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Explorations in Well-Rounded Lattices

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Abstract

Lattices are discrete subgroups of Euclidean spaces. Analogously to vector spaces, they can be described as spans of collections of linearly independent vectors, but with integer (instead of real) coefficients. Lattices have many fascinating geometric properties and numerous applications, and lattice theory is a rich and active field of theoretical work. In this thesis, we present an introduction to the theory of Euclidean lattices, along with an overview of some major unsolved problems, such as sphere packing. We then describe several more specialized topics, including prior work on well-rounded ideal lattices and some preliminary results on the study of planar deep hole lattices.

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Acknowledgments

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Chapter 1

Introduction

Welcome reader! The thesis you hold deals with the mathematical objects known as lattices. You may be a mathematician yourself, and be extremely familiar with lattices already, but even if you are not, I am sure you have experienced them in your day to day life. If you've ever seen a set of oranges, or other circular objects, stacked on top of each other, you've encountered an instance of the face-centered cubic lattice. If you've ever seen a sheet of graph paper, the points at every intersection form a lattice. In this introduction, we will explain what, mathematically, these objects actually are, and introduce some of their basic vocabulary. Then, we will introduce some of the most important applications of lattices, and from there progress to a discussion of some specific areas of active research on 2-dimensional lattices. If you wish to skip this more conceptual introduction and get right to the raw mathematics, advance to Chapter 2.

A lattice of *rank* k is the integer span of a group of k linearly independent vectors in $\mathbb{R}^n$, with $k \leq n$. Those vectors form the *basis* of the lattice. For example, the basis vectors $(0,1)$ and $(1,0)$ generate the lattice consisting of all integer-valued points of $\mathbb{R}^2$. This is called the integer lattice, which you can see in Figure 1.1.

Lattices are *full-rank* when $k = n$. In this thesis, our lattices will all be full-rank unless specified otherwise. Two lattices A and B are *similar* to each other if A can be produced by rotating and scaling B, or vice versa. This is an equivalence relation.

The field of the geometry of numbers was introduced in Minkowski (1910), which provided the initial basis for much of today's lattice theory. However, the idea of a lattice, and the kinds of problems that lattices can be used to solve, have been around for much longer.

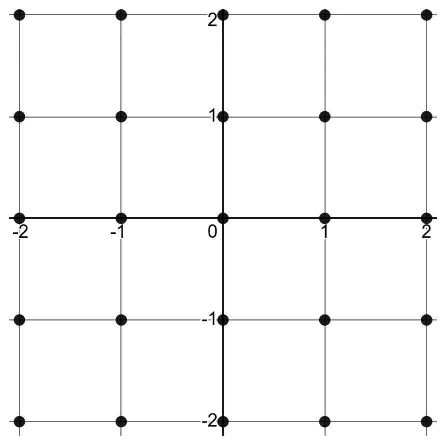

Figure 1.1 A portion of the two-dimensional integer lattice.

The most classic example of an application of lattices is the sphere packing problem. The problem is simple to describe: how densely can identical spheres be packed together in n dimensions with no overlap? Currently, the densest possible sphere configurations are only known for dimensions 2, 3, 8, and 24. All these packings are *lattice packings*, which are achieved by centering spheres at points of a fixed lattice. In 2 dimensions, the optimal packing comes from drawing a circle around each point of the hexagonal lattice, with basis vectors $(1, 0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, so that each circle touches six other circles, as in Figure 1.2.

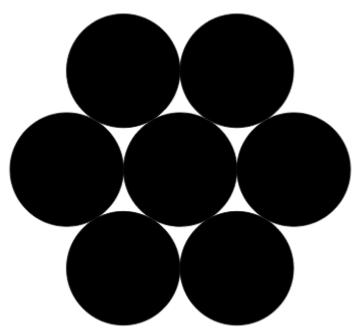

Figure 1.2 Seven optimally packed two-dimensional spheres.

Joseph-Louis Lagrange proved that this was the best lattice packing in 1773, but it was only shown to be the best two-dimensional packing overall in Tóth (1950). For 3 dimensions, 8 dimensions and 24 dimensions, the optimal packing is achieved by centering spheres on each point of the face-centered cubic lattice, the E_8 lattice and the Leech lattice respectively. The three-dimensional case was conjectured by Johannes Kepler in 1611 and only proven, after much controversy, in Hales et al. (2017). The latter two lattice packings were proven to be optimal in Viazovska (2017) and Viazovska et al. (2017), a discovery which Maryna Viazovska received a Fields medal for.

There are two other standard geometric problems which lattices are uniquely suited for. The first is the sphere covering problem, where the goal is to place overlapping identical spheres in $\mathbb{R}^n$ so that every point in $\mathbb{R}^n$ is inside at least one sphere, but also inside as few spheres as possible. There is also the kissing number problem: in n dimensions, how many spheres can touch a central sphere with no overlap? In 2 dimensions, both of these problems are solved by the hexagonal lattice; it provides the least dense sphere covering and demonstrates how six spheres (and no more) can touch a central one.

The optimal sphere covering is only known in 2 dimensions; the kissing number is known in dimensions 2, 3, 4, 8, and 24. In all known cases it can be obtained by placing the spheres on a lattice. However, for all three problems, it is not known if a lattice provides the solution in every dimension. For example, in 11 dimensions, the current best known sphere packing is not a lattice packing.

Despite this, we do know what properties a lattice must have in order to potentially provide a solution to any of these problems. For this introduction, the most important part is that the lattice must be *well-rounded*. This will be explained in detail in the next chapter. Thus, people looking for optimal lattice packings only need to focus on well-rounded lattices.

However, an infinite number of well-rounded lattices exist in all dimensions. There are additional requirements needed for lattices to solve the sphere packing problem, but even then there are still millions of possible candidates in higher dimensions. Manually sorting through all of those would be extraordinarily inefficient. Thus, having some way to generate well-rounded lattices in interesting ways would be useful.

Luckily, this exists. Number fields are objects that look like the rational numbers, $\mathbb{Q}$, but expanded to contain some algebraic number. For instance, $\mathbb{Q}(i)$, the set of all complex numbers $a + bi$ with rational coefficients is a number field. All number fields look like $\mathbb{Q}(\alpha)$ for some algebraic α. Specific ideals of number fields, when mapped into $\mathbb{R}^n$, create lattices, which are known as *ideal lattices*.

There are several known results on when these ideal lattices are well-rounded. For example, if $\mathbb{Q}(\alpha)$ is cyclotomic, i.e. if α is a primitive root of unity, then the ring of integers of the number field is well-rounded. If α is the square root of a positive or negative integer, then it is also fully known when the number field generates well-rounded ideal lattices.

Now for another, partially-related topic. If we restrict our scope to 2-dimensional, or planar, lattices, then, even though all the aforementioned geometric problems are solved, there are still interesting active areas of research.

