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Permutations, Representations, and Partition Algebras: A Random Walk through Algebraic Statistics

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Abstract

My thesis examines a class of functions on the symmetric group called permutation statistics using tools from representation theory. In 2014, Axel Hultman gave formulas for computing expected values of permutation statistics sampled via random walks. I present analogous formulas for computing variances of these statistics involving Kronecker coefficients – certain numbers that arise in the representation theory of the symmetric group. I also explore deep connections between the study of moments of permutation statistics and the representation theory of the partition algebras, a family of algebras introduced by Paul Martin in 1991. By harnessing these partition algebras, I derive a new polynomial describing the mean statistic of the 2^{nd} moment of the number of inversions of a permutation.

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Chapter 1

Introduction

1.1 A problem

Suppose you are shuffling a deck of n cards in the following way. Choose any two cards uniformly at random, and swap them. Repeat this process over and over. It's reasonable to suspect that this procedure will eventually shuffle the cards uniformly, but how long will this take? There are many other natural questions about the permutation of cards after some number of swaps. For example, how many fixed points do we expect it to have? How many pairs of cards will be out of order? These are the kinds of questions I investigate in this thesis.

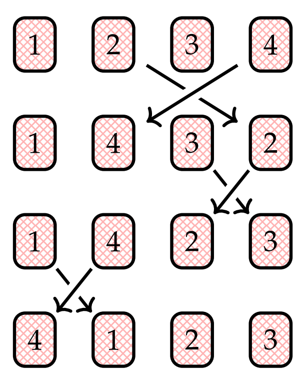


Figure 1.1 An example application of the pairwise shuffle on to a deck of 4 cards.

Functions that capture information about permutations (like, for example, the number of fixed points) are called *permutation statistics*. I am interested in characterizing the distributions of permutation statistics after various random walks (i.e., shuffling procedures), like the one above. One permutation statistic of particular interest is the number of inversions in a permutation; an *inversion* is a pair of cards that are reversed, i.e., a pair $i < j$ where card j comes before card i. To motivate our approach to tackling problems like this, we will first consider a simpler method of shuffling cards.

Suppose that at each step, we either move the top card to the bottom of the deck, or the bottom card to the top of the deck, each with probability $1/2$, as shown in figure 1.2. We'll call this a *cyclic shuffle*, since at each step we are cycling the cards one position up or down. What is the expected number of inversions in the resulting permutation?

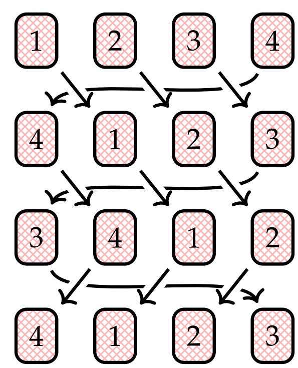


Figure 1.2 An example application of the cyclic shuffle on to a deck of 4 cards.

When shuffling the cards in this way, the permutation of the cards is entirely determined by where card 1 is: if card 1 is in the k^{th} spot, then card j will be in position $k + j \pmod{n}$. So, we can simplify our analysis by only keeping track of the position of card 1.

Suppose card 1 is in the k^{th} spot. Then, each of the cards in the first $k-1$ spots is greater than every card in the last $n-k+1$ spots. Hence, each pair containing one of the first $k-1$ cards and one of the last $n-k+1$ is an inversion. Also, the cards in the first $k-1$ spots are in ascending order, as are the cards and the last $n-k+1$ spots, so there are no other inversions. Thus the number of inversions in such a permutation is $(k-1)(n-k+1)$.

Let $v_t(k)$ denote the probability that card 1 is in position k after t steps. By our reasoning above, the expected number of inversions after t steps (which we will denote by $\mathbb{E}(\textsc{Inv}, t)$) is

$$\mathbb{E}(\textsc{Inv}, t) = \sum_{k=1}^{n} (k-1)(n-k+1)v_t(k).$$

We can collect the values $v_t(k)$ into a single vector which we will denote by $\vec{v}_t$:

$$\vec{v}_t = \begin{bmatrix} v_t(1) \\ v_t(2) \\ \vdots \\ v_t(n) \end{bmatrix}.$$

If we define the vector $\vec{s}$ whose k^{th} entry is $(k-1)(n-k+1)$, then we can write the above equation for the expected number of inversions as

$$\mathbb{E}(\textsc{Inv}, t) = \vec{s} \cdot \vec{v}_t.$$

To be at position k at time t, card 1 must have been at either position $k-1 \pmod{n}$ at or position $k+1 \pmod{n}$ at time $t-1$. In either case, there is a 50% chance card 1 will move to position k at step t. Hence, we have the recurrence relation

$$v_t(k) = \frac{v_{t-1}(k-1) + v_{t-1}(k+1)}{2},$$

where $k+1$ and $k-1$ are computed modulo n. This recurrence may be summarized as a matrix equation $\vec{v}_t = A\vec{v}_{t-1}$, where the $n \times n$ matrix A is given by

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 & \cdots & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & \cdots & 0 & 0 \\ 0 & 0 & 1/2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & \cdots & 1/2 & 0 \end{bmatrix}.$$

The recurrence relation implies $\vec{v}_t = A^t\vec{v}_0$. where the initial vector $\vec{v}_0$ is

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

since the first card starts in position 1. Hence, the expected number of inversions after t steps is

$$\mathbb{E}(\textsc{Inv}, t) = \vec{s} \cdot (A^t \vec{v}_0). \tag{1.1}$$

To compute this quantity, it will be helpful switch to an basis of eigenvectors of A. The eigenvectors $\{\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{n-1}\}$ of A are given by

$$\vec{x}_j = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \vdots \\ \omega^{(n-1)j} \end{bmatrix}$$

where $\omega = e^{2\pi i/n}$. We have included the factor $\frac{1}{\sqrt{n}}$ so that the set $\mathcal{B} = \{\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{n-1}\}$ forms an orthonormal basis. Since it's orthonormal, the dot product in equation (1.1) is unchanged if we compute it with respect to the basis $\mathcal{B}$ rather than the standard basis. Using this trick, we can compute the expected numbers of inversions after t steps of this cyclic shuffle.

