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On Solving Equations, Negative Numbers, and Other Absurdities: Part II

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7. NEGATIVE NUMBERS

Yet the uneasiness won't go away. The history of mathematics is full of "impossible" objects that later became common, so much so that we wonder at the blindness of our ancestors. Irrational ratios horrified the Pythagoreans, but were quite understandable to the school of Plato. Imaginary numbers were simply not there at all, that is, there were no "numbers" x such that $x^2+1 = 0$, until such numbers somehow turned up "temporarily" in some of the cases of Cardano's sixteenth century solution of the cubic equation. Proper combination of such imaginary numbers turned out to deliver genuine, "real" answers, that "checked" in every detail.

Negative numbers, too, have had such a history, and that not so long ago. Even today, while we teach children the number line, positives to the right and negatives to the left (or positives up and negatives down, as the y-axis is marked in the Cartesian plane), and while we feel quite superior to those of our ancestors who said you couldn't subtract 9 from 7 (*We* know the answer to be -2; don't we?), let us consider our algorithm for the more difficult subtractions that we teach in the third or fourth grade:

We subtract 19 from 57; how? We can't take 9 from 7 so we regroup: Instead of subtracting 10+9 from 50+7, we subtract 10+9 from 40+17. Now 9 from 17 is 8 and 10 from 40 is 30, and our answer is 8+30 or 38. In my day this was called "borrowing:" we borrowed the "1" — really 10 — from the 5 (really 50), and so on, with a certain way of placing the borrowed digit on the page. In effect, we replace the array

$$\begin{array}{r} 5\ 7 \\ -\ 1\ 9 \\ \hline \end{array}$$

by the new arrangement

$$\begin{array}{r} 4(17) \\ -\ 1\ \ 9 \\ \hline \end{array}$$

before performing the operation that produces 3 8 as the answer.

But this whole scheme is predicated on the notion that "you can't take 9 from 7," surely nothing other than the quaint prejudice we have a minute earlier been priding ourselves on having overcome! We can take 9 from 7 if we have the courage of our convictions. Damn the torpedoes; let us take 9 from 7 and get -2, and then take 10 from 50 and get 40, and then combine -2 with 40 to get 38, by golly, the correct answer! Here is the layout:

$$\begin{array}{r} 5\ 7 \\ -\ 1\ 9 \\ \hline \end{array}$$

4(-2), i.e. 40-2, or 38.

Is there anything wrong with this?

Yet with no sense of inconsistency, teachers who tell children about negative numbers on the number line in Grade 2 say that "you can't take 9 from 7" in Grade 3, to introduce the apparent necessity for "borrowing". One can give a good reason for all this, in that the "regrouping" or "borrowing" scheme can be chained in a convenient manner for a longish problem, while combining the positive and negative differences over a multi-digit subtraction might prove more tedious, but this is probably not why we have the algorithm we do. Our 'regrouping' scheme as written above was invented four or five hundred years ago, in the early years of the European adoption of the decimal system, and in that era subtracting large from small numbers was suspect. Try it on an abacus, for example, which historically preceded the written algorithm but uses the same idea. In fact, there is not an arithmetic book in the Western world that shows how to subtract 866 from 541 by placing the figures in this form

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 591
-866
----
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and then going through the "borrowing" ritual to the bitter end. Where on earth is the last "loan" to come from? Our schoolbooks even today evade the question by merely announcing that the difference b-a is the negative of the difference a-b, and telling us to solve the subtraction problem printed above by computing the opposite difference

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 866
-591
----
 275
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in the approved manner, finally changing the sign of the positive result, to -275, to answer the original problem.