The first, and most important fact about planar lattices is this: Figure 1.3 describes all planar lattices.

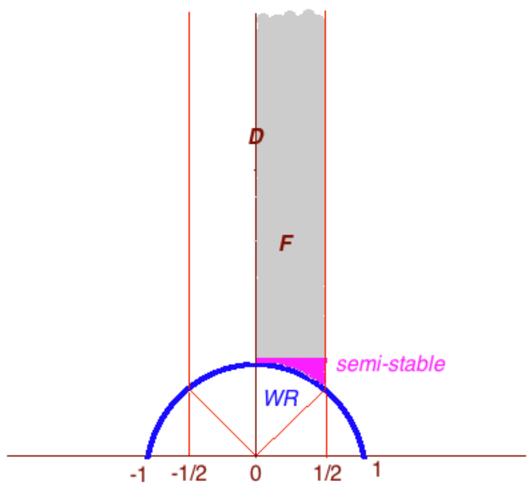

Figure 1.3 A description of all planar lattices.

Every planar lattice is similar to a lattice where one of its basis vectors is $(1, 0)$, and the other corresponds to a point in that strip $\mathcal{F}$, the set $\{(a, b) \mid 0 \leq a \leq 1/2, \sqrt{a^2 + b^2} \geq 1\}$. Since similarity is an equivalence relation, we can specify any single equivalence class of lattices just with one point $(x, y) \in F$.

Among the lattices represented by points in $\mathcal{F}$, the only well-rounded lattices are those where (x, y) is on the arc $x^2 + y^2 = 1$. Well-roundedness is preserved under scaling and rotation, so the points on this arc describe every well-rounded planar lattice up to similarity. In addition, if $y \leq 1$, the lattice is *semi-stable*; the exact definition of this is irrelevant at this point, and will be given later.

Keeping this framework in mind, we turn our attention to *deep hole lattices*. For a given lattice Λ, the *deep holes* of Λ are the set of points that are as far away from the points in Λ as possible.

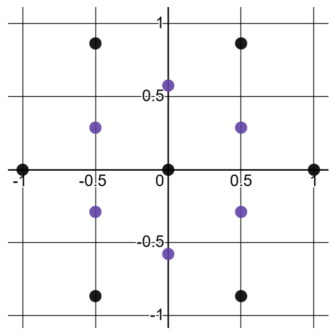

Figure 1.4 Six deep holes (purple) of the hexagonal lattice (black).

Take a planar lattice Λ with $\mathbf{x}_1$ and $\mathbf{x}_2$, $|\mathbf{x}_1| \leq |\mathbf{x}_2|$ as its shortest possible basis vectors, with the angle θ between $\mathbf{x}_1$ and $\mathbf{x}_2$ being in the interval $[\pi/3, \pi/2]$ (this can always be achieved). There is a single deep hole of Λ in the triangle whose sides are $\mathbf{x}_1$ and $\mathbf{x}_2$. Call the vector to that hole $\mathbf{z}$. We can create a new lattice with basis vectors $\mathbf{z}$ and $\mathbf{x}_1 - \mathbf{z}$; this is the deep hole lattice of Λ.

As we show in Chapter 5, if we look at some Λ with representative point (x, y) in $\mathcal{F}$, then the deep hole lattice of Λ will have a smaller y-value. If we take the deep hole lattice of that deep hole lattice, the y-value will shrink even more, until we end up with a point either on or, more likely, below the arc. If it is on the arc, we have a well-rounded lattice. If it is below the arc, it is possible to show that it is still a well-rounded lattice, which is then similar to a lattice with its representative point on the arc. We can see an example of this process in Figure 5.1.

So, repeatedly constructing the deep hole lattices gives us a sequence of points in $\mathbb{R}^2$. This sequence has not been exhaustively studied, but it may have interesting properties.

Chapter 2

Vocabulary and Fundamentals

As mentioned before, a lattice of *rank* k is the integer span of a collection of k linearly independent vectors in $\mathbb{R}^n$, with $k \leq n$. Those vectors are a *basis* of the lattice. Lattices are considered *full-rank* when $k = n$. In this paper, our lattices will be full-rank unless specified otherwise. Lattices can also be represented as a *basis matrix* multiplied by $\mathbb{Z}^n$, such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2,$$

the lattice consisting of all integer-valued points in two dimensions (see Figure 1.1). The basis vectors form the columns of the basis matrix. Basis vectors and matrices are not unique; if a lattice Λ has a basis matrix A, and there exists another matrix B such that $B = AU$ for an integral matrix U with determinant ± 1, then B is also a basis matrix for Λ.

Lattices can also be thought of as groups. In one sense, if we view $\mathbb{R}^n$ as a group whose action is vector addition, then lattices are the subgroups of $\mathbb{R}^n$ that are generated by their basis vectors. In another sense, any subgroup of $\mathbb{R}^n$ that is both *discrete* and *co-compact* in $\mathbb{R}^n$ is a full-rank lattice. (A discrete co-compact subgroup of an r-dimensional subspace of $\mathbb{R}^n$ is a lattice of rank r in $\mathbb{R}^n$). A subset Γ of $\mathbb{R}^n$ is discrete if there exists some constant C such that for every $x, y \in \Gamma$, $||x - y|| > C$. A discrete subset Γ of a subspace V of $\mathbb{R}^n$ is co-compact in V if there exists a compact 0-symmetric subset U of V such that

$$V = \bigcup \{U + x : x \in \Gamma\},$$

i.e. you can cover V with copies of U centered on each point of Γ.

Lattices have some important invariants and properties. The *determinant* of a full rank lattice is the absolute value of the determinant of its basis matrix A. For lattices that are not full rank, their basis matrices are not square, so their determinant is defined as $\sqrt{\det(A^\top A)}$. The determinant of a lattice does not depend on the choice of basis matrix; any basis matrix for the same lattice produces the same determinant. This is because for two basis matrices B and A, we have $B = AU$ and $|U| = 1$, so $|B| = |A|$.

The determinant of a lattice also has some geometric properties. A *fundamental domain* of a full rank lattice Λ in $\mathbb{R}^n$ is a convex set $F \subseteq \mathbb{R}^n$ containing 0, so that

$$\mathbb{R}^n = \bigcup_{x \in \Lambda} (F + x),$$

and for every $x \neq y \in \Lambda$, $(F + x) \cap (F + y) = \emptyset$. In other words, a fundamental domain of Λ is a full set of coset representatives of Λ in $\mathbb{R}$. No matter which fundamental domain of a lattice you choose, they all have the same volume, and that volume is the same as the determinant of the lattice.

There is a special shape related to fundamental domains called a *Voronoi cell*. This is defined by

$$\mathcal{V}(\Lambda) = \{x \in \mathbb{R}^n : ||x|| \leq ||x - y|| \ \forall y \in \Lambda\}.$$

This is the closure of, in a non-formal sense, the "roundest" fundamental domain of Λ.