Example 1.1. Suppose the deck has $n = 4$ cards. The basis $\mathcal{B}$ is now

$$\left\{ \vec{x}_0 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} 1/2 \\ i/2 \\ -1/2 \\ -i/2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1/2 \\ -i/2 \\ -1/2 \\ i/2 \end{bmatrix} \right\}.$$

With respect to the basis $\mathcal{B}$, the matrix A is

$$[A]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vectors $\vec{v}_0$ and $\vec{s}$ decompose in the basis $\mathcal{B}$ as

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\vec{x}_0 + \frac{1}{2}\vec{x}_1 + \frac{1}{2}\vec{x}_2 + \frac{1}{2}\vec{x}_3,$$

and

$$\vec{s} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 3 \end{bmatrix} = 5\vec{x}_0 - 2i\vec{x}_1 + 1\vec{x}_2 + 2i\vec{x}_3.$$

Hence we write

$$[\vec{v}_0]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad [\vec{s}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2i \\ 1 \\ 2i \end{bmatrix}.$$

Now, for all $t \geq 1$ we have

$$[\vec{v}_t]_{\mathcal{B}} = [A]_{\mathcal{B}}^t [\vec{v}_0]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ 0 \\ (-1)^t/2 \\ 0 \end{bmatrix}.$$

Hence, the expected number of inversions is

$$\mathbb{E}(\textsc{Inv}, t) = [\vec{s}]_{\mathcal{B}} \cdot [\vec{v}_t]_{\mathcal{B}} = \frac{5}{2} + \frac{(-1)^t}{2}.$$

1.2 The bigger picture

Now is a good time to take a step back and look at what we've done here. Notice that in the previous example, card 1 has the same probability of moving left or right, regardless of its position. We may consider the possible positions of card 1 as elements of the group $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n under addition. In this framework, the property we have just described may be summarized as

$$\Pr(\text{step from } k \text{ to } k + l) = \Pr(\text{step from } k' \text{ to } k' + l) \quad \text{for any } k, k' \in \mathbb{Z}/n\mathbb{Z}.$$

In this example, the above probability is 0 whenever $l \neq \pm 1$, but this need not always be the case.

This can naturally be generalized to an arbitrary finite group G via the equation

$$\Pr(\text{move from } a \text{ to } ag) = \Pr(\text{move from } a' \text{ to } a'g) \quad \text{for any } a, a' \in G.$$

In other words, the probability of moving from a to ag depends only on g, not a. Hence, there is a single function p on G governing the whole random walk; at any step, the probability of moving from a to ag is $p(g)$. For this to make sense, p must be a probability distribution on G, meaning $p(g) \geq 0$ for all g and $\sum_{g \in G} p(g) = 1$. A random walk constructed from a probability distribution $p\colon G \to [0,1]$ in this way is called a *random walk on the group G*. The problem we tackled in the previous section can be viewed as a question about the random walk on the group $\mathbb{Z}/n\mathbb{Z}$ governed by the probability distribution

$$p(g) = \begin{cases} 1/2 & g = 1 \text{ or } n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Our initial question about shuffling cards pairwise can be viewed as a question about a random walk on the symmetric group S_n. The corresponding probability distribution on S_n is

$$p(\sigma) = \begin{cases} \frac{2}{n(n-1)} & \sigma \text{ is a transposition,} \\ 0 & \text{otherwise.} \end{cases}$$

Given such a distribution p, how can we compute the probabilities involving more than one step of the random walk? For simplicity, assume the walk starts at the identity $e \in G$. In this case, the probability of being at a after one step is simply $p(a)$. To compute probability of ending up at a after two steps, we must sum the probabilities of all possible "paths" from e to a. That is, we multiply the probability of going from e to some b (i.e. $p(b)$) with the probability of going from b to a (i.e. $p(b^{-1}a)$), and sum this over all $b \in G$. This gives the expression

$$\sum_{b \in G} p(b)p(b^{-1}a).$$

More generally, if after some number of steps the position is described by a probability distribution q (meaning $q(b)$ is the probability of being at b), then after one additional step, the probability of being at a is given by the expression

$$\sum_{b \in G} q(b)p(b^{-1}a).$$

We have stumbled upon the following definition.

Definition 1.1. Let G be a finite group. Given two functions $f, g\colon G \to \mathbb{C}$, their *convolution* is the function $f * g\colon G \to \mathbb{C}$ given by

$$(f * g)(a) = \sum_{b \in G} f(b) g(b^{-1}a).$$

Note that convolution is an associative operation, but it is not always commutative. It will also help to give a name to the space in which we're working.

Definition 1.2. The *function algebra* of a finite group G is the set $L(G) = \{\text{functions } f\colon G \to \mathbb{C}\}$ endowed with the operations of pointwise addition and convolution.

With this new terminology, we can say that if the state of the random walk is described by a distribution q, then after a step, the state is described by the distribution $p * q$. Inductively, we can see that if the walk starts at the identity, its distribution after t steps is attained by convolving p with itself t times:

$$p^{*t} := \underbrace{p * p * \cdots * p}_{t}.$$

When we considered the random walk given by cyclically shuffling a deck of cards, we were able to reason about the behavior of p^{*t} using tools from linear algebra. In the following chapter, we will make those methods concrete and generalize them to all finite groups. This collection of methods comes from *representation theory*, a field of math aimed at using linear algebra to analyze algebraic objects like groups. This will give us the tools we need to tackle the more difficult problem about card shuffling. Specifically, our goal will be to compute second-order and higher information about permutation statistics sampled via random walks on S_n.