8.1. DE MORGAN'S RESERVATIONS ABOUT NEGATIVES

Now, so recent and illustrious a mathematician as Augustus DeMorgan, while willing to go so far as to make temporary use of so ridiculous a notion as "-2" as we did in the earlier subtraction of 19 from 57, was still unwilling to grant a negative number a real final existence. In his 1831 book, *On the Study of Mathematics* (reprinted in 1898 by the Open Court publishing company in La Salle, Illinois), Chapter IX is named "On the Negative Sign, etc." Here (p.103) De Morgan cautions the beginner in algebra to beware of negatives:

> If we wish to say that 8 is greater than 5 by the number 3,we write this equation 8-5 = 3. Also to say that a exceeds b by c, we use the equation a-b = c. As long as some numbers whose value we know are subtracted from others equally known, there is no fear of our attempting to subtract the greater from the less; of our writing 3-8, for example, instead of 8-3. But in prosecuting investigations in which letters occur, we are liable, sometimes from inattention, sometimes from ignorance as to which is the greater of two quantities, or from misconception of some of the conditions of a problem, to reverse the quantities in a subtraction, for example to write a-b when b is the greater of two quantities, instead of b-a. Had we done this with the sum of two quantities, it would have made no difference, because a+b and b+a are the same, but this is not the case with a-b and b-a. For example, 8-3 is easily understood; 3 can be taken from 8 and the remainder is 5; but 3-8 is an impossibility; it requires you to take from 3 more than there is in 3, which is absurd. If such an expression as 3-8 should be the answer to a problem, it would denote either that there was some absurdity inherent in the problem itself, or in the manner of putting it into an equation. Nevertheless, as such answers will occur, the student must be aware what sort of mistakes give rise to them, and in what manner they affect the process of investigation...

I caution the reader here that De Morgan is not naive, and that he is making a philosophical point from which he wishes to derive the usual rules of algebra as we know and use them, including"negatives," and that his general idea, as we shall see, is that playing with absurdities like 3-8 AS IF they made sense can be made to lead to correct final conclusions. It takes him a full chapter to explain this.

It would be wise for present-day teachers to have some appreciation of the philosophical problem involved here, and its clever modern solution by "negative numbers" defined as equivalent pairs of such "impossible" subtraction pairings. But this process,which mathematicians call "embedding a commutative semigroup in a group," while logically satisfying and consistent, does not really attack the problem of what the new numbers mean in applications to the world of apples and gardens. However, this is not something for the 9th grade to elucidate. The mere representation of negative numbers as they appear in practical life, debts as against credits, past as against future, and so on, will usually do the job without needing such sophistication.

De Morgan observes this himself later in the same chapter. He has set up a problem in which the answer has turned out to be -c, and the surprise is that we suddenly discover that c is positive. What are we to make of the absurd answer, -c? On page 55 he gives a simple example:

"A father is 56 and his son 29 years old. When will the father be twice the age of the son?"

Putting x a time when this *will* happen, i.e. in the future, he arrives at the equation $2(29+x) = 56+x$, i.e. twice the age of the son x years from now will equal the father's age x years from now. The solution is $x = -2$. It checks in the equation, but what does it mean? Unlike the problem of the rectangular garden above, this negative number is the *only* answer. Can it mean that the problem has no solution?

Today we would immediately construe this solution to mean that it was two years *ago* that the son *was* half the age of the father, and we would be done with it. To De Morgan this needed more explanation. It was a mistake, he explains, to have begun the algebraic formulation of the problem by putting the date in the future. The negative sign, an absurdity, tells us we have made such a mistake and have asked an impossible problem. We should instead let x be the number of years into the *past* that the doubling of age occurred. then $2(29-x) = 56-x$, i.e. twice the age of the son x years *ago* equals the father's age x years *ago*. The solution is $x = 2$, and De Morgan is philosophically satisfied.

