\mathcal{L}_p$. Note that when we find $\mathcal{L}_p^{-1}$, we can ignore the calculation of the final element in each row because the final element of $\mathbf{s}_p$ is 0, and our goal is to find $\mathbf{v}_p = \mathcal{L}_p^{-1}\mathbf{s}_p$. We can think of each row as an array of size $1 \times n$, or more simply an array of size $1 \times (n-1)$ since we ignore the final element.

Base Case: The base case corresponds to the second part of the calculations in Appendix C.

After simplifying the matrix we begin with, we can normalize row 1 by dividing by $(n-1)p$. Thus for $\mathcal{L}_p^{-1}$, row 1 has $\frac{1}{(n-1)p+1}$ in the first position, and zeros everywhere else, which we can see in Equation C.3. For the Gaussian elimination on row 2, add a linear combination of the first and second rows: $R_2 \to R_1 + \frac{1}{p-1}R_2$. Then, normalize by multiplying by $\frac{p-1}{(n-2)p+2}$. The second row of $\mathcal{L}_p^{-1}$ will subsequently have $\frac{p-1}{((n-1)p+1)((n-2)p+2)}$ in the first position, $\frac{1}{(n-2)p+2}$ in the second position, and zeros everywhere else, as in Equation C.4. As we would expect, the left side of our augmented matrix (originally $\mathcal{L}_p$) is reduced to the first two rows of an $n \times n$ identity matrix. We can summarize the starting point for our Gaussian elimination with modifications in row 1 as

$$(\mathcal{L}_p)^{-1}_{1j} = \begin{cases} \dfrac{1}{(n-1)p+1} & j = 1 \\ 0 & 1 < j < n \end{cases}$$

and row 2 as

$$(\mathscr{L}_p)^{-1}_{2j} = \begin{cases} \dfrac{p-1}{((n-1)p+1)((n-2)p+2)} & j = 1 \\ \dfrac{1}{(n-2)p+2} & j = 2 \\ 0 & 2 < j < n. \end{cases}$$

Inductive hypothesis: Assume that for rows 2 through i, the entries in positions 1 through $i-1$ of $\mathscr{L}_p^{-1}$ contain $\frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)}$, entry i contains $\frac{1}{(n-i)p+i}$, and there are zeros from entry $i+1$ to $n-1$. Since we are doing Gaussian elimination, the left side of our augmented matrix is reduced to the first i rows of an $n \times n$ identity matrix. In summary, for row i, we have

$$(\mathscr{L}_p)^{-1}_{ij} = \begin{cases} \dfrac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & j < i \\ \dfrac{1}{(n-i)p+i} & j = i \\ 0 & i < j < n. \end{cases}$$

Inductive step: We wish to show that performing Gaussian elimination on row $i+1$ yields the right hand side $\mathscr{L}_p^{-1}$ containing $\frac{p-1}{((n-i)p+i)((n-i-1)p+i+1)}$ in entries 1 through i, contains $\frac{1}{(n-i-1)p+i+1}$ in entry $i+1$, and contains zeros from entry $i+2$ through $n-1$. Recall that at this point, we have reduced the previous i rows on the left side. Thus, we can copy the reduced entries for row i according to the inductive hypothesis, and read off the two sides of row $i+1$ within the augmented matrix from Equation C.4. On the left side, we have

$$\begin{array}{c} \\ i \\ i+1 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -(n+1-i)p-(i-1) & (n-i-1)p+i+1 & 0 & \cdots & 0 \end{bmatrix}, \end{array}$$

and on the right side, we have

$$\begin{array}{c} \\ i \\ i+1 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n-1 & n \end{array} \\ \begin{bmatrix} \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \cdots & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \frac{1}{(n-i)p+i} & 0 & \cdots & \cdots & 0 & \star \\ 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}. \end{array}$$

Now, replace row $i+1$ with a linear combination of row i and row $i+1$. Perform the row reduction $R_{i+1} \to R_{i+1} + R_i \cdot ((n+1-i)p + (i-1))$. Then, we end with the left side of the augmented matrix being

$$\begin{array}{c|cccccccc} & 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ \hline i & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ i+1 & 0 & \cdots & 0 & 0 & (n-i-1)p+i+1 & 0 & \cdots & 0 \end{array},$$

and the right side being

$$\begin{array}{c|ccccccccc} & 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n-1 & n \\ \hline i & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \cdots & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \frac{1}{(n-i)p+i} & 0 & \cdots & \cdots & 0 & \star \\ i+1 & \frac{(p-1)((n+1-i)p+(i-1))}{((n-i+1)p+i-1)((n-i)p+i)} & \cdots & \frac{(p-1)((n+1-i)p+(i-1))}{((n-i+1)p+i-1)((n-i)p+i)} & \frac{(n+1-i)p+(i-1)}{(n-i)p+i} - \frac{(n-i)p+i}{(n-i)p+i} & 1 & 0 & \cdots & 0 & \star \end{array}$$

$$= \begin{array}{c|ccccccccc} & 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n-1 & n \\ \hline i & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \cdots & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \frac{1}{(n-i)p+i} & 0 & \cdots & \cdots & 0 & \star \\ i+1 & \frac{p-1}{((n-i)p+i)} & \cdots & \frac{p-1}{((n-i)p+i)} & \frac{p-1}{(n-i)p+i} & 1 & 0 & \cdots & 0 & \star \end{array}.$$

Finally, normalize the left side by multiplying row $i+1$ by $\frac{1}{((n-i-1)p+(i+1))}$. On the left, we have the $(i+1)$th row of an $n \times n$ identity matrix. On the right, we find the equation that we desired based on the inductive hypothesis! That is, the entries in row $i+1$ are described by

$$(\mathscr{L}_p)^{-1}_{i+1,j} = \begin{cases} \dfrac{p-1}{((n-i)p+i)((n-i-1)p+i+1)} & j < i+1 \\ \dfrac{1}{(n-i-1)p+i+1} & j = i+1 \\ 0 & i+1 < j < n. \end{cases}$$

Conclusion: Hence, we can conclude that the inductive hypothesis holds, which means our equation for $\mathscr{L}_p^{-1}$ in Equation 4.11 is valid up to row $n-1$!

Determining row n: At this stage, we have a row of ones in the nth row, and the first $n-1$ rows of an $n \times n$ identity matrix above. So, we subtract rows 1 through $n-1$ from row n to conclude our Gaussian elimination. To determine the final row, it turns out we can also use induction. Working backward, we can set up our base cases, the $n-1$ and $n-2$ elements. The $n-1$ element simply comes from the diagonal element of the $n-1$ row, so it is

$$(\mathscr{L}_p)^{-1}_{n,n-1} = -\frac{1}{p+n-1}.$$

The $n-2$ element comes from row $n-1$ and the diagonal element of row $n-2$, and it is

$$\begin{aligned}(\mathcal{L}_p)^{-1}_{n,n-2} &= -\frac{p-1}{((n-n+2)p+n-2)((n-n+1)p+n-1)} - \frac{1}{(n-(n-2))p+n-2} \\ &= -\frac{1}{p+n-1}\end{aligned}$$

To gain intuition before the inductive step, we can try calculating one more element. The $n-3$ element comes from row $n-1$, $n-2$, and the diagonal element of row $n-3$. However, we can also define it recursively in terms of $(\mathcal{L}_p)^{-1}_{n,n-2}$.

$$\begin{aligned}(\mathcal{L}_p)^{-1}_{n,n-3} = (\mathcal{L}_p)^{-1}_{n,n-2} &+ \frac{1}{(n-(n-2))p+n-2} \\ &- \frac{p-1}{((n-n+3)p+n-3)((n-n+2)p+n-2)} \\ &- \frac{1}{(n-(n-3))p+n-3} \\ = -\frac{1}{p+n-1} &+ \frac{1}{2p+n-2} - \frac{p-1}{(3p+n-3)(2p+n-2)} \\ &- \frac{2p+n-2}{(3p+n-3)(2p+n-2)} \\ = -\frac{1}{p+n-1}&\end{aligned}$$

If we follow this step of recursively defining the elements of $\mathcal{L}_p^{-1}$ in row n, we can determine all the values inductively! We can exactly use our calculations for $(\mathcal{L}_p)^{-1}_{n,n-3}$ above, substituting $j+1$ for 3 and j for 2 in our inductive step.