1.3 Overview

In chapter 2, we introduce relevant background about representation theory, discrete Fourier analysis, and the symmetric group. In chapter 3, we make our first attempt to compute second-order information about random walks by extending methods used by Axel Hultman in 2014. This approach yields some interesting results, but they are difficult to apply in practice and are only applicable to a narrow range of permutation statistics. In chapter 4, we introduce partition algebras, a family of associative algebras whose representation theory is closely intertwined with the symmetric group's. Next, in chapter 5, we use the theory of the partition algebra to compute variances of permutation statistics that we wouldn't have been able to compute with the methods of chapter 3. Finally, in chapter 6 we discuss some questions that remain open and offer directions for future work.

To the reader, I would like to emphasize that the journey is more fun than the destination. I concede that I am not especially invested in modeling the mechanics of shuffling a deck of cards. However, our quest to compute higher-order information about card shuffling will take us on a tour through a wide range of topics in representation theory and algebraic combinatorics, which is — in my opinion — where the real fun lies.

Chapter 2

Background

2.1 Representations of finite groups

Representation theory is a broad discipline of mathematics aimed at understanding mathematical objects by understanding the ways they can act on other objects. In our case the objects in question will be finite groups, and those "other objects" will be finite-dimensional vector spaces. The natural way to act on a vector space is via a linear transformation, which may be described by a matrix. Consequently, some say that the representation theory of finite groups is the study of homomorphisms from finite groups to groups of matrices.

Representation theory is a vast domain of study that one could devote their entire career to. Rather than aiming to be comprehensive, I will attempt to provide a brief and friendly introduction to the core concepts necessary to understand the methods in the later sections of this document. As such, it won't be possible to include proofs of all the theorems mentioned. I urge the interested reader to see Sagan (2001) and James and Liebeck (2001) for accessible explications of many of these results.

Definition 2.1. Let G be a finite group. A *representation* of G is a pair (π, V) consisting of a finite-dimensional vector space V and a group homomorphism π from G to $GL(V)$, the group of invertible linear transformations from V to itself. V is called the *representation space*, and π is called the *representation map*. The dimension of V is called the *degree* of the representation.

All of our representation spaces will be assumed to be vector spaces over $\mathbb{C}$. We can think of a representation of a group G as a way for G to *act* *on* a vector space V. With this in mind, we often denote $\rho(g)v$ by $g \cdot v$ to emphasize that g is thought of as acting on vectors $v \in V$.

Example 2.1. Consider the cyclic group $C_4 = \langle x \mid x^4 = 1 \rangle$. Below is a representation of g with $V = \mathbb{C}^2$.

g	1	x	x^2	x^3
$\pi(g)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

For example, the action of x on the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is

$$x \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}.$$

Hence, x acts by rotating vectors 90 degrees counterclockwise around the origin. Similarly, x^2 rotates vectors 180 degrees, and x^3 rotates them 270 degrees.

Definition 2.2. Given a finite group G written multiplicatively, the *group algebra* of G, denoted $\mathbb{C}G$, is the set of formal linear combinations of elements of G with coefficients in $\mathbb{C}$ under the operations of addition and multiplication. If $a = \sum_{g \in G} a_g g$ and $b = \sum_{g \in G} b_g g$, then their sum and product are defined to be

$$a + b = \sum_{g \in G} (a_g + b_g) g$$

and

$$ab = \sum_{gh \in G} (a_g b_h) gh.$$

We have actually seen this algebra before. The function algebra $L(G)$ (as in definition 1.2) is actually isomorphic to $\mathbb{C}G$, since the identification

$$\sum_{g \in G} a_g g \longleftrightarrow \left\{ \begin{aligned} f : G &\to \mathbb{C} \\ g &\mapsto a_g \end{aligned} \right\}$$

preserves addition and turns multiplication into convolution. It's helpful to keep both pictures of the group algebra in mind and pass between them freely, since in some circumstances we'll want to be thinking about functions on the group, while in other circumstances it will be more natural to consider formal sums of group elements.

Although a representation map π is defined only on group elements $g \in G$, it can be extended linearly to all elements of the group algebra. If $a = \sum_{g \in g} a_g g \in \mathbb{C}G$, we define

$$\pi(a) := \sum_{g \in G} a_g \pi(g).$$

Analogously, for functions $f \in L(G)$ we define

$$\pi(f) := \sum_{g \in G} f(g)\pi(g).$$

In other words, we can extend π to obtain a map from $\mathbb{C}G$ to $\operatorname{End}_{\mathbb{C}} V$, the set of all $\mathbb{C}$-linear maps from V to itself. Hence, we can think of a representation as a vector space on which $\mathbb{C}G$ can act. In the hope of making this idea concrete, the following algebraic object will be helpful.

Definition 2.3. Let R be a ring with identity. An R-*module* is an abelian group $(M, +)$ together with an action of R on M, denoted $\cdot\colon R \times M \to M$, such that for all $r, s \in R$ and $m, n \in M$,

(1) $(rs) \cdot m = r \cdot (s \cdot m)$,

(2) $(r + s) \cdot m = r \cdot m + s \cdot m$,

(3) $r \cdot (m + n) = r \cdot m + r \cdot n$,

(4) $1 \cdot m = m$.

We can think of an R-module as a generalization a vector space; elements of M play the role of vectors and the action of R on M is analogous to scalar multiplication. The collection of scalars for a vector space is required to be a field, meaning scalar multiplication is commutative and every nonzero scalar has a multiplicative inverse. When we consider modules over a general ring R, both of those conditions are no longer required.

With this terminology, we can describe group representations more abstractly. As we noted above, a representation of G can be viewed as a vector space on which $\mathbb{C}G$ can act. In other words, a representation of G is precisely a $\mathbb{C}G$-module. The correspondence goes the other way as well; any

$\mathbb{C}G$-module can be thought of as a representation of G with representation map $\pi\colon G \to GL(V)$ defined by

$$\pi(g)v = g \cdot v$$

for all $g \in G$. This correspondence between G-representations and $\mathbb{C}G$-modules is one of the central ideas of representation theory.

Definition 2.4. Two representations (π, V) and (ρ, W) are said to be *equivalent* if there exists an invertible linear transformation $T\colon V \to W$ such that

$$\pi(g) = T^{-1}\rho(g)T$$

for all $g \in G$.