Just the same, this kind of thing happens so often that there must be a simpler way to interpret what has happened. De Morgan announces his principle, his justification for the use of absurd numbers, on page 121:

> ...When such principles as these have been established, we have no occasion to correct an erroneous solution by recommencing the whole process, but we may, by means of the form of the answer [by 'form' he means negative or positive], set the matter right at the end. The principle is, that a negative solution indicates that the nature of the answer is the very reverse of that which it was supposed to be in the solution; for example, if the solution supposes a line measured in feet in one direction, a negative answer, such as -c, indicates that c feet must be measured in the opposite direction; if the answer was thought to be a number of days after a certain epoch, the solution shows that it is c days before that epoch; if we supposed that A was to receive a certain number of pounds, it denotes that he is to pay c pounds, and so on. In deducing this principle we have not made any supposition as to what -c is; we have not asserted that it indicates the subtraction of c from 0; we have derived the result from observations only, which taught us first to deduce rules for making that alteration in the result which arises from altering +c into -c at the commencement; and secondly, how to make the solution of one case of a problem serve to determine those of all the others...reserving all metaphysical discussion upon such quantities as +c and -c to a later stage, when [the pupil] will be better prepared to understand the difficulties of the subject.

8.2. DE MORGAN'S RESERVATIONS AS TO IMAGINARY NUMBERS

From this point onwards, De Morgan uses negative numbers without much shame, stating for example that a positive number has two square roots, one of them negative. On the other hand, he still does not use negatives entirely freely. In discussing the quadratic equation a few pages later he distinguishes six cases, viz.

$$ax^2+b = 0$$
$$ax^2-b = 0$$
$$ax^2+bx+c = 0$$
$$ax^2-bx+c = 0$$
$$ax^2+bx-c = 0$$
and $$ax^2-bx-c = 0.$$

This is to say that he is loath to permit a, b, or c to be negative, since, after all, there is no need. Whatever we today might call the signs of the coefficients is taken care of by letting the letters always represent positive numbers but having the equation take on the appropriate one of the six forms listed. This all leads to an analysis of the sign of the discriminant, b^2-4ac in some cases and of b^2+4ac in others, all very correct and difficult to remember. (In many American school algebra books of a hundred years ago students were asked to memorize the analysis of all six cases, and whether the roots in each case would be positive, negative, etc.) But worse is to come: When the discriminant is negative, a wholly new problem emerges: imaginary numbers.

De Morgan was writing in 1831, but in an insular England that was largely ignorant of recent developments in Continental mathematics. The Argand diagram for complex numbers had been known for 35 years, and Gauss and Cauchy had developed a science of complex numbers almost to the point of view taken today, but De Morgan makes no attempt in his book to develop a philosophy of their interpretation equivalent to what he has done for negatives. Perhaps he understood more than he was saying, but in this book, designed for teachers of children, he refrained from its elaboration. On page 151 he writes:

> We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility. It depends upon the fact, which must be verified by experience, that the common rules of algebra may be applied to these expressions without leading to any false results...

Complex numbers are absurd if construed as line segments — or are they?

Despite these pleasant features, he denies them any sense. He proposes two problems to distinguish his meanings: The first is the problem of the ages of father and son described above, where a *negative* answer can be made to yield up some sense, either as a guide to a restatement of the problem, or by the device of interpreting such a number as the same as its positive opposite, taken in an opposite direction. The equivalence of the two devices is of algebraic and practical importance. But his second example, he thinks, yields no such practical interpretation. Here it is: "It is required to divide **a** into two parts, whose product is **b**. The resulting equation is x^2-ax+b = 0..., the roots of which are imaginary when **b** is greater than $a^2/4$." Try as he may, he cannot get out of this one. If he replaces x by -x in the problem the roots are still imaginary when **a** is too small. (For De Morgan,"imaginary" means what we call complex.) He concludes that there is an essential difference between mere negative numbers, which can be repaired by a reinterpretation of the problem, and imaginary numbers, which for all that they obey the usual algebraic rules, cannot be made to represent anything sensible.