For our inductive hypothesis, assume that $(\mathcal{L}_p)^{-1}_{nj} = -\frac{1}{p+n-1}$.

To prove this last row by induction, we wish to show that $(\mathcal{L}_p)^{-1}_{n,j+1} = -\frac{1}{p+n-1}$. In general, the $j+1$ element comes from row $n-1, n-2, \ldots n-j$, and the diagonal element of row $n-(j+1)$, and it can be defined recursively as

$$
\begin{aligned}
(\mathscr{L}_p)^{-1}_{n,n-(j+1)} &= (\mathscr{L}_p)^{-1}_{n,n-j} + \frac{1}{(n-(n-j))p+n-j} \\
&\quad - \frac{p-1}{((n-n+(j+1))p+n-(j+1))((n-n+j)p+n-j)} \\
&\quad - \frac{1}{(n-(n-(j+1)))p+n-(j+1)} \\
&= -\frac{1}{p+n-1} + \frac{1}{jp+n-j} - \frac{p-1}{((j+1)p+n-(j+1))(jp+n-j)} \\
&\quad - \frac{jp+n-j}{((j+1)p+n-(j+1))(jp+n-j)} \\
&= -\frac{1}{p+n-1} + \frac{1}{jp+n-j} - \cancelto{\frac{-1}{jp+n-j}}{\frac{(j+1)p+n-(j+1)}{((j+1)p+n-(j+1))(jp+n-j)}} \\
&= -\frac{1}{p+n-1}
\end{aligned}
$$

Hence, the last row is also as we expect in Equation 4.11, and that equation has now been fully verified.

4.4 Ratio for Sensitivity of the Family

Now that we are confident in our equation for $\mathbf{v}_p$, we can conduct more preliminary analysis on the ratios between ratings for the perfect season. Inspired by Vaziri (2016), we study the sensitivity by studying the ratio of the smallest to the largest increment in rating. Thus, let the sensitivity ratio be

$$R_s = \frac{v_1 - v_2}{v_{n-1} - v_n}.$$

First, calculate the numerator, $v_1 - v_2$, as

$$
\begin{aligned}
v_1 - v_2 &= \frac{n-1}{np-p+1} - \frac{(p-1)(n-1)}{(np-p+1)(np-2p+2)} - \frac{n-3}{np-2p+2} \\
&= \frac{2n}{(np-p+1)(np-2p+2)}.
\end{aligned}
$$

Next, calculate the denominator as

$$\begin{aligned} v_{n-1} - v_n &= \frac{(p-1)(n-2)2}{(2p+n-2)(p+n-1)} + \frac{-n+3}{p+n-1} - \frac{(p-1)(n-1)}{n(p+n-1)} - \frac{-n+1}{n} \\ &= \frac{2np}{(2p+n-2)(p+n-1)}. \end{aligned}$$

Thus, the ratio is

$$R_s = \frac{\cancel{2n}}{(np-p+1)(np-2p+2)} \cdot \frac{(2p+n-2)(p+n-1)}{\cancel{2n}p} \tag{4.15}$$

$$= \frac{n^2 - n(3+3p) + (2p^2 - 4p + 2)}{n^2p^3 - n(3p^3 + 3p^2) + (2p^3 - 4p^2 + 2p)}, \tag{4.16}$$

and if we take the limit as n approaches ∞, we have the sensitivity ratio R_s is

$$\lim_{n\to\infty} R_s = \frac{1}{p^3}. \tag{4.17}$$

Notice that if $p = 0$, as in the Markov method, $\lim_{n\to\infty} R_s = \infty$. Thus especially for larger n, the increments in the tail of the rating vector become very small. Since only a small increment is needed to jump to the next ranking spot, we can conclude that the tail of the Markov method is extremely sensitive, which we can also confirm experimentally. On the other hand, if $p = 1$ as in the m-Colley method, the rating vector is much more stable. Even without taking the limit, notice the numerator and denominator are the same if we take $p = 1$ in Equation 4.16, which means $R_s = 1$ for all n. In other words, the rating vector should be evenly spaced, evenly stepping from the first to the last rating.

Chapter 5

Laplacian Family Sensitivity

In this chapter, we will study how the ratings of methods within the family are affected when upsets occur. First, we will study the extreme case of maximal upsets. Then, we will build up the relevant equations to mathematically describe any upset between two teams (rank-one upsets). We will also return to our canonical five team example to build intuition for this problem and share some insights about the sensitivity of ranking methods based on the effects of these rank-one upsets.

But first, what exactly do we mean by an upset?

Definition 5.1 (Upset, Maximal Upset)**.** *Suppose there are n alternatives, and we have a perfect season as in Definition 3.1. Now, let an additional ϵ of a game be played that is unexpected based on the current ranking, so that Team j has an additional ϵ of a win against Team i for $j > i$. We call this game an upset* (*in other words, a lone perturbation to the perfect season*)*. If $i = 1$ and $j = n$, then there is a maximal worst-beats-best upset, which we will call the maximal upset.*

We first defined an upset as a perturbation to the perfect season involving two teams. However, we can generalize this definition to involve n teams.

Definition 5.2 (Rank-n Upset)**.** *In a rank-n upset, there are $n - 1$ upsets between distinct pairs of teams. For example, if Team 5 beats Team 1* (*or it beats Team 1 ten times*)*, there is a rank-one upset.*

Even after an upset, we will refer to teams by their original rank from the perfect season, which means that Team 5 may start in fifth (last) place, then move up to rank 1 after winning against Team 1 (the team originally ranked first) five times, but we will still refer to it as Team 5.

5.1 Maximal Upset

Ranking methods differ in how they weigh the strength of the schedule and the number of points or games attributed to wins and losses. In this section, we ask about how rankings change by applying the maximal upset, which is a question that is also answered in Devlin and Treloar (2018). We replace the win of Team 1 against Team n with a loss. By investigating this extreme case of maximal upsets on the perfect season, we will gain intuition for which ranking methods in our family are ideal for a variety of applications. Across the ranking family, we can see how rankings change when the maximal upset is applied in Figure 5.1.

We see the same trends in Figures 5.1a, 5.1b, 5.1c, and 5.1d, but with the number of teams varying from $n = 10$ to $n = 25$ to $n - 50$ to $n = 100$. In all the subfigures within Figure 5.1, Team n is tied for first place at $p = 0$, and is tied for last place at $p = 1$, which is a dramatic shift!

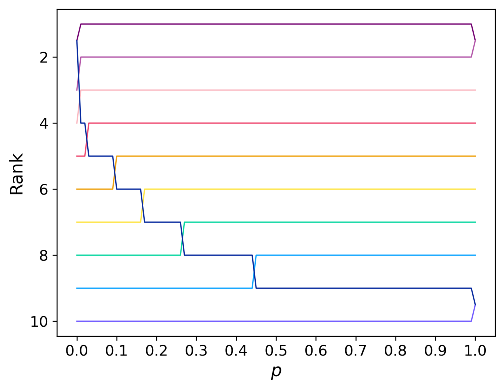
Remark 1: There is a steep drop in the rank of the Team n after p increases from 0, and this effect is even more pronounced as n increases. The fast decrease in rank can be attributed to the same fast decrease in rating in the perfect season (see Figure 4.1).