Note that since $T\colon V \to W$ is invertible, V and W must be isomorphic vector spaces. So, an equivalent definition is that (π, V) and (ρ, V) are equivalent representations if and only if there's some change of basis that transforms $\pi(g)$ into $\rho(g)$ for all g.

Example 2.2. Consider the representation ρ of C_4 given by

g	1	x	x^2	x^3
$\rho(g)$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}-i & 0\\0 & i\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}i & 0\\0 & -i\end{bmatrix}$

We claim that ρ is equivalent to the representation π from example 2.1. Indeed, if we consider the matrix $T = \begin{bmatrix}1 & 1\\i & -i\end{bmatrix}$, we see that $\rho(g) = T^{-1}\pi(g)T$ for all $g \in C_4$.

Given two representations (π, V) and (ρ, W) of G, we can construct a larger representation called their *direct sum*, which has representation space $V \oplus W$ and representation map

$$(\pi \oplus \rho)(g) = \left[\begin{array}{c|c}\pi(g) & 0\\\hline 0 & \rho(g)\end{array}\right].$$

If a representation is equivalent to a a direct sum of lower-degree representations, we call it *decomposable* or *reducible*.[1] Otherwise, we say the

[1]These terms are actually subtly different, but for our objects of study (i.e., finite-dimensional complex representations of finite groups) they happen to coincide exactly, so we will use them interchangeably here.

representation is *indecomposable* or *irreducible*. For example, the representation ρ of example 2.2 is reducible since it is a direct sum of two degree 1 representations ρ_1 and ρ_2, given by

$$\begin{aligned}
&\rho_1(1) = 1, \quad \rho_1(x) = -i, \quad \rho_1(x^2) = -1, \quad \rho_1(x^3) = i,\\
&\rho_2(1) = 1, \quad \rho_2(x) = i, \quad \rho_2(x^2) = -1, \quad \rho_2(x^3) = -i.
\end{aligned}$$

Consequently, π of example 2.1 is reducible, since it is equivalent to ρ.

A central result of representation theory states that every finite group G has finitely many inequivalent irreducible representations. The irreducible representations of G are the building blocks of all representations, since any representation can be decomposed into a direct sum of irreducibles. Because of this, in order to understand all representations of G, it suffices to understand just the finite collection of irreducible ones.

We conclude this section by introducing R-algebras (often called *associative algebras*), which can be thought of as R-modules in which elements can be multiplied.

Definition 2.5. Let R be a commutative ring with unity. An *R-algebra* is an R-module A with an additional binary operation of multiplication such that for all $a, b \in A$ and $r \in R$,

$$r \cdot (ab) = (r \cdot a)b = a(r \cdot b).$$

Equivalently, we can think of an R-algebra as an object that simultaneously has the structure of a ring and an R-module. All algebras we will encounter will have base ring $R = \mathbb{C}$. We have seen one notable example so far: the group algebra $\mathbb{C}G$ is a $\mathbb{C}$-algebra, since elements can be added and multiplied, and the action of $\mathbb{C}$ given by scaling satisfies the condition above. Another notable example of a $\mathbb{C}$-algebra is $\operatorname{End}_{\mathbb{C}} V$, the set of $\mathbb{C}$-linear maps from V to itself, called the *endomorphism algebra* of V. Given $f, g \in \operatorname{End}_{\mathbb{C}}(V)$ and $r \in \mathbb{C}$, the endomorphisms $r \cdot f$, $f + g$, and fg are defined by

$$(r \cdot f)(v) = rf(v), \quad (f + g)(v) = f(v) + g(v), \quad \text{and} \quad (fg)(v) = f(g(v)).$$

We can extend our notion of a representation to all complex algebras.

Definition 2.6. Given a $\mathbb{C}$-algebra A, a *representation of A* is a pair (π, V) where V is a complex vector space and $\pi\colon A \to \operatorname{End}_{\mathbb{C}}(V)$ is a $\mathbb{C}$-algebra homomorphism (i.e., a $\mathbb{C}$-linear ring homomorphism).

This definition is chosen so that if $A = \mathbb{C}G$ for a group G, then representations of A are just representations of G in the original sense. Alternatively, recalling that a representation of G can be thought of as a module over $\mathbb{C}G$, we can think of any module over A as a representation of A. In chapters 4 and 5, we will dive deep into the representation theory certain algebras called *partition algebras*, where this more general notion of representations will be necessary.

2.2 Fourier analysis on finite groups

A famous theorem of Wedderburn asserts that for any finite group G, the function algebra $L(G)$ is isomorphic to a direct sum of matrix rings:

$$L(G) \cong \bigoplus_i M_{d_i}(\mathbb{C}).$$

Moreover, if $\{\pi_1, \ldots, \pi_k\}$ is a complete set of inequivalent irreducible representations of G, then the map $F\colon L(G) \to \bigoplus_i M_{d_i}(\mathbb{C})$ given by

$$F(f) = \bigoplus_{i=1}^{k} \pi_i(f)$$

is an isomorphism. We call this map the *Fourier transform* on G.

For abelian groups, all irreducible representations are degree 1. Hence, the Fourier transform on an abelian group maps every element of the group algebra to a diagonal matrix. This is precisely what we saw in section 1.1 when we diagonalized our transition matrix. We were able to convert our probability distribution $p \in L(C_n)$ into a diagonal matrix $[A]_{\mathcal{B}}$, and then used $[A]_{\mathcal{B}}^t$ to compute p^{*t}. Using the group Fourier transform, we can do a similar procedure on S_n to tackle the first problem we presented, although the resulting matrix is not guaranteed to be diagonal, so understanding its powers may be more difficult.

Two elements g and g' of a group G are said to be *conjugate* if there exists an element $h \in G$ such that $g' = hgh^{-1}$. The set of all elements of G conjugate to some element g (including g itself) is called the *conjugacy class* of g, and is denoted C_g. A function $f \in L(G)$ is said to be a *class function* if it is constant on conjugacy classes, meaning that if g and g' are conjugate then $f(g) = f(g')$.