Of course, he has a physical prejudice in the back of his mind here. The problem of dividing **a** into two parts whose product is **b** is an ancient one, Babylonian but put into geometric form in Euclid, where it is construed as asking for a segment of length **a** to be partitioned into two segments which are sides of a rectangle of given area. (We would say "of given area," whereas Euclid remains purely geometric, and exhibits as the datum "**b**" a triangle to which he wants the resulting rectangle to be equivalent in his own sense of "equals." There are no numbers at all, hence no "areas" in our sense, in Euclid's formulation of such problems.)

Euclid's theorems provide a *construction* by which the point of partition may be found, but he notes a limitation: If the triangle **b** is larger than the square built on a/2 (i.e. half the segment **a**), then the necessary point of partition cannot be found. And that's the end of it: impossible. Euclid's "impossibility condition" is precisely our criterion concerning the discriminant, as it turns out. It says that the given length **a** is simply too short to accomplish the asked-for job, no matter where you divide it.

Neither Euclid nor De Morgan construes this problem in any other way; it is plain that the number **a**, which is to be partitioned in De Morgan's problem, looks to him like a line segment, and that there is plainly no solution, not even one that can be reinterpreted as an "opposite" when it turns out negative, when **b** is larger than the square upon a/2. Yet today, we often take a different point of view.

To us, to "divide a into two parts" when a is a *number*, means nothing other than to find two *numbers* whose sum is **a**, and this can be done in such a way that the product is any given *number* (not area) **b** is easy, when complex numbers are allowed as answers. Complex numbers are absurd if construed as line segments — or are they? Remember, -10 was also absurd, when construed as a length.

9. THE NEGATIVE ROOT IS NOT ABSURD!

But this is not the only interpretation of the number -10 that turns up in our gardening problem.* Ah, how much wiser we are, or think we are, than our forefathers! Let us return to the problem of the garden, whose area is to be 600 square yards, and one side of which is 50 more than the other. We put x for the

**Introduced in Part I.*

"length" of the garden, and found that x had to be 60 or -10, if anything. We rejected -10 as absurd, and solved the problem: 60 was the length, 10 the width.

Now where is this garden to be located? Here: One corner of it is under my feet, and the length is to be taken to the east, the width to the north. We can walk around the garden by walking 60 yards east, ten yards north, 60 back to the west and 10 south again and here we are. What about -10? Suppose we use that absurd solution as De Morgan, poor, simple De Morgan, suggested. We now surround what piece of land? Well, x = -10 and the "width" is 50 yards less, or -60, so: We walk -10 yards east, i.e. +10 yards *west*, then -60 yards north, i.e. 60 yards *south*, then back ten yards and back 60 yards and here we are at the origin (original corner). It is a totally different piece of land, to be sure, lying in the fourth (Cartesian) quadrant rather than the first. Its east-west dimension is of absolute value 10 rather than 60, so that "length" might be considered a strange description of that part of the boundary; but it, with the "width" of absolute value 60, satisfies all criteria of the problem. Its length is — as a *number* — indeed fifty more than its width (-10 is greater than -60 by 50, is it not?), and its area is 600, if "area" is the product of the numbers that describe the sides.

The answer the teacher expected was then 60 yards east by 10 yards north. But the stupid kid who insisted on "checking" the impossible answer x = -10, and got it to "check" at an area of 600 had just as good an answer, only his garden had a different orientation and position. I wonder what a Babylonian would have said to that.

One lesson that comes from all this is summarized by the title of a famous paper by the physicist Eugene P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" (*Comm. in Pure & Appl. Math*. v.13 (1960), 1-14). The present example, interpreting the 'absurd' second solution of the quadratic equation, is trivial compared to the sort of thing Wigner mainly had in mind, but it is of the same nature: the equations arrived at by scientists to describe some part of the physical world often seem to contain more information than the inventors thought they had put into it, and that it is one of the wonders of the life of science to discover such a thing in practice. But one also has to know how to look.