Remark 2: At $p = 0$ (the Markov method), there is a tie between Teams 1 and n. Using the rank diffusion interpretation of Chapter 3, recall that for $p = 0$ there is only rank flowing from wins. Since Team 1 is undefeated in the perfect season, and we then replace its win against Team n with a loss, all of the rank from Team 1 flows to Team n. Thus, Team 1 and Team n are tied at $p = 0$. This definition of the Markov method at $p = 0$ follows the description of the Markov method in Chartier et al. (2011).

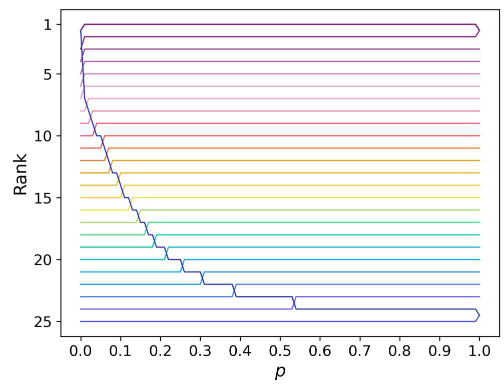
Remark 3: At $p = 1$ (the m-Colley method), recall from Chapter 3 that the flow from wins and losses is the same, which means that only the records matter. With the maximal upset, Team n and Team $n - 1$ have the same record: one win and $n - 2$ losses. Since the record for the bottom two teams is the same (and so is the number of games they have played), they have the same external infusion of rank and hence the same rank. The same logic holds for the rank of Team 1 and Team 2, which is why they also tie at $p = 1$.

After seeing the effects of the maximal upset, we can place more faith in our conclusions from the perfect season about the sensitivity of this Laplacian ranking family, and how the ranking methods with larger p deviate much less from the rankings of the perfect season when upsets are applied (in other words, they are more stable).

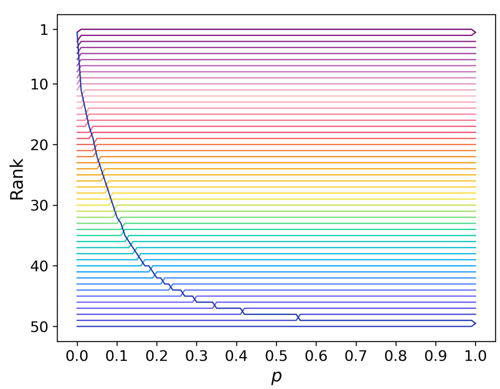
In this section, we ask how rankings change after the maximal upset is



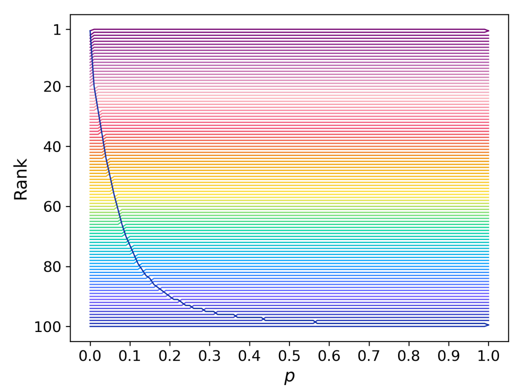
a. $n = 10$



b. $n = 25$



c. $n = 50$



d. $n = 100$

Figure 5.1 Ranks of teams across the range where $p \in [0, 1]$, where there is a maximal upset such that Team n unexpectedly beats Team 1 in their game, swapping the direction of an edge in the perfect season.

applied as in Devlin and Treloar (2018). However, we end up with slightly different results, especially at $p = 0$. In Figure 5.1, our rankings at $p = 0$ align with the Markov method defined in Chartier et al. (2011). However, they do not fully match up with the definition of $p = 0$ in the family of ranking methods defined by Devlin and Treloar (2018). For example, Devlin and Treloar (2018) finds that Team $n = 100$ is ranked 11th when $n = 100$, but in Figure 5.1d, we can see that Team $n = 100$ is tied for first place when we use the definition of the Markov method from Chartier et al. (2011). One direction of future work involves better integrating the Markov method into this family of ranking methods.

5.2 Perturbation

If pairwise comparison data is altered, how do rankings change? This question is crucial to many applications. In sports, you might wonder "would my team have placed higher if they'd won the game against the Dastardly Demons by five more points?"

Our motivating question in this section, derived from Chartier et al. (2011), is as follows: given the number of teams n, an upset of Team j winning over Team i where $i < j$, and a desired k for the new rank of Team j, what ϵ is needed to propel Team j to rank k? In contrast to Section 5.1 where we altered the result of a game in the perfect season, in this section we keep all the games in the perfect season and add additional games. To answer the motivating question, we will begin by studying how our matrix equation changes. We will denote perturbed matrices and vectors with a tilde symbol.

Author's Note

Pull out your cheat sheet for this section! The equation numbers match, but everything is listed for reference in Appendix A.

Recall that $\mathscr{L}_p$ is essentially $W + pL$[1] from Equation 4.3, and W and L are summarized in Equation 2.12 and Equation 2.13. If Team j beats Team i by an additional ϵ, then there are four consequences that are reflected by $\mathscr{L}_p$:

(1) Team i has an additional ϵ of a loss (total),

(2) Team j has an additional ϵ of a win (total),

[1]The last row of $\mathscr{L}_p$ is replaced by all ones, but the other rows are all the same as $W + pL$.

(3) Team i has an additional ϵ of a loss against Team j, and

(4) Team j has an additional ϵ of a win against Team i.

Based on their definitions, (1) means that W_{ii} increases by ϵ, (2) means that L_{jj} increases by ϵ, (3) means that L_{ij} decreases by ϵ, and (4) means that W_{ji} decreases by ϵ. Thus,

$$\widetilde{W} = W + \epsilon(\mathbf{e}_i - \mathbf{e}_j)\mathbf{e}_i^\top$$
$$\text{and } \widetilde{L} = L - \epsilon(\mathbf{e}_i - \mathbf{e}_j)\mathbf{e}_j^\top,$$

which means that based on Equation 4.3,

$$\widetilde{\mathscr{L}_p} = \mathscr{L}_p + \epsilon(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - p\mathbf{e}_j)^\top.$$

There is one exception to this equation: the case of $j = n$. Recall that $\overline{\mathscr{L}_p}$ is not full rank, and the information it contains can be described without the last row. We replaced the last row of $\overline{\mathscr{L}_p}$ with all ones and the last entry in $\overline{\mathbf{s}_p}$ with a zero to specify that all ratings must sum to one to create $\mathscr{L}_p$ and $\mathbf{s}_p$. Thus even when we modify the original data, all the necessary information is still contained in the first $n - 1$ rows, and we do not want to modify the final row. Since (2) and (4) alter the last row, we need to specify that they should only come into play for $j \neq n$. To do so, we modify the previous equation by introducing the Kronecker delta (first mentioned in Section 3.5):

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases},$$

and we have

$$\widetilde{\mathscr{L}_p} = \mathscr{L}_p + \epsilon(\mathbf{e}_i - (1 - \delta_{jn})\mathbf{e}_j)(\mathbf{e}_i - p\mathbf{e}_j)^\top, \tag{5.1}$$

where row j is only modified for $j \neq n$ because $1 - \delta_{jn}$ is only 1 if $j \neq n$.