In many circumstances, it will be convenient to convert a statistic (i.e., a function from G to $\mathbb{C}$) into a class function.

Definition 2.7. Let $s\colon G \to \mathbb{C}$ be a statistic. The associated *mean statistic* is the function $\overline{s}\colon G \to \mathbb{C}$ given by

$$\overline{s}(g) = \frac{1}{|G|}\sum_{h\in G} s(hgh^{-1}),$$

where C_g is the conjugacy class containing g.

By construction, the mean statistic $\overline{s}$ is a class function for any statistic s. It can be viewed as the best class function approximation to a statistic. As we will see in chapter 3, $\overline{s}$ often encodes all of the information we would want to know about s.

Definition 2.8. Let (π, V) be a representation of V. The *character* of this representation is the map $\chi\colon G \to \mathbb{C}$ given by

$$\chi(g) = \operatorname{tr}(\pi(g)).$$

To emphasize the representation a character comes from, we often denote the character of (π, V), by χ_π or χ_V. A character χ is said to be irreducible if its corresponding representation is irreducible.

Characters are incredibly important objects in representation theory, because in some sense they encode everything one would want to know about a representation. Firstly, equivalent representations have equal characters, since if $\rho(g) = T^{-1}\pi(g)T$, then

$$\chi_\rho(g) = \operatorname{tr}(\rho(g)) = \operatorname{tr}(T^{-1}\pi(g)T) = \operatorname{tr}(TT^{-1}\pi(g)) = \operatorname{tr}(\pi(g)) = \chi_\pi(g)$$

where we have used the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ with $A = T^{-1}\pi(g)$ and $B = T$. In fact, two representations are equivalent *if and only if* their corresponding characters are equal (see 1.9.4 in Sagan (2001)).

A similar computation shows that characters are always class functions. If $g' = hgh^{-1}$, then

$$\begin{aligned}\chi_\pi(g') &= \operatorname{tr}(\pi(g')) = \operatorname{tr}(\pi(h)\pi(g)\pi(h^{-1}))\\ &= \operatorname{tr}(\pi(h^{-1})\pi(h)\pi(g)) = \operatorname{tr}(\pi(g)) = \chi_\pi(g).\end{aligned}$$

In fact, the set of irreducible characters forms a basis for the set of class functions on G. Moreover, with respect to the inner product on $L(G)$ given be

$$\langle a, b\rangle = \frac{1}{|G|}\sum_{g\in G} a(g)b(g)^*,$$

the irreducible characters form an orthonormal set (where * denotes complex conjugation). Consequently, any class function a can be written in the form

$$a = \sum_{\chi} \langle a, \chi \rangle \chi,$$

where the sum ranges over all irreducible characters χ of G.

2.3 The symmetric groups

For each positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Definition 2.9. Let n be a positive integer. The *symmetric group on n letters*, denoted S_n, is the group of all bijective functions from $[n]$ to itself under function composition.

An element of the symmetric group is called a *permutation*. A natural way to write a permutation $\sigma \in S_n$ is by listing where it sends each element of $[n]$, as shown below.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{bmatrix}$$

This is called *two-line notation*. In practice, the first line is often omitted, since it contains no information about σ. In this context it's customary to write σ_n to denote $\sigma(n)$. Hence we denote the permutation σ by the sequence $\sigma_1\sigma_2\sigma_3 \cdots \sigma_n$. This is called *one-line notation*.

Given some subset $\{i_1, \ldots, i_k\}$ of $[n]$, the permutation that maps i_j to i_{j+1} for all $j < k$, maps i_k to i_1, and fixes all other elements of $[n]$ is called *cyclic*, and is denoted

$$(i_1\, i_2\, i_3 \cdots i_k).$$

A cyclic permutation of k letters is called a *k-cycle*. Every permutation may be written as a product of disjoint cycles. For example, the permutation $\sigma = 3472561 \in S_7$ may be written as

$$\sigma = (1\,3\,7)(2\,4)(5)(6).$$

Often we omit 1-cycles, with the convention that numbers that don't appear are fixed. Hence we could write $\sigma = (1\,3\,7)(2\,4)$. This is called the *cycle decomposition* of σ, and is the primary way we will represent permutations.

Note that a single permutation may be written as a product of disjoint cycles in multiple different ways. For example, the permutation σ may be written in any of the equivalent ways below.

$$\sigma = (1\,3\,7)(2\,4) = (2\,4)(1\,3\,7) = (4\,2)(3\,7\,1).$$

In particular, reordering the cycles or cyclically permuting the numbers within a cycle does not change the permutation. However, every disjoint cycle decomposition of σ has the same number of 1-cycles, the same number of 2-cycles and so on. The list of the sizes of the cycles in σ is called the *cycle structure*. For example, the cycle structure of σ is $(3, 2, 1, 1)$, since σ has one 3-cycle, one 2-cycle, and two 1-cycles. Importantly, two permutations in S_n are conjugate to each other if and only if they have the same cycle structure (see Sagan (2001)).

As we noted above, the order of the cycles in σ doesn't matter, so it's customary to write the list of sizes in decreasing order when writing σ's cycle structure. Note that since σ is a permutation of size n, the sizes of all of the cycles in σ must sum to n.

Definition 2.10. An *integer partition* λ is a non-increasing finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$. If $\lambda_1 + \cdots + \lambda_l = n$, we say λ is a *partition of* n, and write $\lambda \vdash n$ or $|\lambda| = n$.

By our observations above, the cycle structure of a permutation in S_n is a partition of n, and the conjugacy classes of σ are indexed by the partitions of n. We write C_λ for the conjugacy class of permutations with cycle structure λ.

For example, the table below lists the partitions of 5, the corresponding conjugacy class, and gives an example element from that conjugacy class.