Another set of important data points associated with a given lattice are its *successive minima* and *inhomogeneous minimum.* For a compact convex 0-symmetric set $M \subseteq \mathbb{R}^n$ and a lattice $\Lambda \subseteq \mathbb{R}^n$, the first successive minimum of Λ with respect to M is

$$\lambda_1 = \inf\{\lambda \in \mathbb{R}_{>0} : \lambda M \cap \Lambda \text{ contains a nonzero point}\},$$

where λM is just the set consisting of every point in M scaled by λ. The ith successive minimum of Λ with respect to M is

$$\lambda_i = \inf\{\lambda \in \mathbb{R}_{>0} : \dim(\mathrm{span}_{\mathbb{R}}(\lambda M \cap \Lambda)\} \geq i.$$

Thus, a lattice of rank k has exactly k successive minima. When we do not specify a set M, and talk only about "the successive minima of Λ," then M is assumed to be the unit ball in $\mathbb{R}^n$.

Because of the properties of M, we know that $\lambda_1 < \infty$. This is due to a result known as Minkowski's Convex Body Theorem, from Hermann

Minkowski (1864-1909), a pioneer in the field of the geometry of numbers. The theorem states that for any compact convex 0-symmetric set M and lattice Λ in $\mathbb{R}^n$,

$$0 < \lambda_1 \leq 2\left(\frac{\det(\Lambda)}{\mathrm{Vol}(M)}\right)^{1/n}.$$

There is an accompanying result known as Minkowski's Successive Minima Theorem, which says that, for the same objects as before,

$$\frac{2^n \det(\Lambda)}{n!\mathrm{Vol}(M)} \leq \lambda_1 \cdot \lambda_2 \ldots \lambda_{n-1} \cdot \lambda_n \leq \frac{2^n \det(\Lambda)}{\mathrm{Vol}(M)}.$$

This guarantees that $\lambda_i < \infty$ for all $1 \leq i \leq n$.

Next, given a compact convex 0-symmetric set M and a lattice Λ, we have that the inhomogenous minimum of M with respect to Λ is

$$\mu = \inf\{\lambda \in \mathbb{R}_{>0} : \lambda M + \Lambda = \mathbb{R}^n\},$$

in other words, the amount you have to scale M by so that it will cover $\mathbb{R}^n$ if we make a copy of it on every point of Λ. There are some known bounds on this value. If we take λ_i to be the ith successive minimum for M and Λ, then there is a result from Vojtěch Jarník (1897-1970) that

$$\mu \leq \frac{1}{2}\sum_{i=1}^{n} \lambda_i.$$

Jarník also proved that

$$\mu \geq \frac{\lambda_n}{2}.$$

There are three more important traits of lattices; they can be *well-rounded*, *perfect*, and *eutactic*. Well-rounded lattices are lattices where

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n.$$

The *minimal norm* of a lattice Λ is $|\Lambda|$, the length of its shortest nonzero vector under the normal Euclidean metric. Note that $|\Lambda| = \lambda_1$. Then, the *minimal vectors* of Λ are the vectors $x \in \Lambda$ such that $||x|| = |\Lambda|$. For instance, the minimal norm of the 2-dimensional integer lattice is 1, and its minimal vectors are $(0,1)$, $(1,0)$, $(0,-1)$, and $(-1,0)$. A well-rounded lattice must have at least $2n$ minimal vectors, since it must have at least n linearly independent vectors, along with their additive inverses.

Define $x_1, x_2, \ldots, x_m$ as the set of minimal vectors of a lattice, written as column vectors. A lattice is perfect if

$$\{x_i x_i^\top : 1 \leq i \leq m\},$$

the set of symmetric matrices generated by its minimal vectors, spans the set of real $n \times n$ symmetric matrices. A lattice is eutactic if there exist positive constants $c_1, c_2, \ldots, c_m$ such that

$$||v||^2 = \sum_{i=1}^{m} c_i (v^\top x_i)^2$$

for all vectors v in $\text{span}_{\mathbb{R}} \Lambda$. If a lattice is both perfect and eutactic, it is called *extreme*. Note that if a lattice is not well-rounded, it has $2m < 2n$ minimal vectors, so $\{x_i x_i^\top : 1 \leq i \leq m\}$ contains at most $2m < 2n$ linearly independent matrices, so it cannot span the set of real $n \times n$ symmetric matrices, which is $\frac{n^2+n}{2}$-dimensional. Thus, only well-rounded lattices can be perfect or extreme.

Finally, lattices can be *similar* to each other. For a lattice Λ_1 with basis matrix A, and a lattice Λ_2 with basis matrix B, the two lattices are similar, which is represented as $\Lambda_1 \sim \Lambda_2$, if there exists an orthogonal $n \times n$ matrix U and a constant $\alpha \in \mathbb{R}$ such that $A = \alpha U B$. In geometric terms, this means that we can create Λ_1 by rotating and scaling Λ_2, and vice versa. This is an equivalence relation on all full-rank lattices in $\mathbb{R}^n$, so it partitions them into disjoint equivalence classes. The properties of being well-rounded, perfect, and eutactic are all preserved under similarity.

Now that we know all the relevant facts about lattices, what is it that we use them for? As mentioned in the introduction, there are three major geometrical problems where lattices play an important part in their solutions. Now we will go over them again, with more detail. The first is the *sphere packing* question. In $\mathbb{R}^n$, how densely can you pack spheres of the same radius? In general, density refers to the ratio of volume covered by spheres to volume not covered by spheres. However, since $\mathbb{R}^n$ is infinite, both of those volumes are infinite, so the density is defined in more specific ways. For *lattice packings*, where the spheres are centered on point of a fixed lattice

Λ, the density of a packing is called $\Delta(\Lambda)$, and looks like

$$\Delta(\Lambda) := \frac{r^n \omega_n}{\det(\Lambda)}$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$ and r is the largest possible radius of the spheres without overlap. What's more, $r = \lambda_1/2$, half the length of the shortest vector in Λ. This means for a lattice, the packing density can be written instead as

$$\Delta(\Lambda) := \frac{\lambda_1^n \omega_n}{2^n \det(\Lambda)},$$

so it depends fully on the determinant and the first successive minimum. Why is the density defined that way? This is actually a formula for the volume of the largest sphere which can be inscribed into the Voronoi cell of Λ. So, we could think of this way of measuring density as partitioning $\mathbb{R}^n$ into copies of the Voronoi cell of Λ and seeing what fraction of each of those cells is occupied by a sphere. This fraction will always be the same, since the lattice is periodic. Viewing density this way also lets us see that the "rounder" the Voronoi cell of a lattice is, the more dense its associated sphere packing is.