Similarly, we need to modify $\mathbf{s}_p$. From Equation 4.4, we can see that we need to subtract ϵ from the ith entry and add ϵ to the jth entry of $\mathbf{s}_p$. Again, we should only modify the last entry if $j \neq n$. Thus, we have

$$\widetilde{\mathbf{s}_p} = \mathbf{s}_p - \epsilon(\mathbf{e}_i - (1 - \delta_{jn})\mathbf{e}_j). \tag{5.2}$$

To find $(\widetilde{\mathscr{L}_p})^{-1}$, we need to apply the Sherman-Morrison formula for a rank-one update:

$$(A + \mathbf{u}\mathbf{v}^*)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^*A^{-1}}{1 + \mathbf{v}^*A^{-1}\mathbf{u}}. \tag{5.3}$$

Let $\mathbf{u} = \epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)$ and $\mathbf{v}^* = (\mathbf{e}_i - p\mathbf{e}_j)^\top$ in Equation 5.3. Then, $(\widetilde{\mathscr{L}_p})^{-1}$ is equivalent to

$$(\widetilde{\mathscr{L}_p})^{-1} = (\mathscr{L}_p + \mathbf{u}\mathbf{v}^*)^{-1} = (\mathscr{L}_p)^{-1} - \frac{\mathscr{L}_p^{-1}\epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)(\mathbf{e}_i - p\mathbf{e}_j)^\top \mathscr{L}_p^{-1}}{1 + (\mathbf{e}_i - p\mathbf{e}_j)^\top \mathscr{L}_p^{-1}\epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)}.$$

To simplify this complicated expression, we will say that $(\widetilde{\mathscr{L}_p})^{-1} = \mathscr{L}_p^{-1} - \beta H$, where β is the constant term from the denominator and a factor of ϵ from $\mathbf{u}$ in the numerator such that

$$\beta = \frac{\epsilon}{1 + (\mathbf{e}_i - p\mathbf{e}_j)^\top \mathscr{L}_p^{-1}\epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)},$$

and H is the matrix from the numerator without the factor of ϵ, so

$$H = \mathscr{L}_p^{-1}(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)(\mathbf{e}_i - p\mathbf{e}_j)^\top \mathscr{L}_p^{-1}.$$

We can examine these equations for β and H further to simplify the expressions. For β, there are constant factors, but we will study the second term in the denominator, where we essentially have an inner product, but in two steps. First, there is $(\mathbf{e}_i - p\mathbf{e}_j)^\top \mathscr{L}_p^{-1}$, which results in a $1 \times n$ vector representing row i of $\mathscr{L}_p$ minus row j of $\mathscr{L}_p$ weighted by p. Second, we have an inner product where this row difference is multiplied by $\epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)$. Thus, we have the ith minus the jth entry of this row difference as a result, which is a constant! For $j = n$, we merely return the ith entry of the row difference. If we denote this constant c, then $\beta = \frac{\epsilon}{1+\epsilon c}$. To simplify the notation, we will create an alias for $\mathscr{L}_p^{-1}$, so that the matrix $A = \mathscr{L}_p^{-1}$, with columns $\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_n$ and entries a_{nm} to make it easier to reference. In summary,

$$\beta = \frac{\epsilon}{1 + \epsilon(a_{ii} - pa_{ji} - (1-\delta_{jn})(a_{ij} - pa_{jj}))}. \tag{5.4}$$

In H, we have an outer product. The first term in the outer product is $\mathscr{L}_p^{-1}(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)$, which is a difference of columns i and j (or just column i if $j = n$), and results in a $n \times 1$ vector. The second term is $(\mathbf{e}_i - p\mathbf{e}_j)^\top \mathscr{L}_p^{-1}$, which is row i of $\mathscr{L}_p$ minus row j of $\mathscr{L}_p$ weighted by p, and results in a $1 \times n$ vector. Their product is an outer product of these column and row differences to determine H. So, we can conclude that the elements of H are

$$H_{nm} = (a_{ni} - (1-\delta_{jn})a_{nj})(a_{im} - pa_{jm}). \tag{5.5}$$

Moving back to the rating vector, we can conclude that the perturbed rating vector is $\widetilde{\mathbf{v}_p} = (\widetilde{\mathscr{L}_p})^{-1}\widetilde{\mathbf{s}_p} = (\mathscr{L}_p^{-1} - \beta H)(\mathbf{s}_p - \epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j))$, which we can expand to

$$\widetilde{\mathbf{v}_p} = \mathbf{v}_p - \epsilon\mathscr{L}_p^{-1}(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j) - \beta H\mathbf{s}_p + \epsilon\beta H(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j). \tag{5.6}$$

In the first term, we already know that $\mathscr{L}_p^{-1}\mathbf{s}_p = \mathbf{v}_p$, and we proved that

$$\mathbf{v}_p = \left[\frac{(n-i+1)(n-i)p - i(i-1)}{(((n-i+1)p+i-1))((n-i)p+i)}\right] \qquad \text{(Equation 4.13)}$$

in Chapter 4! Now we need to make sense of the remaining three terms. To start to understand what $\widetilde{\mathbf{v}_p}$ is, we will return to our canonical example and see what intuition we can glean from the five-team system.

Recall that we previously found $\mathscr{L}_p^{-1}$ for five teams in Equation 4.6.

Author's Note

If you wish to gain more intuition about H and β (beyond the notes about the inner and outer products in the previous section) try doing the computations for the five team system!

We know H from the previous section and $\mathscr{L}_p^{-1}$ from Equation 4.11, and we can substitute $n = 5, j = 5$ to determine that

$$\beta = \frac{\epsilon(4p+1)(p+4)}{(4p+1)(p+4) + 2\epsilon(2p^2+p+2)}.$$

Hence multiplying, we obtain that for an upset of Team i against Team 5, the perturbed rating vector is

$$\widetilde{\mathbf{v}_p} = \mathbf{v}_p + \left[-\epsilon + \left(\frac{1}{4p+1} + \frac{p}{p+4}\right)\frac{\epsilon(4p+1)(p+4)}{(4p+1)(p+4)+2\epsilon(2p^2+p+2)}(\epsilon - 4)\right]\mathbf{a}_i. \tag{5.7}$$

Now that we know what the perturbed rating vector for each of the five teams would be, we can ask about how rankings change after upsets occur. We will primarily ask about exactly what upset is required for the rankings to shift and result in a tie between any two teams.

5.3 Epsilon Changes

Suppose there are n teams, and Team i is initially higher ranked than Team j. In this section, we are motivated by the question "how many wins (ϵ) are needed of Team j against Team i in order for Team j to surpass the ranking of Team i?" Mathematically, we are asking what ϵ is needed to have $\tilde{v}_j > \tilde{v}_i$?

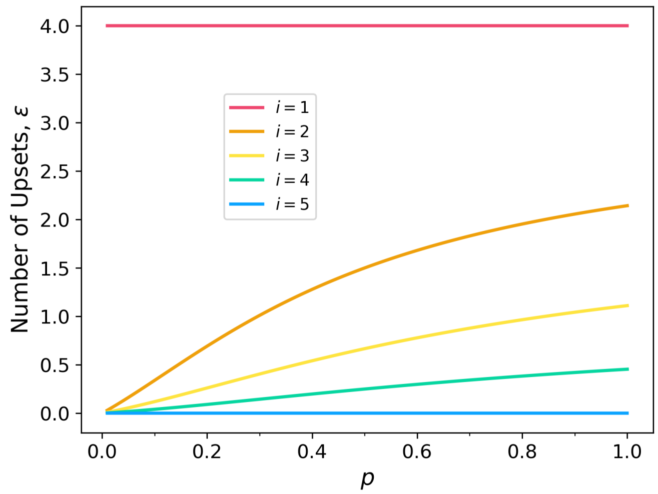
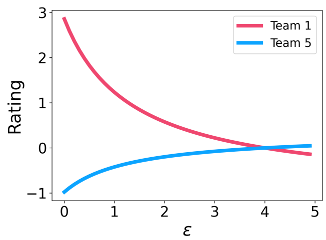


Figure 5.2 For Team j playing against Team $i < j$, the ϵ of an upset or upsets needed in order for Team j to surpass Team i in ranking.