Cycle structure	Cycle sizes	Example element from C_λ
(5)	$(*\,*\,*\,*\,*)$	(1 2 3 4 5)
(4,1)	$(*\,*\,*\,*)$	(1 2 3 4)
(3,2)	$(*\,*\,*)(*\,*)$	(1 2 3)(4 5)
(3,1,1)	$(*\,*\,*)$	(1 2 3)
(2,2,1)	$(*\,*)(*\,*)$	(1 2)(3 4)
(2,1,1,1)	$(*\,*)$	(1 2)
(1,1,1,1,1)	$(*)$	(1)

To write partitions more compactly, we will often write a^k to indicate that the entry a occurs k times. For example, we can write the partition $(3, 2, 2, 2, 1, 1, 1, 1)$ more compactly as $(3, 2^3, 1^4)$.

2.3.1 Permutation statistics

A permutation statistic is a function from the group S_n to $\mathbb{C}$ that encodes something meaningful about permutations. Below we give names to some characteristics of permutations of interest.

Definition 2.11. Let $\sigma \in S_n$ be a permutation.

- A *fixed point* of σ is an integer that $i \in [n]$ that is mapped to itself by σ, meaning $\sigma(i) = i$.
- An *exceedance* is an integer i such that $\sigma(i) > i$.
- An *inversion* is a pair $i < j$ such that $\sigma(i) > \sigma(j)$.
- A *descent* is a pair $\{i, i+1\}$ such that $\sigma(i) > \sigma(i+1)$.

For each of these terms, we can define a permutation statistic counting its occurrences.

$$\begin{aligned} \textsc{Fix}(\sigma) &:= \text{ number of fixed points in } \sigma, \\ \textsc{Exc}(\sigma) &:= \text{ number of exceedances in } \sigma, \\ \textsc{Inv}(\sigma) &:= \text{ number of inversions in } \sigma, \\ \textsc{Des}(\sigma) &:= \text{ number of descents in } \sigma. \end{aligned}$$

We can also define

$$\textsc{Cyc}_k(\sigma) := \text{ number of } k\text{-cycles in cycle decomposition of } \sigma.$$

The following class of permutation statistics generalizes many well-studied statistics.

Definition 2.12 (see Gill (2013))**.** Let $\sigma \in S_n$ be a permutation. A collection of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is said to be an occurrence of a (classical) pattern $\phi \in S_k$ if the sequence $(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))$ has the same relative order as the sequence $(\phi(1), \phi(2), \ldots, \phi(k))$, by which we mean for any pair j and l, $\sigma(i_j) < \sigma(i_l)$ if and only if $\phi(j) < \phi(l)$. We call k the *length* of the pattern.

For example, in the permutation $\sigma = 3\underline{4}1\underline{26}5$, the collection of underlined indices is an occurrence of the classical pattern (2-1-3). Many people are concerned with the problem of finding and enumerating permutations that *avoid* certain patterns, meaning they have no occurrences of that pattern (see, for example, Currie (2005)).

Definition 2.13 (see Gill (2013))**.** A *vincular* pattern v of length k is a permutation $\phi \in S_k$ written in one-line notation, enclosed in either brackets or parentheses, possibly with dashes between adjacent entries. An occurrence of v in a permutation $\sigma \in S_n$ is an occurrence of the classical pattern ϕ that satisfies the following additional conditions:

- If v begins with a bracket, i_1 must be 1,
- If v ends with a bracket, i_k must be n, and
- If two entries of v are not separated by a dash, the corresponding indices must be adjacent.

For example, if we consider $\sigma = 415362$ and $v = [2\text{-}31)$, the entries $\underline{4}1\underline{53}62$ are an occurrence of v. The entries $41\underline{5}3\underline{62}$ are not an occurrence of v despite the fact that they are in the correct relative order, since the opening bracket indicates that the occurrence must start from the first entry of σ. Similarly, the entries $\underline{4}1\underline{5}36\underline{2}$ are not an occurrence of v, since the absence of a dash between the last two entries of v indicates that the last two entries of the occurrence must be next to each other in σ.

Given a classical or vincular pattern v we can define a statistic counting the number of occurrences of v in a permutation:

$$\textsc{Pat}_v(\sigma) := \text{ number of occurrences of } v \text{ in } \sigma.$$

With this notation, it's clear that $\textsc{Inv} = \textsc{Pat}_{(2-1)}$, and $\textsc{Des} = \textsc{Pat}_{(21)}$.

When doing representation theory on the symmetric group, it is desirable to have some notion of the frequency of a function. low-frequency functions should not be drastically affected by "small" changes to the input, in some reasonable sense. One way to capture this intuition is given in the following definition due to Hamaker and Rhoades (2022).

Definition 2.14. Let $i, j \in [n]$, and consider the function $\mathbf{1}_{i \to j}$ on S_n by given by

$$\mathbf{1}_{i \to j}(\sigma) = \begin{cases} 1 & \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

We call a function of this form a *1-indicator*, and we say a statistic is 1-*local* if it can be written as a linear combination of 1-indicators.

Intuitively, a statistic is 1-local if its value only depends on what the permutation does to individual letters, as opposed to pairs or larger groups of letters. For example, the statistics Fix and Exc are 1-local, since they only depend on what the permutation does to individual letters.

$$\textsc{Fix} = \sum_{i=1}^{n} \mathbf{1}_{i\to i}, \qquad \text{and} \qquad \textsc{Exc} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{1}_{i\to j}.$$

Extending this definition, we can define analogs of higher-frequency functions.

Definition 2.15. Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ be two lists of indices. We call function $\mathbf{1}_{I\to J}$ given by

$$\mathbf{1}_{I\to J}(\sigma) = \begin{cases} 1 & \sigma(i_l) = j_l, \text{ for } l = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

a *k-indicator*. A statistic is said to be *k-local* if it may be written as a linear combination of k-indicators.