For example, in two dimensions, the optimal packing is to arrange the circles so that each circle touches 6 others. This is equivalent to placing the centers of the circles on the points of the hexagonal lattice Λ_h, which looks like

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \mathbb{Z}^2.$$

Moreover, the Voronoi cell of the hexagonal lattice is itself a regular hexagon, which is, intuitively, a very "round" shape.

Solutions to the sphere packing problem are currently known in dimensions 2, 3, 8, and 24, and in all these cases they are given by lattice packings, though it is not known if lattice packings are always the optimal ones.

The second major problem is the *sphere covering* question. In this case, we are still placing spheres in $\mathbb{R}^n$, but now instead of maximizing coverage without overlap, we want to cover the entire space of $\mathbb{R}^n$ with the minimum possible overlap of spheres. This is also a problem where lattice packings, or in this case *lattice coverings*, give promising results. The only known optimal sphere covering is the hexagonal lattice in dimension 2, and it is not known if lattice coverings are optimal or merely very good in higher dimensions. We measure the *thickness* of a lattice, which we want to minimize, as

$$\Theta(\Lambda) := \frac{R^n \omega_n}{\det(\Lambda)},$$

with ω_n as before and R being the smallest possible radius such that the lattice covering does in fact cover $\mathbb{R}^n$. We notice immediately that $R = \mu$, the inhomogenous minimum of Λ, so we can rewrite this as

$$\Theta(\Lambda) = \frac{\mu^n \omega_n}{\det(\Lambda)}.$$

In this case, similar to the sphere packing case, this is the formula for the volume of the smallest sphere that can contain an entire Voronoi cell of Λ inside of it. So, we are again measuring thickness as the amount of overlap of spheres inside a single Voronoi cell of Λ, and we can see again that "rounder" cells are likely to be more optimal.

The third problem is the *kissing number* question: given a set of spheres of the same radius, how many spheres can you arrange touching a central sphere? In 2 dimensions, the kissing number is 6, as you can see by considering, again, spheres centered on the hexagonal lattice. In 3 dimensions, the kissing number is 12. This case was another hotly debated topic for centuries; for instance, Isaac Newton and another mathematician, David Gregory famously disagreed on it, with Newton proposing 12 was the maximum and Gregory certain that a 13^{th} sphere could be made to fit. The kissing number is also known in dimensions 4, 8, and 24.

In general, by taking a lattice and placing spheres centered on each of its minimal vectors, you create a potential candidate for the kissing number in each dimension. In other words, for spheres arranged on a lattice, the number of spheres touching a central one is exactly the same as the number of minimal vectors in the lattice. We do not know if the optimal kissing number in all dimensions can be achieved by a lattice arrangement, though for all known cases it is possible.

Note that in two of these problems, it is certain that only well-rounded lattices are possible candidates for solutions. For the kissing number problem, we see instantly that in any dimension well-rounded lattices have more minimal vectors than lattices that are not well-rounded. For the sphere packing problem, there is a theorem from Voronoi (1908) that a lattice packing is only optimal if it is extreme, which requires it to be well-rounded. There are no results that we are aware of on whether sphere coverings must be well-rounded to be optimal, but intuitively it does seem likely.

Now we turn to another question: where do we get lattices, especially well-rounded lattices, from? Of course, you can simply generate sets of basis vectors, but there are other ways. For this, we need to introduce some concepts from abstract algebra.

A *number field* is a finite algebraic extension of $\mathbb{Q}$. For example, the field $\mathbb{Q}(\sqrt{2})$ contains all elements $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$, and the field $\mathbb{Q}(\sqrt[4]{5})$ contains all elements of the form $a + b\sqrt[4]{5} + c(\sqrt[4]{5})^2 + d(\sqrt[4]{5})^3$ with $a, b, c, d \in \mathbb{Q}$. In general, an algebraic extension of $\mathbb{Q}$ with regard to some set S of algebraic numbers is the smallest field that contains all of $\mathbb{Q}$ and all of S. It is known, in a result called the Primitive Element Theorem, that every number field is simple, i.e. it can be represented as $\mathbb{Q}(\alpha)$ where α is a single algebraic number.

An *algebraic number* is a number that is the root of a polynomial with coefficients in $\mathbb{Q}$. Each algebraic number has a single polynomial of smallest degree (up to multiplication by a constant) that it is a root of; that polynomial is its *minimal polynomial*. For example, the minimal polynomial of $\sqrt{2}$ is $x^2 - 2$. An *algebraic integer* is an algebraic number whose minimal polynomial is monic with coefficients in $\mathbb{Z}$. So, $\sqrt{2}$ is an algebraic integer, but $\sqrt{2}/2$ is not, as its monic minimal polynomial is $x^2 - \frac{1}{4}$. The set of all algebraic integers in a number field K is a ring called that number field's *ring of integers*, $\mathcal{O}_K$.

Number fields can be *embedded* into $\mathbb{R}$ by *embeddings* written as σ_i. For some $\mathbb{Q}(\alpha)$, these embeddings map α to other roots of its minimal polynomial, known as *algebraic conjugates* of α. To continue our example, the embeddings of $\mathbb{Q}(\sqrt{2})$ are

$$\sigma_1(a + b\sqrt{2}) \to a + b\sqrt{2}$$

and

$$\sigma_2(a + b\sqrt{2}) \to a - b\sqrt{2},$$

i.e. they map $\sqrt{2}$ to $\pm\sqrt{2}$, the roots of $x^2 - 2$. *Real* embeddings are embeddings whose range is within $\mathbb{R}$, and *complex* embeddings are embeddings whose range is not totally within $\mathbb{R}$ (though it is still always within $\mathbb{C}$). Complex embeddings come in pairs; a complex embedding σ_i always implies another one $\overline{\sigma_i}$, where $\overline{\sigma_i}(x) = \overline{\sigma_i(x)}$ and $\overline{x}$ is, as standard, the complex conjugate of x.

The *degree* of a number field K is its dimension as a vector field over $\mathbb{Q}$. If we represent the number field as $K = \mathbb{Q}(\alpha)$, then the degree of K is also the degree of α in K, the number of algebraic conjugates of α, and the number of embeddings of K. We can organize them into r real embeddings $\sigma_1, \dots, \sigma_r$ and $2s$ complex embeddings $\sigma_{r+1}, \overline{\sigma_{r+1}}, \dots, \sigma_{r+s}, \overline{\sigma_{r+s}}$. We use this to define a map $\Sigma : K \to \mathbb{R}^r \times \mathbb{C}^s$, where

$$\Sigma(x) = (\sigma_1(x), \sigma_2(x), \dots, \sigma_r(x), \sigma_{r+1}(x), \sigma_{r+2}(x), \dots, \sigma_{r+s-1}(x), \sigma_{r+s}(x)).$$

Note that we only pick one embedding from each conjugate pair for this. Then, if we identify $\mathbb{C}^s$ with $\mathbb{R}^{2s}$, we have the map

$$\Sigma(x) = (\sigma_1(x), \dots, \sigma_r(x), \Re(\sigma_{r+1}(x)), \Im(\sigma_{r+1}(x)), \dots, \Re(\sigma_{r+s}(x)), \Im(\sigma_{r+s}(x))).$$

This is a vector in $\mathbb{R}^{r+2s} = \mathbb{R}^d$. This map is called the *Minkowski embedding*, and for any ideal I of O_K, $\Sigma(I)$ is a full rank lattice in $\mathbb{R}^d$. Lattices generated this way are called *ideal lattices*. The lattice generated by $\Sigma(O_K)$ is a *principal ideal lattice*.