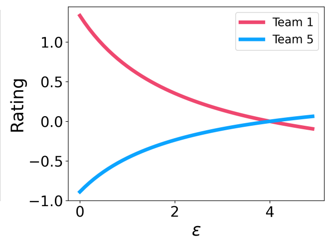
Within Figure 5.2, we observe the ϵ needed for Team 5 to tie with various Teams i across the range where $p \in (0, 1]$. Starting with the lowest rank team, $i = 5$ has $\epsilon = 0$ for all p values, which should be true because no additional games need to be played for Team 5 to have the same rating as Team 5 itself. For $i = 2, i = 3$, and $i = 4$, the ϵ needed for Team 5 to start to overtake the other team's ranking increases as p increases. So, our observations in Chapter 4 are reflected quite clearly, since higher p values correspond with more stable ranking methods which require larger ϵ upsets to affect the overall rankings. Finally, for $i = 1$, ϵ is constant at 4 across all p values. Why does this occur only for $i = 1$? To answer this question, we will study the perturbed ratings for Team $i = 1$ and Team $j = 5$ in Figures 5.3a, 5.3b, and 5.3c.

We can gain an intuition for why the $i = 1$ line in Figure 5.2 has an ϵ of 4, independent of the ranking method in the Laplacian family by considering Figures 5.3a, 5.3b, and 5.3c. As we might expect, ϵ is higher for higher-ranked teams i, as it takes more wins for Team 5 to tie with a higher-ranked team.

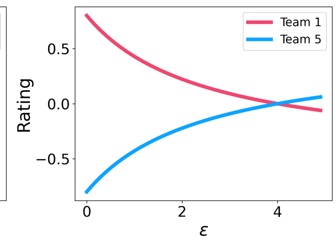
Remark 1: Looking at the ratings of Team 1 and Team 5, we notice that rating magnitudes are higher and there is a larger range of ratings at lower



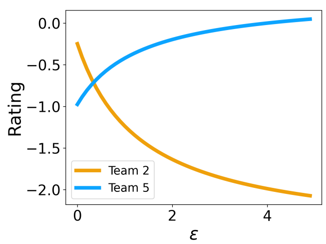
a. $i = 1, j = 5, p = 0.1$



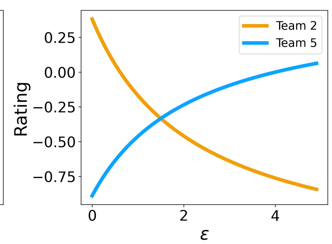
b. $i = 1, j = 5, p = 0.5$



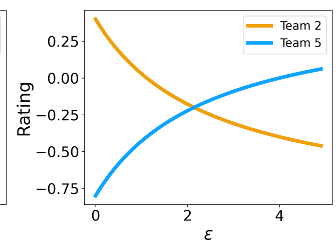
c. $i = 1, j = 5, p = 1$



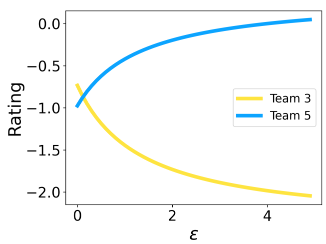
d. $i = 2, j = 5, p = 0.1$



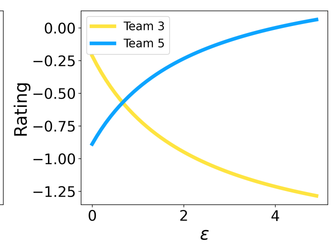
e. $i = 2, j = 5, p = 0.5$



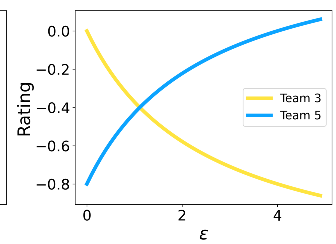
f. $i = 2, j = 5, p = 1$



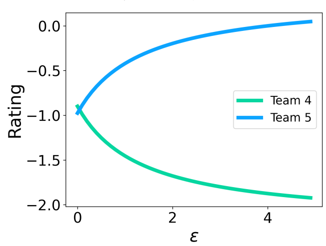
g. $i = 3, j = 5, p = 0.1$



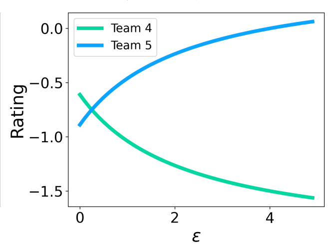
h. $i = 3, j = 5, p = 0.5$



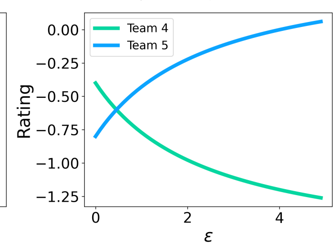
i. $i = 3, j = 5, p = 1$



j. $i = 4, j = 5, p = 0.1$



k. $i = 4, j = 5, p = 0.5$



l. $i = 4, j = 5, p = 1$

Figure 5.3 The change in ratings for Team i and Team $j = 5$, after Team j has ϵ of an upset against Team i. Within the range where $p \in (0, 1]$, the points $p = 0.1, p = 0.5$, and $p = 1$ are studied. Each row has a different Team i, and each column corresponds to a different parameter p in the family.

p values for the perfect season with $\epsilon = 0$ as we mentioned in Chapter 4. In Figure 5.3a, this large range in rating magnitudes is most apparent at $\epsilon = 0$.

Remark 2: As ϵ increases from 0, the initial decrease in rating for Team 1 is steeper at lower p values, since low p values are sensitive to upsets due to the uneven distribution of ratings we highlighted in Chapter 4. On the other hand, for ranking methods corresponding to larger p values, ratings are more evenly spaced and the rankings are more stable so they require a larger ϵ of upsets to change the rankings. Thus the rate of decrease for the rating of Team 1 is much faster between $\epsilon = 0$ to $\epsilon = 4$ in Figure 5.3a (from about 2.8 to 0) than in Figure 5.3b (from about 1.3 to 0) and 5.3c (from 0.8 to 0). In light of Chapter 3, we know that for lower p values, losses do not have as much of an effect on the rank diffusion, which is why Team 5's rank increases slower than Team i's rank decreases.

The faster decrease in rating compensates for the higher ratings at lower p values from Figure 5.3a to 5.3b to 5.3c, and this contrasting effect between Remark 1 and Remark 2 leads to a tie between Team 1 and Team 5 after $\epsilon = 4$ upsets in all three cases. Next, we can look at the ratings of Teams i and j for not just the maximal upset to deepen our understanding of Figure 5.2, which leads us to the remainder of Figure 5.3.

Remark 3: For $i = 2, 3$, and 4, we see similar effects when ϵ and p increase as in $i = 1$. As per Remark 1, rating magnitudes and ranges are higher at lower p values. However, this effect is much less pronounced for $i > 1$, as we notice in Figure 4.1a from the perfect season ratings. As per Remark 2, increasing p leads to a slower decrease in ratings when ϵ is increased.

For $i > 1$, as p increases, the ϵ required for Team j to tie with Team i is not constant. Since the effect of Remark 1 is weaker for $i > 1$, the steeper decrease in rating for Team i (compared to the slower increase in rating for Team 5) at low p values means that the ϵ required to tie is smaller.