10. THE ANALYTIC GEOMETRY OF THE GARDEN

How did anyone ever think of that second solution to the garden problem? It sounds like a stretching of the meaning of "-10" to suddenly start talking east and west, north and south, but in truth we do talk that way in the 20th century all the time. Here is a reformulation of the garden problem which will *automatically* make sense of the "absurd" solution as well as the usual one. The word "analysis" was used above to describe the process of algebra we were using; well, the reformulation has to do with analytic geometry. Any child can do it:

<u>Problem</u>: Let a rectangle in the plane have one corner at (0,0) andthe opposite corner at (x,y), where y = x-50. Find all the corners if the area is to be 600.

<u>Answer</u>: Notice the problem does not insist on (x,y) being in the first quadrant. The area is clearly the absolute value of xy, whatever quadrant (x,y) is in. Since y = x-50, we set |x(x-50)| = 600 and hope such an x can be found, as above. Then either x(x-50) = 600 or -x(x-50) = 600, according to whether the number inside the absolute value signs turns out to be positive or negative. The first of the two equations gives x = 60 or x = -10, as earlier, and produces the corners (60,10) and (-10,-60) to define two rectangles (whose opposite corners are at the origin) that do the job. How easy! Of *course* x = -10 has a meaning, once we set the thing up on the coordinate plane.

But wait, what about the other equation, "-x(x-50) = 600"? This one has solutions, too, and they are x = 30 and x = 20, producing opposite-corner points (30,-20) and (20,-30), either of which, with the origin, sure enough forms a rectangle of area 600. Goodness, the more we want to make sense of the problem, the more answers turn up! But if you look at these last two "solutions," do they "check" when we try to prove they satisfy the conditions of the problem? They do: they give the correct area, and y = x-50 as demanded. The trouble here is that we probably have stated the problem badly.

If all we wanted was that the *number* that is the y-coordinate of the opposite-corner point should be 50 less than the *number* that is the x-coordinate of that point, these last two solutions check out in every detail. But surely this is a poor statement of the original problem, where the architect doubtless intended the *length* of one side of the garden to be 50 more than the *length* of the other side. The condition "y = x-50" is not a statement of that condition, while $|y| = |x|-50$ is the point (either that, or $|x| = |y|-50$).

With this restatement we can go back over the whole problem and find that the third and fourth "solutions" do *not* check. On the other hand, the new conditions on length, expressed in terms of absolute value, give rise to some new possibilities, and it will perhaps surprise nobody that there are eight solutions, with the"opposite-corners" at (10,60), (60,10), (10,-60), (60,-10), (-10,60), (-60,10), (-10,-60), and (-60,-10), that is, all the possible ways you can place a sixty-by-ten rectangle with one corner at the origin and sides parallel to the axes.

> ***The process of associating these invented numbers with some scientific or architectural use is not as simple or obvious as it might seem when they are presented axiomatically***

Pandora's Box is now open: what if the rectangles are not parallel to the axes? There are answers to that one, too, but they go beyond simple algebraic equations and their meaning. It were best now to cut our losses and go back to the beginning: "Sixty by ten" is doubtless the best answer. But intellectually we have found something out: negative numbers, just as De Morgan said, *can* be made to mean something valid. We have found something else out, too, just as De Morgan said, which is that we must understand that we are *making* them mean something, and that the process of associating these invented numbers with some scientific or architectural *use* is not as simple or obvious as it might seem when they are presented axiomatically. Logic is not only a matter of reasoning from axioms for a field, it is also a matter of reasoning from life.

11. EVEN IMAGINARY SOLUTIONS ARE NOT NECESSARILY ABSURD

Finally, let us return to the partition of a segment of (positive) length **a** into two pieces forming adjacent sides of a rectangle of area **b**. (This discussion will be rather condensed, compared to what has gone before.)

We suppose x is a length that does the job, i.e. x and a-x are the two side-lengths. We blindly set up the quadratic equation x(a-x) = b and find two solutions (both of which check in the equation if not the problem): They are

$$(a/2) + \sqrt{\left(\frac{a}{2}\right)^2 - b} \text{ and } (a/2) - \sqrt{\left(\frac{a}{2}\right)^2 - b}.$$

So, if there is a solution it has to be one of these two numbers. (Actually, since these solutions add up to a, this pair of numbers is the only possible solution, i.e. if x is the first, a-x is the second, and if x is the second, a-x is the first.)