Chapter 3

Well-Rounded Ideal Lattices

There is a general question we can ask about ideal lattices: given a number field, what are the properties of the lattices it generates? More specifically, when does it generate well-rounded lattices? There has been much work done on this in recent years.

The question was first raised in Fukshansky and Petersen (2012), where they showed that there exist infinitely many quadratic number fields – fields of degree 2 – whose rings of integers contain ideals that map to well-rounded lattices. At this point, we will stretch the notation slightly, and call an ideal itself well-rounded if it maps to a well-rounded lattice. Furthermore, the authors proved:

Theorem 1. *Let K be a number field of degree $d \geq 2$ and $I \subseteq \mathcal{O}_K$ a nonzero ideal. Then $|\Sigma(I)| \geq (r+s)N(I)^{\frac{1}{r+s}}$, where $N(I) := |\mathcal{O}_K/I|$ is the norm of I. Moreover, $|\Sigma(\mathcal{O}_K)| = r+s$,*

$$S(\Sigma(\mathcal{O}_K)) = \{\Sigma(x) : x \in \mathcal{O}_K \text{ is a root of unity}\},$$

where $S(\Lambda)$ is the set of minimal vectors of Λ, and $\Sigma(\mathcal{O}_K)$ is WR if and only if K is a cyclotomic field, i.e., $K = \mathbb{Q}(\zeta_k)$ for some primitive k-th root of unity ζ_k, $k \geq 2$. If this is the case, then

$$|\Sigma(\mathcal{O}_K)| = s = \frac{d}{2} = \frac{\varphi(k)}{2},$$

where φ is the totient function.

In their paper, they provide some examples of ideal lattices and their minimal vectors, which we have reproduced in Tables 3.1 and 3.2. For these, $\langle a, b\rangle$ is an ideal generated by a and b. Here we will also give an example of an ideal lattice of a cyclotomic field.

Take $K = \mathbb{Q}(\zeta_3)$. Then its ring of integers is $\mathbb{Z}(\zeta_3)$. The only algebraic conjugate of $\zeta_3 = \frac{-1+i\sqrt{3}}{2}$ is $\zeta_3^2 = \frac{-1-i\sqrt{3}}{2}$, so there are 2 complex embeddings, the identity and the complex conjugate. We choose the identity map as our complex embedding, so

$$\Sigma(a + b\zeta_3 + c\zeta_3^2) = (a - b/2 - c/2, b\sqrt{3}/2 - c\sqrt{3}/2) \in \mathbb{R}^2$$

We use Theorem 1 to see that the minimal vectors of this lattice are $\Sigma(1)$, $\Sigma(\zeta_3)$, and $\Sigma(\zeta_3^2)$, which are $(1, 0)$, $(-1/2, \sqrt{3}/2)$, and $(-1/2, -\sqrt{3}/2)$, respectively. These vectors are linearly independent, as desired, and they all have the same length of 1, so $\Sigma(O_K)$ is a well-rounded ideal lattice. In fact, this is the hexagonal lattice again! Also note that $1 = s = d/2 = \varphi(3)/2$, as the theorem predicts.

Let's also calculate an ideal lattice that is *not* well-rounded. Take $K = \mathbb{Q}(\sqrt{-7})$. In this case, its ring of integers is $\mathbb{Z}(\sqrt{7})$. As we will soon see, this ring contains *no* well-rounded ideal lattices, but for now we will just show one example. An ideal of this ring is $3O_K$. The two embeddings are the identity and the map from $\sqrt{7}$ to $-\sqrt{7}$, so

$$\Sigma(a + b\sqrt{7}) = (a + b\sqrt{7}, a - b\sqrt{7}).$$

Then $|\Sigma(a + b\sqrt{7})| = \sqrt{2a^2 + 14b^2}$. This is minimized, for values of a and b within $3O_K$, at $a = 3$ and $b = 0$, so $\lambda_1 = 6$ and the only minimal vectors of $\Sigma(3O_K)$ are the pair $\pm(3\sqrt{2}, 3\sqrt{2})$. These are obviously not linearly independent, so let's find another basis vector. In order to be linearly independent with $(3\sqrt{2}, 3\sqrt{2})$, we need $b \neq 0$, so the smallest vector fulfilling that is $\Sigma(3\sqrt{7}) = (3\sqrt{7}, -3\sqrt{7})$. This has magnitude $3\sqrt{14}$, so $\lambda_2 = 3\sqrt{14} \neq \lambda_1$. Thus, $\Sigma(3O_K)$ is not a well-rounded ideal lattice.

Soon after, Fukshansky et al. (2013) improved on the 2012 result, and were able to describe some specific quadratic fields that give rise to WR ideal lattices. A positive squarefree integer D satisfies the *3-nearsquare* condition if it has a divisor d such that $\sqrt{D/3} < d < \sqrt{D}$.

Theorem 2. *If D satisfies the 3-nearsquare condition, then the rings of integers of quadratic number fields $K = \mathbb{Q}(\sqrt{\pm D})$ contain WR ideals; the statement becomes if and only if when $K = \mathbb{Q}(\sqrt{-D})$. This in particular implies that a positive proportion (more than $1/5$) of real and imaginary quadratic number fields contain*

WR ideals, more specifically

$$\liminf_{N\to\infty} \frac{\left|\left\{K = \mathbb{Q}(\sqrt{\pm D}) : K \text{ contains a WR ideal}, 0 < D \leq N\right\}\right|}{\left|\left\{K = \mathbb{Q}(\sqrt{\pm D}) : 0 < D \leq N\right\}\right|} \geq \frac{\sqrt{3}-1}{2\sqrt{3}}. \tag{3.1}$$

Moreover, for every D satisfying the 3-nearsquare condition the corresponding imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-D})$ contains only finitely many WR ideals, up to similarity of the corresponding lattices, and this number is

$$\ll \min\left\{2^{\omega(D)-1}, \frac{2^{\omega(D)}}{\sqrt{\omega(D)}}\right\}, \tag{3.2}$$

where $\omega(D)$ is the number of prime divisors of D and the constant in the Vinogradov notation $\ll$ does not depend on D.