Remark 4: At $p = 1$ in the network diffusion interpretation from Chapter 3 (see Figure 3.8), wins and losses both cause rank to diffuse with equal weights of 1, which is why the rating increase for Team i and rating increase for Team 5 are the same. Hence the curves in Figures 5.3c, 5.3f, 5.3i, and 5.3l are symmetric and reflected over the horizontal lines $y = 0$, $y = -0.25$, $y = -0.4$, and $y = -0.625$ respectively.

Now, we have fully investigated Figure 5.2, and we can precisely determine the number of upsets needed for a lower-ranked team to tie with a higher-ranked team.

Chapter 6

Conclusion

Finally, we will summarize the work done in this thesis and suggest future directions.

6.1 Summary

Ultimately, we can now select methods within the Laplacian ranking family of Devlin and Treloar (2018) that are more suitable for problems of partial ranking and ranking using data with high variability. We draw these conclusions by studying the ratings and rankings of the perfect season, then applying rank-one perturbations such as the maximal upset under the lens of a network diffusion interpretation.

To summarize, we first investigated the network diffusion interpretation for the Laplacian family of ranking methods from Devlin and Treloar (2018). The parameter p can be visualized as the amount of backward flow resulting from losses. The ranking method at $p = 0$ corresponds to the Markov method, where there is a reverse random walk, and the ranking method at $p = 1$ corresponds to the m-Colley method where ratings are found via a least squares approximation.

We then proved a formula for the rating for n teams in the perfect season using induction along with a version of Gaussian elimination. The ratings near $p = 0$ have a large range in ratings, and top-ranked teams have especially high magnitude ratings, which is conducive to partial ranking problems. As p increases, the rating range and magnitudes decrease. The ratio of the difference between the ratings of the top two and last two ranked teams is proportional to p^3. Then, at $p = 1$, ratings are evenly spaced, which means this method (m-Colley) is much more stable and suitable for ranking with high variability data.

After examining the perfect season, we tested the effect of perturbations on the resulting rankings, starting with the maximal upset. We found that our results from the perfect season were still valid when upsets were applied, and ranking methods with higher p values were much more stable.

Finally, we studied the effect of any upset on the rankings and derived a formula for the number of wins a lower-ranked team would need to tie with a higher-ranked team. Upon closer examination, we found that two main factors contributed to the number of upsets: the rating of the team in the perfect season and the stability of the ranking method. Interestingly enough, for the first and last ranked team to tie, a constant number of wins was required for all methods in the family. In this case, the two factors were balanced out. However, for upsets between middle-ranked teams and last-ranked team, a larger number of upsets was needed to tie with larger p values because the ratings of the middle teams were not high enough in the perfect season in low p values (when compared to the first ranked team) to compensate for the steep decrease in rating due to the less stable ranking method at low p values.

6.2 Future Directions

This family can be modified by using point differentials as input data instead of the number of wins and losses for each team, as in the Massey method. Using points allows for more intricacy and precision in the ranking data, as there are more data to rank based on than simple wins and losses. On the other hand, ignoring the margin of victory eliminates bias (see Section 3.2). In addition, we can expand and adapt the family in other ways. As mentioned in Subsection 2.5.1, we can add the number of virtual games k as a parameter to the family. Finally, as mentioned in Section 5.1, one important remaining task is to better integrate the Markov method into the family of Laplacian ranking methods.

Another future direction of this work involves expanding the rank-one update to a rank-n update using the Woodbury formula. We proved the formula for a perturbed rating vector using the Sherman-Morrison formula (see Equation 5.3) for inverting a matrix after a rank-one update, but the Sherman-Morrison formula is a special case of the Woodbury formula. The Woodbury formula gives the inverse of a matrix after a rank-n update, and we could use this formula to study how upsets involving more than two teams affect rankings in a perfect season.

Besides upsets, we can study different types of perturbations on the pairwise comparison data. In particular, we can test the effects of spoilers, where a node is removed (Myatt, 2007). In voting theory, a spoiler would mean a candidate drops out of an election, which may change the election results. This change is a violation of the independence of irrelevant alternatives condition in Arrow's impossibility theorem (Arrow, 1950). However, for different applications, it may be desirable to violate this condition. In sports, for example, the styles and various strengths of different teams mean that a team dropping out of the league could change the rankings. How do spoilers affect ranking methods in the Laplacian family?

Even more generally, we can try to measure the tractability of data for ranking problems to see how different ranking methods handle messy data. This measure is called rankability, and it is based on the distance from perfectly rankable data δ and the number of optimal rankings ρ that are δ away from the input data (Anderson et al., 2019, 2021). In this thesis, we only study the effect of rank-one upsets on the perfect season. For rank-n upsets, do our findings still hold as n increases and the rankability decreases?

Finally, we found that ranking methods with low p are better suited towards ranking using data with high variability, and ranking methods with high p are better suited towards partial ranking problems. Are there any other use cases or properties that shift as p varies between 0 and 1? Depending on the ranking method used, it would be interesting to study how properties from Vaziri et al. (2018) and González-Díaz et al. (2014), as well as other factors like fairness (Pitoura et al., 2022; Kuhlman and Rundensteiner, 2020), vary in this interval for p.

Author's Note

Hope you enjoyed reading and learned something new! Thank you for joining me on this journey.

Appendix A

Key Equations

A.1 Variables

Ranking Problem Variables	
Variable	Definition
n	Number of teams/candidates/alternatives
p	The ranking family parameter
$\mathbf{v}_p$	The rating vector as a function of p
W_i	The total number of wins for team i
W_{ij}	The number of wins for team i against team j
L_i	The total number of losses for team i
L_{ij}	The number of losses for team i against team j

Table A.1 A summary of definitions for variables that emerge in the key equations.

A.2 Ranking Family Key Equations

The ranking family equation is

$$\mathscr{L}_p \mathbf{v}_p = \mathbf{s}_p. \qquad \text{(Equation 4.2)}$$

The formulation for the modified Laplacian is

$$(\mathscr{L}_p)_{ij} = \begin{cases} W_{ij} + pL_{ij} & i \leq n-1 \\ 1 & i = n \end{cases}. \qquad \text{(Equation 4.3)}$$

with the second to last row taken out, where W and L are defined entrywise as

$$W_{ij} = \begin{cases} -w_{ij} & i \neq j \\ L_i & i = j \end{cases}, \qquad \text{(Equation 2.12)}$$

and

$$L_{ij} = \begin{cases} -l_{ij} & i \neq j \\ W_i & i = j. \end{cases} \qquad \text{(Equation 2.13)}$$

Alternatively, we can define $\mathscr{L}_p$ piecewise as

$$(\mathscr{L}_p)_{ij} = \begin{cases} -1 & i \neq n, i < j \\ (n-i)p + (i-1) & i \neq n, i = j \\ -p & i \neq n, i > j \\ 1 & i = n \end{cases}, \qquad \text{(Equation 4.9)}$$

or write it in matrix form as

$$\mathscr{L}_p = \begin{array}{r} \\ 1 \\ 2 \\ 3 \\ \\ i \\ \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & & i & n-1 & n \end{array} \\ \begin{bmatrix} (n-1)p & -1 & \cdots & \cdots & \cdots & -1 \\ -p & (n-2)p+1 & -1 & \cdots & \cdots & -1 \\ -p & -p & (n-3)p+2 & -1 & \cdots & -1 \\ \vdots & \ddots & & \ddots & & \vdots \\ -p & \cdots & -p & (n-i)p+(i-1) & -1 \cdots & -1 \\ \vdots & & & & \ddots & \vdots \\ -p & \cdots & \cdots & -p & p+(n-2) & -1 \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} \end{array}. \qquad \text{(Equation 4.10)}$$