Extending our intuition for 1-local functions, k-local statistics are the ones that only depend on what a permutation does to k-tuples of letters, but not larger collections. For example, for each k the statistic $\textsc{Cyc}_k$ is k-local, since

$$\textsc{Cyc}_k = \frac{1}{k} \sum_{(i_1,\dots,i_k)} \mathbf{1}_{(i_1,i_2,\dots,i_k)\to(i_2,i_3\dots,i_k,i_1)},$$

where the sum ranges over all k-tuples $(i_1, \dots, i_k)$ of distinct integers from 1 to n. The statistics Inv and Des are each 2-local, since

$$\textsc{Inv} = \sum_{i_1<i_2} \sum_{j_1>j_2} \mathbf{1}_{(i_1,i_2)\to(j_1,j_2)}, \qquad \text{and} \qquad \textsc{Des} = \sum_{i=1}^{n-1} \sum_{j_1>j_2} \mathbf{1}_{(i,i+1)\to(j_1,j_2)}.$$

These results can be viewed as consequences of the following fact.

Proposition 2.16. *If v is a vincular pattern of length k, $\textsc{Pat}_v$ is k-local.*

Proof. Given a vincular pattern v of length k, let ϕ_v denote the corresponding permutation in S_k. Let CI_v denote the set of lists of indices $i_1 < i_2 < \cdots < i_k$ that satisfy the vincular conditions associated with v; that is,

$$CI_v = \left\{ (i_1, \ldots, i_k) \in [n]^k : \begin{array}{c} i_1<i_2<\cdots<i_k, \\ i_1=1 \text{ if } v \text{ starts with a bracket,} \\ i_k=n \text{ if } v \text{ ends with a bracket,} \\ i_{l+1}=i_l+1 \text{ if } l^{\text{th}} \text{ and } (l+1)^{\text{st}} \text{ entries of } v \text{ aren't separated by a dash.} \end{array} \right\}.$$

Also let O_v denote the set of tuples of distinct indices $(j_1, \ldots, j_k)$ that are in the same relative order as ϕ_v:

$$O_v = \left\{ (j_1, \ldots, j_k) \in [n]^k : j_l < j_m \text{ if and only if } \phi_v(l) < \phi_v(m) \right\}.$$

Then

$$\textsc{Pat}_v = \sum_{\substack{(i_1,\ldots,i_k)\in CI_v \\ (j_1,\ldots,j_k\in O_v}} \mathbf{1}_{(i_1,\ldots,i_k)\to(j_1,\ldots,j_k)},$$

so $\textsc{Pat}_v$ is k-local. □

Note that this proposition also applies to classical patterns, since a classical pattern can be viewed as a vincular pattern which starts and ends with parentheses and each of whose pairs of adjacent entries is separated by a dash.

The following proposition characterizes the way pointwise products interact with k-locality.

Proposition 2.17 (Prop. 4.3 in Hamaker and Rhoades (2022))**.** *If $s\colon S_n \to \mathbb{C}$ is k-local and $s'\colon S_n \to \mathbb{C}$ is k'-local, their pointwise product ss' is $(k + k')$-local.*

Proof. Let $I, J \in [n]^k$ and $I', J' \in [n]^{k'}$. Notice that the pointwise product of the k-indicator $\mathbf{1}_{I\to J}$ and the k'-indicator $\mathbf{1}_{I\to J}$ is given by

$$\mathbf{1}_{I\to J}\mathbf{1}_{I'\to J'} = \mathbf{1}_{(I+\!\!+I')\to(J+\!\!+J')}.$$

where $+\!\!+$ denotes concatenation of tuples: $I +\!\!+ I' = (i_1, i_2, \ldots, i_k, i'_1, i'_2, \ldots, i'_{k'})$. Hence the pointwise product of a k-indicator and a k'-indicator is a $(k + k')$-indicator. Using this fact, if we have two statistics

$$s = \sum_{I\in[n]^k} c_{I,J}\mathbf{1}_{I\to J} \qquad \text{and} \qquad s = \sum_{I\in[n]^{k'}} c'_{I',J'}\mathbf{1}_{I'\to J'},$$

then their pointwise product is

$$ss' = \sum_{I\in[n]^k} \sum_{I'\in[n]^{k'}} c_{I,J}c'_{I',J'}\mathbf{1}_{(I+\!\!+I')\to(J+\!\!+J')}.$$

Hence ss' is a linear combination of $(k + k')$-indicators, so is $(k + k')$-local. □

If $s\colon S_n \to \mathbb{C}$ is a statistic, the d^{th} moment of s is the statistic $s^d\colon S_n \to \mathbb{C}$ given by

$$s^d(\sigma) := s(\sigma)^d.$$

In other words, s^d can be obtained by taking the pointwise product of s with itself d times. The previous proposition now implies the following result.

Corollary 2.18. *If $s\colon S_n \to \mathbb{C}$ is k-local and $d \in \mathbb{N}$, then $s^d\colon S_n \to \mathbb{C}$ is dk-local.*

The mean statistics of many permutation statistics of interest are surprisingly well behaved.

Theorem 2.19 (Rodrigues (1839))**.** *Let $\sigma \in S_n$ be a permutation, and let m_i be the number of i-cycles in σ. Then*

$$\overline{\textsc{Inv}}(\sigma) = \frac{3n^2 - n - m_1^2 - 2m_1 n + m_1 + 2m_2}{12}.$$

Note that since $\overline{\textsc{Inv}}$ is a class function, it must only depend on the cycle structure of σ. However, it's surprising that $\overline{\textsc{Inv}}$ depends only on the number of 1-cycles and 2-cycles in σ, and not the number of k-cycles for any $k > 2$. It's even more surprising that the value of $\overline{\textsc{Inv}}$ on all symmetric groups is described by a single quadratic polynomial in the variables n, a_1 and a_2. Similar phenomena have been observed for many other statistics (see Levet et al. (2023)). For example, the following theorem gives a similar characterization of $\textsc{Des}$.

Theorem 2.20 (Fulman (1998))**.** *Let $\sigma \in S_n$ be a permutation, and let m_i be the number of i-cycles in σ. Then*

$$\overline{\textsc{Des}}(\sigma) = \frac{n^2 - n + 2m_2 - m_1^2 + m_1}{2n}.$$

Gaetz and Ryba (2021) proved the following characterization of mean statistics of moments of pattern statistics.