Finally, Srinivasan (2019) completed the documentation of ideal lattices of quadratic number fields. In their paper, they showed the following:

Theorem 3. *Let $K = \mathbb{Q}(\sqrt{D})$ with squarefree positive D be a real quadratic field. A primitive ideal $I = a\mathbb{Z} + \frac{b-\sqrt{D}}{2}\mathbb{Z}$ in the ring of integers is well-rounded if and only if $b = a$, where a is a positive integer that divides D and satisfies $\sqrt{\frac{D}{3}} < a < \sqrt{3D}$. Moreover if $d = 4d_1$ and I is well-rounded then $d_1 \equiv 3 \pmod 4$.*

Since all ideals in $\mathcal{O}_K$ are integer multiples of primitive ideals, which are well-rounded iff their corresponding primitive ideal is well-rounded, this describes whether or not any ideal in $\mathcal{O}_K$ is well-rounded. Combining this with Theorem 2 allows us to always determine, for positive or negative D, whether or not $Q(\sqrt{D})$'s ring of integers contains well-rounded ideal lattices.

Of course, there are many ideal lattices that are not quadratic. For those, work is still ongoing.

Table 3.1 Examples of ideals in imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ that give rise to WR lattices.

$-D$	Ideal $I \subset \mathcal{O}_K$	Minimal elements
-15	$\left\langle 2, \frac{1-\sqrt{-15}}{2} \right\rangle$	$\pm 2, \pm\frac{1-\sqrt{-15}}{2}$
-55	$\left\langle 4, \frac{3-\sqrt{-55}}{2} \right\rangle$	$\pm 4, \pm\frac{3-\sqrt{-55}}{2}$
-119	$\left\langle 6, \frac{5-\sqrt{119}}{2} \right\rangle$	$\pm 6, \pm\frac{5-\sqrt{119}}{2}$
-207	$\left\langle 8, \frac{7-\sqrt{207}}{2} \right\rangle$	$\pm 8, \pm\frac{7-\sqrt{207}}{2}$

Table 3.2 Examples of ideals in real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ that give rise to WR lattices.

D	Ideal $I \subset \mathcal{O}_K$	Minimal elements
21	$\left\langle 7, \frac{7-\sqrt{21}}{2} \right\rangle$	$\pm\frac{7\pm\sqrt{21}}{2}$
165	$\left\langle 15, \frac{15-\sqrt{165}}{2} \right\rangle$	$\pm\frac{15\pm\sqrt{165}}{2}$
285	$\left\langle 19, \frac{19-\sqrt{285}}{2} \right\rangle$	$\pm\frac{19\pm\sqrt{285}}{2}$
957	$\left\langle 33, \frac{33-\sqrt{957}}{2} \right\rangle$	$\pm\frac{33\pm\sqrt{957}}{2}$

Chapter 4

Deep Hole Lattices

This remainder of this thesis is based on a joint paper with L. Fukshansky, currently in preparation.

In this chapter, we discuss the objects called deep hole lattices, and examine deep holes of planar lattices. First, though, we need to introduce some more terminology. As you may have noticed, the results in the previous chapter were almost entirely about planar lattices. This has a simple explanation: planar lattices are easier to work with than lattices of higher dimensions. There is one major reason for this.

As mentioned before, similarity partitions all lattices into equivalence classes. There is a specific region of $\mathbb{R}^2$ that contains exactly one representative of every equivalence class of planar lattices. Let's identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $\mathbb{H} = \{\tau = a + bi : b \geq 0\} \subset \mathbb{C}$ be the upper half-plane, and let

$$\mathcal{D} := \{\tau = a + bi \in \mathbb{H} : -1/2 < a \leq 1/2, |\tau| \geq 1\}.$$

Define

$$\mathcal{F} := \{\tau = a + bi \in \mathbb{H} : 0 \leq a \leq 1/2, |\tau| \geq 1\},$$

so, loosely speaking, $\mathcal{F}$ is "half" of $\mathcal{D}$. $\mathcal{F}$ is shown in Figure 4.1. Every point $\tau = a + bi \in \mathcal{F}$ can be identified with a lattice

$$\Lambda_\tau := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \tag{4.1}$$

in $\mathbb{R}^2$, i.e. a lattice with one basis vector equal to $(0, 1)$ and one basis vector defined by a point in $\mathcal{F}$. Then, every planar lattice is similar to exactly one lattice of this form.

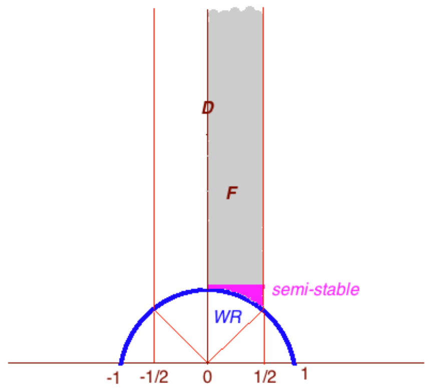

Figure 4.1 Space of lattices in $\mathbb{R}^2$ with well-rounded and semi-stable subregions marked by colors.

In addition, for lattices of this form, their first successive minimum λ_1 equals 1 and their second successive minimum λ_2 equals $|\tau|$. This means that if τ is on the arc of the circle, then $\lambda_2 = \lambda_1$, so Λ_τ is well-rounded. Conversely, if τ is not on the arc then Λ_τ is not well-rounded. There is also a weaker condition related to the placement of τ; if $\tau = a + bi$ has $b \leq 1$, then Λ_τ is *semi-stable*, which means that $\lambda_1 \geq \det(\Lambda_\tau)^{1/2}$.

We now turn to deep holes. For a lattice $\Lambda \subset \mathbb{R}^n$, a *deep hole* of Λ is a point in $\mathbb{R}^n$ that is as far away as possible from all points of Λ. The minimum distance from a deep hole $\mathbf{z}$ to any point of Λ is precisely the inhomogenous minimum μ of Λ.

Every lattice has infinitely many deep holes; we would like to create a way to identify specific ones. Consider a lattice Λ with shortest possible basis vectors $\mathbf{y}_1$ and $\mathbf{y}_2$, i.e. $|\mathbf{y}_1| = \lambda_1$ and $|\mathbf{y}_2| = \lambda_2$. It is always possible, from $\pm\mathbf{y}_1$ and $\pm\mathbf{y}_2$, to choose two basis vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ with $|\mathbf{x}_1| = \lambda_1$ and $|\mathbf{x}_2| = \lambda_2$, with the angle θ between $\mathbf{x}_1$ and $\mathbf{x}_2$ in $[\pi/3, \pi/2]$. This angle is an invariant of Λ.

Then, there is exactly one deep hole of Λ in the center of the triangle with vertices $\mathbf{0}$ and the endpoints of $\mathbf{x}_1$ and $\mathbf{x}_2$. Call the vector to this hole $\mathbf{z}$.