The RHS vector in general is

$$\mathbf{s}_p = \begin{cases} \left[\left[W_i - L_i\right]_{(n-1)\times 1} \;\middle|\; 0\right]^\top & p > 0 \\ \mathbf{0} & p = 0, \end{cases} \qquad \text{(Equation 4.4)}$$

and for the perfect season, it is

$$(\mathbf{s}_p)_i = \begin{cases} n - 2i + 1 & \text{if } p > 0, 1 \leq i \leq n-1 \\ 0 & \text{if } p > 0, i = n \text{ or } p = 0 \end{cases}. \qquad \text{(Equation 4.5)}$$

$\mathscr{L}_p^{-1}$ in the perfect season is

$$(\mathscr{L}_p)^{-1}ij = \begin{cases} \dfrac{-1}{p+n-1} & i = n, i \neq j \\ \dfrac{p}{p+n-1} & i = n = j \\ 0 & i \neq n, i < j \\ \dfrac{1}{(n-i)p+i} & i \neq n, i = j \\ \dfrac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & i \neq n, i > j. \end{cases}$$

(Equation 4.11)

The rating vector in the perfect season is

$$\mathbf{v}_p = \left[\frac{(p-1)(i-1)(n-i+1)}{((n-i+1)p+i-1)((n-i)p+i)} + \frac{n-2i+1}{(n-i)p+i} \right]_{n\times 1},$$

(Equation 4.12)

or we can simplify it as

$$\mathbf{v}_p = \left[\frac{(n-i+1)(n-i)p - i(i-1)}{(((n-i+1)p+i-1))((n-i)p+i)} \right]. \quad \text{(Equation 4.13)}$$

With a rank-one upset where $p \neq 0$, $\widetilde{\mathscr{L}_p}$ is

$$\widetilde{\mathscr{L}_p} = \mathscr{L}_p + \epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j)(\mathbf{e}_i - p\mathbf{e}_j)^\top, \quad \text{(Equation 5.1)}$$

and the right hand side vector $\widetilde{\mathbf{s}_p}$ is

$$\widetilde{\mathbf{s}_p} = \mathbf{s}_p - \epsilon(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j). \quad \text{(Equation 5.2)}$$

Thus, the perturbed rating vector for $p \neq 0$ is

$$\widetilde{\mathbf{v}_p} = \mathbf{v}_p - \epsilon\mathscr{L}_p^{-1}(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j) - \beta H\mathbf{s}_p + \epsilon\beta H(\mathbf{e}_i - (1-\delta_{jn})\mathbf{e}_j), \quad \text{(Equation 5.6)}$$

where β is

$$\beta = \frac{\epsilon}{1 + \epsilon(a_{ii} - pa_{ji} - (1-\delta_{jn})(a_{ij} - pa_{jj}))}, \quad \text{(Equation 5.4)}$$

and the (n, m)th element of H is

$$H_{nm} = (a_{ni} - (1-\delta_{jn})a_{nj})(a_{im} - pa_{jm}). \quad \text{(Equation 5.5)}$$

A.3 Examples for $n = 5$

$\mathscr{L}_p$ for five teams is

$$\mathscr{L}_p^+ = \begin{bmatrix} 4p & -1 & -1 & -1 & -1 \\ -p & 3p+1 & -1 & -1 & -1 \\ -p & -p & 2p+2 & -1 & -1 \\ -p & -p & -p & p+3 & -1 \\ -p & -p & -p & -p & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \qquad \text{(Equation 4.1)}$$

$\mathscr{L}_p^{-1}$ for five teams is

$$\mathscr{L}_p^{-1} = \begin{bmatrix} \frac{1}{4p+1} & 0 & 0 & 0 & \frac{1}{4p+1} \\ \frac{p-1}{(4p+1)(3p+2)} & \frac{1}{3p+2} & 0 & 0 & \frac{5p}{(4p+1)(3p+2)} \\ \frac{p-1}{(3p+2)(2p+3)} & \frac{p-1}{(3p+2)(2p+3)} & \frac{1}{2p+3} & 0 & \frac{5p}{(3p+2)(2p+3)} \\ \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{p-1}{(2p+3)(p+4)} & \frac{1}{p+4} & \frac{5p}{(2p+3)(p+4)} \\ -\frac{1}{p+4} & -\frac{1}{p+4} & -\frac{1}{p+4} & -\frac{1}{p+4} & \frac{p}{p+4} \end{bmatrix}. \qquad \text{(Equation 4.6)}$$

$\mathbf{s}_p$ for five teams is

$$\mathbf{s}_p = \begin{bmatrix} 4 & 2 & 0 & -2 & 0 \end{bmatrix}^\top. \qquad \text{(Equation 4.7)}$$

The rating vector $\mathbf{v}_p$ for five teams is

$$\mathbf{v}_p = \begin{bmatrix} \frac{4}{4p+1} & \frac{12p-2}{(4p+1)(3p+2)} & \frac{6(p-1)}{(3p+2)(2p+3)} & \frac{2(p-6)}{(2p+3)(p+4)} & \frac{-4}{p+4} \end{bmatrix}^\top. \qquad \text{(Equation 4.8)}$$

With a rank-one upset where $p \neq 0$, the perturbed rating vector $\widetilde{\mathbf{v}_p}$ for five teams is

$$\widetilde{\mathbf{v}_p} = \mathbf{v}_p + \left[-\epsilon + \left(\frac{1}{4p+1} + \frac{p}{p+4} \right) \frac{\epsilon(4p+1)(p+4)}{(4p+1)(p+4) + 2\epsilon(2p^2+p+2)} (\epsilon - 4) \right] \mathbf{a}_i. \qquad \text{(Equation 5.7)}$$

Appendix B

Matrix Form of $(\mathscr{L}_p)^{-1}$

In this appendix, we write out the expression for $\mathscr{L}_p^{-1}$ from Equation 4.11 in matrix form. For the sake of readability, we split $\mathscr{L}_p^{-1}$ into two parts: one containing columns 1 through 3, and the other containing the remaining columns, up to column n. You may picture moving the second matrix up to lie on the right side of the first matrix, piecing the two together.

$$
\mathscr{L}_p^{-1} =
$$

$$
\begin{array}{c|cccc|}
i = & 1 & 2 & 3 & \\
1 & \frac{1}{(n-1)p+1} & 0 & \cdots & \cdots \\
2 & \frac{p-1}{((n-1)p+1)((n-2)p+2)} & \frac{1}{(n-2)p+2} & 0 \cdots & \cdots \\
3 & \frac{p-1}{((n-2)p+2)((n-3)p+3)} & \frac{p-1}{((n-2)p+2)((n-3)p+3)} & \frac{1}{(n-3)p+3} & 0 \cdots \\
 & \vdots & & & \ddots \\
i & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \cdots & \cdots & \cdots \\
 & \vdots & & & \\
n-1 & \frac{p-1}{(2p+n-2)(p+n-1)} & \cdots & \cdots & \cdots \\
n & \frac{-1}{p+n-1} & \cdots & \cdots & \cdots
\end{array}
$$

$$
\left.\begin{array}{c|cccc|}
 & i-1 & i & n-1 & n \\
1 & 0 \cdots & \cdots & 0 & \frac{1}{(n-1)p+1} \\
2 & 0 \cdots & \cdots & 0 & \frac{np}{((n-1)p+1)((n-2)p+2)} \\
3 & 0 \cdots & \cdots & 0 & \frac{np}{((n-2)p+2)((n-3)p+3)} \\
 & \ddots & \ddots & \vdots & \vdots \\
i & \frac{p-1}{((n-i+1)p+i-1)((n-i)p+i)} & \frac{1}{(n-i)p+i} & 0 \cdots 0 & \frac{np}{((n-i+1)p+i-1)((n-i)p+i)} \\
 & \vdots & & \ddots & \vdots \\
n-1 & \frac{p-1}{(2p+n-2)(p+n-1)} \cdots & \frac{p-1}{(2p+n-2)(p+n-1)} & \frac{1}{p+n-1} & \frac{np}{(2p+n-2)(p+n-1)} \\
n & \frac{-1}{p+n-1} \cdots & \cdots & \frac{-1}{p+n-1} & \frac{p}{p+n-1}
\end{array}\right. .
$$

(Equation 4.11)

Appendix C

Matrix Reduction

We can use Gaussian elimination on $\mathscr{L}_p$ to show that our formula for $\mathbf{v}_p$ holds for any number of teams $n \geq 3$.