When $(a/2)^2>b$ all is well; we get two positive numbers which add to a and which solve the problem. We can draw a picture of the resulting rectangle, and we have no negative solution to have to interpret. But what happens when $(a/2)^2< b$? Can we, with Wigner, discover the "unreasonable effectiveness of mathematics" by finding that there really is a genuine visible rectangle that solves the problem even when **a** is too small to partition properly, i.e. to produce the sides of a rectangle with desired area **b**? Sure.

Let a four dimensional Euclidean space have its axes labelled x,y,u,v, with the point (x,y,0,0) representing the number x+yi when this is the solution of a quadratic equation using the sign "+" in the quadratic formula, and the point (0,0,u,v) representing thenumber u+vi where this is the solution of the same quadratic equation using the "-" sign in the quadratic formula. Observe that in our problem, where a and b are positive, the numbers x,y,u and v obtained from our quadratic will always be positive when the discriminant forces us into complex roots. Thus x+yi can be pictured as a vector, or rather an arrow with tail at the origin and arrow-head in the first quadrant of the xy plane, and similarly for u+iv in its plane, which is perpendicular to the xy plane. The vectors are (when you disregard the frill of the arrowhead) perpendicular segments in 4-space, and the area of the rectangle they subtend — a genuine, visible rectangle — is their inner product xu+yv. Work it out; it is **b**.

How come? In this problem, **a** was "too small" to admit such a partition, or to put it in other terms, b was “too big” for a rod of length **a** to be broken for the purpose of making a rectangle that big. But what are the lengths of the vectors that made the sides of our rectangle in 4-space? They are $\sqrt{x^2+y^2}$ and $\sqrt{u^2+v^2}$, or $\sqrt{b}$ in each case; hey! — we’ve even got a square, not just a rectangle! Those are pretty long segment lengths, big enough so that the square they build in 4-space is sure enough of area **b**. But we found earlier that long things like that can’t partition a segment of length **a**. Indeed, the sum of these two *lengths* is $2\sqrt{b}$, which is certainly not **a**.

Well, what was the problem? Did we ask for **a** to be partitioned into two pieces whose lengths add to **a**, or did we ask for **a** to be partitioned into two numbers whose sum was **a**? We solved the latter problem, by finding complex numbers whose (complex) sum was **a** but whose lengths were big enough to make a square of size **b**.

Do I hear someone cry fraud?

“Fraud!” cried the maddened thousands, and echo answered fraud; But one scornful look from Casey and the audience was awed.

Partitioning a into *complex* pieces that make, in a suitable geometric interpretation of complex numbers, a suitable real rectangle is no more fraudulent than interpreting the garden problem as one of finding coordinates of a point in the Cartesian plane, rather than lengths of wall, or using negative numbers in the manner of DeMorgan to represent the past instead of a putative future.

We all know there is no date at which the son *will* be half the age of the father; it’s too late for that already. In De Morgan’s time it was still questionable whether using a negative answer amounted to a swindle. Unfortunately, “hardly a man is now alive,” (to quote from another narrative poet) who still appreciates the intellectual effort it took to overcome this natural disinclination to treat mathematical artifices as if they had real significance, and it is a rare teacher who recognizes there is even a problem.

A garden plot with negative sides is really every bit as silly, at first glance, as a square with complex sides. But you can get used to these things after a while. The important thing is to understand just what it is you are getting used to.

Editor’s Note: In the last issue of the Humanistic Mathematics Network Journal Mr. Raimi’s e-mail address was incorrect. It should be: rarm@db1.cc.rochester.edu, with a 1 instead of an l. We apologize for the error.

Mathematics Found in Poetry

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