We call the lattice with the basis vectors $\mathbf{z}$ and $\mathbf{x}_1 - \mathbf{z}$ the *deep hole lattice* of Λ, which is also written as $H(\Lambda)$. The properties of Λ imply some properties of $H(\Lambda)$.

Lemma 4. *If* $\alpha = \lambda_2/\lambda_1 \leq 2\sin(\theta + \pi/6)$*, then* $H(\Lambda)$ *is well-rounded.*

Proof. Let ν be the angle between the vectors $\mathbf{z}, \mathbf{x}_1 - \mathbf{z}$. Since $\mathbf{z}$ is the center of the circle circumscribed about the triangle T, we have $\|\mathbf{z}\| = \|\mathbf{x}_1 - \mathbf{z}\|$ as both of these line segments are radii of this circle. This common value is μ, the covering radius of the lattice L, which is given by the formula

$$\mu = \frac{\sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\theta}}{2\sin\theta},$$

as in Lemma 5.1 of Forst and Fukshansky (2023).

Then the triangle with sides

$$\mathbf{x}_1, \mathbf{z}, \mathbf{x}_1 - \mathbf{z}$$

is isosceles, and so the bisector of the angle ν is perpendicular to $\mathbf{x}_1$ and is the median of that side. Therefore we have

$$\sin(\nu/2) = \frac{\lambda_1/2}{\mu} = \frac{\lambda_1 \sin\theta}{\sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta}} = \frac{\sin\theta}{\sqrt{1 + \alpha^2 - 2\alpha\cos\theta}}. \tag{4.2}$$

If $\nu \in [\pi/3, \pi/2]$, then the lattice is well-rounded, so $\sin(\nu/2) \geq \sin(\pi/6) = 1/2$ is a sufficient condition. Thus, the condition is

$$4\sin^2\theta \geq 1 + \alpha^2 - 2\alpha\cos\theta.$$

Rewriting $\sin^2\theta$ as $1 - \cos^2\theta$, we obtain

$$3 - 3\cos^2\theta = 3\sin^2\theta \geq \alpha^2 - 2\alpha\cos\theta + \cos^2\theta = (\alpha - \cos\theta)^2,$$

which leads to

$$\alpha \leq \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + \pi/6).$$

□

This proof implies some further corrolaries.

Corollary 5. *If Λ is semi-stable, then $H(\Lambda)$ is well-rounded.*

Proof. If Λ is semi-stable, then

$$\alpha = |\tau| \leq \sqrt{5}/4 < \sqrt{3} = \min\{2\sin(\theta + \pi/6) : \theta \in [\pi/3, \pi/2]\},$$

and so the condition of Lemma 4 is satisfied. □

Corollary 6. *If Λ is well-rounded, then $H(\Lambda) \sim \Lambda$.*

Proof. Notice that two well-rounded lattices are similar if and only if they have the same angle. Using the notation and setup of Lemma 4 and applying Equation 4.2 with $\alpha = 1$ (since Λ is well-rounded), we obtain:

$$\sin(\nu/2) = \frac{\sin\theta}{\sqrt{2-2\cos\theta}},\ \cos(\nu/2) = \frac{\sqrt{2-2\cos\theta-\sin^2\theta}}{\sqrt{2-2\cos\theta}}.$$

Therefore

$$\begin{aligned}\cos\nu &= \cos^2(\nu/2)-\sin^2(\nu/2) = \frac{2-2\cos\theta-2\sin^2\theta}{2-2\cos\theta}\\ &= \frac{-\cos\theta(1-\cos\theta)}{1-\cos\theta} = -\cos\theta = \cos(\pi-\theta).\end{aligned}$$

Since $\theta \in [\pi/3, \pi/2]$, $\pi - \theta \in [\pi/2, 2\pi/3]$, and so taking the basis $\mathbf{z}, \mathbf{x}_1 - \mathbf{z}$ for $H(L)$ we see that the angle of $H(L)$ is θ. □

For some subfield K of $\mathbb{C}$ and Λ_τ, if $\tau = a + bi$ where $a, b \in K$, then we say that Λ_τ (and its similarity class) *lies over* K. This implies that there exist infinitely many lattices Λ in this similarity class so that $\Lambda \subset K^2$. (Note that even though K can contain complex numbers, Λ is also still always a subset of $\mathbb{R}^2$). Be careful; the statements "τ is in K", "τ lies over K", and "$\Lambda \subset K^2$, for Λ similar to Λ_τ" all mean different things, and none of them imply each other.

Corollary 7. *If $\Lambda \subset K^2$ for some subfield K of $\mathbb{C}$, then $H(\Lambda) \subset K^2$. Further, if the condition*

$$\lambda_2/\lambda_1 \le 2\sin(\theta + \pi/6) \tag{4.3}$$

of Lemma 4 is satisfied, then the similarity class of $H(\Lambda)$ lies over K.

Proof. Since $\Lambda \subset K^2$, we have $\lambda_j^2 = \|\mathbf{x}_j\|^2 \in K$ for $j = 1, 2$, and so $\alpha^2 \in K$. Let $A = (\mathbf{x}_1\ \mathbf{x}_2) \in \mathrm{GL}_2(K)$ be the corresponding basis matrix. By Lemma 4.1 of Forst and Fukshansky (2023), the deep hole

$$\mathbf{z} = \frac{1}{2}(A^\top)^{-1}\begin{pmatrix}\lambda_1^2\\ \lambda_2^2\end{pmatrix}, \tag{4.4}$$

and hence $H(L) = \mathrm{span}_{\mathbb{Z}}\{\mathbf{x}_1, \mathbf{z}\} \subset K^2$. Further, $\lambda_1\lambda_2\cos\theta \in K$ since it is equal to the dot product of the vectors $\mathbf{x}_1, \mathbf{x}_2 \in L$, and $\lambda_1\lambda_2\sin\theta = \det L \in K$. Then

$$\cos^2\theta = \frac{(\lambda_1\lambda_2\cos\theta)^2}{\lambda_1^2\lambda_2^2} \in K,\ \sin^2\theta = 1-\cos^2\theta \in K,$$

as well as

$$\sin\theta\cos\theta = \frac{(\lambda_1\lambda_2\sin\theta)(\lambda_1\lambda_2\cos\theta)}{\lambda_1^2\lambda_2^2} \in K.$$

With the notation of Lemma 4, we use Equation 4.2 along with a double angle formula for cosine, and obtain:

$$\cos\nu = 1 - \frac{2\lambda_1^2\sin^2\theta}{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\cos\theta} \in K,$$

as well as

$$\sin\nu = \frac{2\lambda_1\lambda_2\sin\theta - 2\lambda_1^2\sin\theta\cos\theta}{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\cos\theta} \in K.$$

Then the angle of $H(\Lambda)$, which we will call ν', is equal to either ν or $\pi-\nu$. Now assume Equation 4.3 is satisfied, then $H(\Lambda)$ is well-rounded by Lemma 4. Taking $\tau = \cos\nu' + i\sin\nu'$, we see that $H(\Lambda)$ is similar to Λ_τ, which completes the proof. □