Notice that the last element of $\mathbf{s}_p$ is 0, which means that we can ignore the calculations for the last column of $(\mathscr{L}_p)^{-1}$. We will use $\star$ to denote these calculations that we don't care about.

Recall from Equation 4.10 the form of $\mathscr{L}_p$. We will set up Gaussian elimination to find $(\mathscr{L}_p)^{-1}$ with $\left[\mathscr{L}_p | I\right] \to \left[I | (\mathscr{L}_p)^{-1}\right]$.

Initially, we begin with

$$\mathscr{L}_p = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & & i & & n-1 & n \end{array} \\ \left[\begin{array}{ccccccc} (n-1)p & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ -p & (n-2)p+1 & -1 & \cdots & \cdots & \cdots & -1 \\ -p & -p & (n-3)p+2 & -1 & \cdots & \cdots & -1 \\ \vdots & \ddots & & \ddots & & & \vdots \\ -p & \cdots & -p & (n-i)p+(i-1) & -1 & \cdots & -1 \\ \vdots & & & & \ddots & & \vdots \\ -p & \cdots & \cdots & \cdots & -p & p+(n-2) & -1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{array}\right] \end{array}. \tag{4.10}$$

Now, set up the augmented matrix to perform Gaussian elimination, so we have

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ \\ i \\ \\ n-1 \\ n
\end{array}
\left[
\begin{array}{cccccc|c}
1 & 2 & & i & n-1 & n & \\
(n-1)p & -1 & \cdots & \cdots & \cdots & -1 & \\
-p & (n-2)p+1 & -1 & \cdots & \cdots & -1 & \\
-p & -p & (n-3)p+2 & -1 & \cdots & -1 & \\
\vdots & \ddots & & \ddots & & \vdots & I \\
-p & \cdots & -p & (n-i)p+(i-1) & -1 \cdots & -1 & \\
\vdots & & & & \ddots & \vdots & \\
-p & \cdots & \cdots & -p & p+(n-2) & -1 & \\
1 & \cdots & \cdots & \cdots & \cdots & 1 &
\end{array}
\right].
$$

First, add the last (nth) row to all the other rows to turn the matrix into a lower triangular matrix. Then, we end up with

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ \\ i \\ \\ n-1 \\ n
\end{array}
\left[
\begin{array}{cccccc|ccccc}
1 & 2 & & i & n-1 & n & & & & & \\
(n-1)p+1 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
1-p & (n-2)p+2 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & 1 \\
1-p & 1-p & (n-3)p+3 & 0 & \cdots & 0 & 0 & 0 & 1 & 0\cdots & 1 \\
\vdots & \ddots & & \ddots & & \vdots & \vdots & & & \ddots & \\
1-p & \cdots & 1-p & (n-i)p+i & 0 \cdots & 0 & & & & & \\
\vdots & & & & \ddots & \vdots & & & & & \\
1-p & \cdots & \cdots & 1-p & p+(n-1) & 0 & 0 & \cdots & 0 & 1 & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}
\right]. \tag{C.1}
$$

Notice that the last row (comprised of all ones) looks different than all the other rows. We will ignore this row for now, and return to it at the end. The next step is to turn $\mathscr{L}_p$ into a bidiagonal matrix with entries only on the diagonal and lower subdiagonal (except the last row). To do so, subtract the previous row from the current row for the third row up to the second to last row. So, for $3 \le i \le n-1$, do $Ri \to Ri - R(i-1)$. Afterward, we obtain

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ \\ i \\ \\ n-1 \\ n
\end{array}
\left[
\begin{array}{ccccccc|ccccc}
1 & 2 & & & i & n-1 & n & & & & & \\
(n-1)p+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & \star \\
-p & (n-2)p+1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \star \\
0 & -(n-1)p-1 & (n-3)p+3 & 0 & \cdots & \cdots & 0 & 0 & -1 & 1 & 0\cdots & 0 \\
\vdots & & & \ddots & & & \vdots & \vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & -(n+2-i)p-(i-2) & (n-i)p+i & 0 \cdots & 0 & 0 & & -1 & 1 & 0\cdots \\
\vdots & & & & & \ddots & \vdots & \vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & -3p-(n-3) & p+(n-1) & 0 & 0\cdots & 0\cdots & -1 & 1 & 0 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}
\right]. \tag{C.2}
$$

Now, we can reduce the first two rows. First, normalize the first row by dividing by $(n-1)p$ to find

$$\begin{array}{r} \\ 1 \\ 2 \\ 3 \\ \\ i \\ \\ n-1 \\ n \end{array}\left[\begin{array}{cccccccc|ccccc}
1 & 2 & & & i & & n-1 & n & & & & & \\
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{(n-1)p+1} & 0 & \cdots & 0 & \frac{1}{(n-1)p+1} \\
-p & (n-2)p+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \star \\
0 & -(n-1)p-1 & (n-3)p+3 & & 0 & \cdots & \cdots & 0 & 0 & -1 & 1 & 0\cdots & 0 \\
\vdots & & & \ddots & & & & \vdots & \vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & -(n+2-i)p-(i-2) & (n-i)p+i & 0 & \cdots & 0 & 0 & & -1 & 1 & 0\cdots \\
\vdots & & & & & \ddots & & \vdots & \vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & -3p-(n-3) & p+(n-1) & 0 & 0\cdots & 0\cdots & -1 & 1 & 0 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}\right]. \tag{C.3}$$

Next, we can normalize $R2$ by adding a linear combination of $R1$ and $R2$: $R2 \longrightarrow (R1 + \frac{R2}{p-1})\frac{p-1}{(n-2)p+2}$. Then we have

$$\begin{array}{r} \\ 1 \\ 2 \\ 3 \\ \\ i \\ \\ n-1 \\ n \end{array}\left[\begin{array}{cccccccc|ccccc}
1 & 2 & & & i & & n-1 & n & & & & & \\
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{(n-1)p+1} & & 0 & \cdots 0 & \star \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{p-1}{((n-1)p+1)((n-2)2)} & \frac{1}{(n-2)p+2} & 0 & \cdots & \star \\
0 & -(n-1)p-1 & (n-3)p+3 & & 0 & \cdots & \cdots & 0 & 0 & -1 & 1 & 0\cdots & 0 \\
\vdots & & & \ddots & & & & \vdots & \vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & -(n+2-i)p-(i-2) & (n-i)p+i & 0 & \cdots & 0 & 0 & & -1 & 1 & 0\cdots \\
\vdots & & & & & \ddots & & \vdots & \vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & -3p-(n-3) & p+(n-1) & 0 & 0\cdots & 0\cdots & -1 & 1 & 0 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}\right], \tag{C.4}$$

which contains first two rows of $(\mathscr{L}_p)^{-1}$! We can determine the remaining rows until $n-1$ using induction.

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