Lemma 8. *Let $\tau = a+bi \in \mathcal{F}$, Λ_τ the corresponding lattice as in Equation 4.1 and suppose that Λ_τ lies over a field $K \subseteq \mathbb{C}$. Then $\gamma \in \mathcal{F}$ representing the similarity class of $H(\Lambda_\tau)$ also lies over K.*

Proof. Let us write

$$\boldsymbol{e}_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \boldsymbol{\tau} = \begin{pmatrix}a\\b\end{pmatrix}$$

for the minimal basis of Λ_τ and $\theta \in [\pi/3, \pi/2]$ for the angle between these two vectors, hence

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}},\ \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Then Equation 4.4 guarantees that the deep hole of Λ_τ is

$$\boldsymbol{\tau}_1 = \begin{pmatrix}1/2\\(a^2+b^2-a)/2b\end{pmatrix}, \tag{4.5}$$

which certainly lies over K and $H(\Lambda_\tau) = \Lambda_{\tau_1}$ for $\tau_1 = \frac{1}{2} + \frac{a^2+b^2-a}{2b}$. If $\tau_1 \in \mathcal{F}$, then $\gamma = \tau_1$ and we are done.

Suppose $\tau_1 \notin \mathcal{F}$. This means that $|\tau_1| < 1$, then the vectors $\boldsymbol{\tau}_1, \boldsymbol{e}_1 - \boldsymbol{\tau}_1 \in \Lambda_{\tau_1}$ have equal length and the angle between them is in the interval

$(\pi/3, 2\pi/3]$: the largest value $2\pi/3$ is attained when $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. On the other hand, $|\tau_1| < 1$ implies that $(a^2 + b^2 - a)/2b < \sqrt{3}/2$, hence

$$\frac{\|\boldsymbol{\tau}\|}{\|\boldsymbol{e}_1\|} = \sqrt{a^2 + b^2} < \frac{\sqrt{3}b + a}{\sqrt{a^2 + b^2}} = \sqrt{3}\sin\theta + \cos\theta,$$

which means that Λ_τ satisfies the condition of Lemma 4. Therefore $H(\Lambda_\tau)$ is well-rounded and its similarity class γ lies over K by Corollary 7. This completes the proof. $\square$

Chapter 5

Next Steps

So, that was a collection of things that you can infer about a deep hole lattice from the lattice that generated it. Now, let's take it one step further; what does the deep hole lattice of a deep hole lattice look like?

Theorem 9. *Let $\tau_0 = a_0 + b_0 i \in \mathcal{F}$ with $a_0, b_0 \in K$ for some subfield $K \subseteq \mathbb{C}$. There exists a finite sequence of numbers $\tau_1, \ldots, \tau_n \in \mathcal{F}$ so that*

$$a_k = \frac{1}{2},\ b_k = \frac{a_{k-1}^2 + b_{k-1}^2 - a_{k-1}}{2b_{k-1}} \in K\ \forall\ 1 \le k \le n,$$

with $\Lambda_{\tau_k} = H(\Lambda_{\tau_{k-1}})$ and Λ_{τ_n} well-rounded, hence $H(\Lambda_{\tau_n}) \sim \Lambda_{\tau_n}$. Furthermore, $n \le \log_2\left(\frac{2b_0}{\sqrt{3}}\right)$.

Proof. Existence of the sequence τ_k with the claimed properties follows by iterative application of Equation 4.5. Further, $a_k \le 1/2$ for each k, so

$$b_k = \frac{b_{k-1}}{2} - \frac{a_{k-1}(1 - a_{k-1})}{2b_{k-1}} \le \frac{b_{k-1}}{2} - \frac{1}{8b_{k-1}} < \frac{b_{k-1}}{2},$$

hence $b_k < \frac{b_0}{2^k}$. If $n \ge \log_2\left(\frac{2b_0}{\sqrt{3}}\right)$, then $b_n < \sqrt{3}/2$, which means that $|\tau_n| < 1$ and hence Λ_{τ_n} is well-rounded. Then Corollary 6 implies that $H(\Lambda_{\tau_n})$ is similar to Λ_{τ_n} and hence the sequence of deep holes has stabilized. $\square$

In other words, if we recursively create deep hole lattices, we move "down" $\mathcal{F}$ until we reach a well-rounded lattice, and we create a corresponding sequence $\{\tau_k\}$ of complex numbers. For example, if we choose $\tau_0 = 0.25 + 3i$, then τ_1 would be $0.5 + 0.146875i$. Then τ_2 would be

$0.5 + 0.649268617021277i$, and τ_3 would be $0.5 + 0.132109986954059i$. Since τ_3 is below $\mathcal{F}$, Λ_{τ_3} must be similar to a well-rounded lattice. We calculate that $\Lambda_{\tau_3} \sim \Lambda_{\tau_f}$, with $\tau_f = 0.255453509374653 + 0.966821340552728i$.

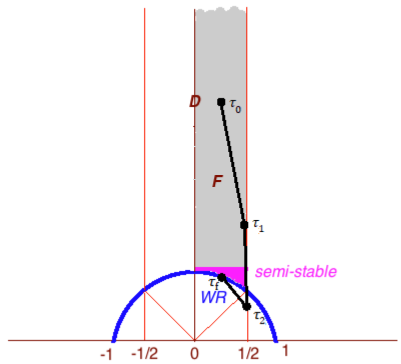

Figure 5.1 Evolution of the initial lattice represented by $\tau = 0.5 + 3i$.

What can we say about these numbers? Very little, currently. For our further work, we are investigating the endomorphism rings of the elliptic curves corresponding to each Λ_{τ_k}.

Every elliptic curve can be represented by quotienting the complex plane with a planar lattice, and likewise every planar lattice generates an elliptic curve. Every elliptic curve has a corresponding endomorphism ring that is at least as large as $\mathbb{Z}$.

A lattice is *arithmetic* if for its basis matrix A, $A^\intercal A$ is a scalar multiple of an interger matrix. Λ_τ is arithmetic iff

$$\tau = \frac{p}{q} + i\sqrt{\frac{s}{t}} \in \mathcal{F}$$

for integer p, q, s, t. If Λ_τ is arithmetic, the endomorphism ring of the associated elliptic curve is $\mathbb{Z}[\tau]$. Furthermore, as we see from Theorem 9, if Λ_τ is arithmetic, then so is its deep hole lattice. Thus the sequence $\{\tau_n\}$ generates a corresponding sequence of endomorphism rings $\mathbb{Z}[\tau_n]$.

Question 1. *Is there anything interesting that can be said about these endomorphism rings $\mathbb{Z}[\tau_n]$? Are they related in some interesting way? They do not appear to be subrings of each other.*

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