Theorem 2.21 (Gaetz and Ryba (2021))**.** *Let $\phi \in S_k$ be a fixed permutation thought of as a classical k-pattern. For a permutation $\sigma \in S_n$, let m_i denote the number of i-cycles in σ. Then as a function on all symmetric groups at once, $\overline{\textsc{Pat}_\phi^d}$ is described by a single polynomial of degree at most dk in the variables $n, m_1, \ldots, m_{dk}$, where n has degree 1 and each m_i has degree i.*

Applied to $\overline{\textsc{Inv}} = \overline{\textsc{Pat}^1_{(2-1)}}$, this tells us that $\overline{\textsc{Inv}}(\sigma)$ is a polynomial of degree at most 2 in n, m_1, and m_2, where n and m_1 are considered to be degree 1 and m_2 is considered to be of degree 2. As a consequence of this degree condition, terms like m_2^2 and nm_2 can't appear, since they would be of degree 4 and 3, respectively. One can check that the polynomial in theorem 2.19 satisfies this condition.

This result says that many of the surprising properties we noticed about $\overline{\text{Inv}}$ actually hold for all moments of classical pattern statistics. In chapter 4, we harness the methods used in the proof of the above theorem to compute an explicit polynomial for $\overline{\text{Inv}^2}$. This yields the following new result.

Proposition 2.22 (S.). *Let $\sigma \in S_n$ be a permutation, and let m_i denote the number of i-cycles in σ. Then*

$$\begin{aligned}\overline{\text{Inv}^2}(\sigma) = \frac{1}{720}(&5m_1^4 + 20m_1^3 n - 14m_1^3 - 12m_1^2 m_2 + 50m_1^2 n^2 - 90m_1^2 n \\ &- 25m_1^2 - 24m_1 m_2 n + 12m_1 m_2 - 24m_1 m_3 + 60m_1 n^3 - 126m_1 n^2 \\ &+ 94m_1 n + 98m_1 + 60m_2^2 - 20m_2 n^2 + 108m_2 n - 124m_2 \\ &- 24m_3 n - 48m_3 - 24m_4 + 45n^4 - 130n^3 + 111n^2 - 98n).\end{aligned}$$

In chapter 5, we use this result to analyze random walks on S_n.

Note that theorem 2.21 only applies to classical patterns, not necessarily all vincular patterns. Indeed, $\text{Des} = \text{Pat}_{(21)}$ is a non-classical pattern statistic, and theorem 2.20 shows that $\overline{\text{Des}}(\sigma)$ is not given by a polynomial but instead a rational function. It would be interesting to attempt to find a generalization of the theorem to all vincular patterns. In chapter 6, we discuss a potential avenue for of finding such a generalization.

2.3.2 Representation theory of S_n

Irreducible representations of the symmetric group are called *Specht modules* and are indexed by integer partitions λ of n. We denote the Specht module corresponding to λ by S^λ and the corresponding character by χ^λ. There exist several equivalent but distinct choices of representation map for S^λ. For this thesis, we will use Young's orthogonal form, which we will denote by ρ^λ. The details of this map are not important for our purposes, but Sagan (2001) gives a nice description of this representation.

Let f_λ denote the degree of ρ^λ. With this notation, the Fourier transform on S_n is given by

$$F\colon L(S_n) \to \bigoplus_{\lambda \vdash n} M_{f_\lambda}(\mathbb{C})$$
$$a \mapsto \bigoplus_{\lambda \vdash n} \rho^\lambda(a).$$

In appendix A, we list Fourier transforms of many of the permutation statistics we've seen so far.

A key reason we care about k-local functions is that their Fourier transforms are what we call *band-limited*, meaning they are 0 for all except the lowest frequency data.

Theorem 2.23 (Hamaker and Rhoades (2022))**.** *A statistic $s \in L(S_n)$ is k-local if and only if $\rho^\lambda(s) = 0$ for every partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ with $\lambda_1 < n - k$.*

This implies that 1-local functions can only take on nonzero values on 2 irreducible representations; $\rho^{(n)}$ and $\rho^{(n-1,1)}$. Similarly, 2-local functions can only attain nonzero values for 4 irreducible representations, and 3-local functions can only attain nonzero values for 7. This justifies our idea that k-local functions are "low-frequency." In classical Fourier analysis, a low-frequency signal will have all Fourier coefficients equal to 0 past a certain threshold, and the lower the frequency, the lower that threshold will be. In light of theorem 2.23, k-local functions exhibit this behavior with respect to the Fourier transform on S_n.

Hultman (2014) presents a method for computing expected values of statistics sampled via random walks on S_n by decomposing the statistic into irreducible characters. If that statistic is k-local for k small, this decomposition will only involve a handful of characters, no matter how big n is. Ultimately, this is the property that makes this approach feasible.

2.4 Expected values of statistics

In his 2014 paper, Hultman gave a method for computing the expected value of a statistic $s\colon S_n \to \mathbb{C}$ after t steps of a random walk governed by a probability distribution $p\colon S_n \to [0, 1]$. Here, we describe these methods in a way that generalizes to all finite groups G. The proof we present of Hultman's main theorem (theorem 2.26) differs from the one in the original paper substantially; this proof is considerably longer, but in my view it sheds more light on the underlying representation theory than the original proof.

Recall that in section 2.2, we introduced the inner product on $L(G)$ given by

$$\langle a, b\rangle := \frac{1}{|G|}\sum_{g\in G} a(g)b(g)^*$$

where * denotes complex conjugation. If a and b happen to be class functions, the expression above can be simplified to

$$\langle a, b\rangle := \frac{1}{|G|}\sum_{i=1}^{r} |C_{g_i}| a(g_i)b(g_i)^*.$$

where $\{g_1, \dots, g_r\}$ is a complete set of representatives for the conjugacy classes of G. As discussed in section section 2.2, with respect to this inner product the set of irreducible characters is an orthonormal basis for the set of class functions on G.

Hultman's proof relies on the following rather simple observation.

Proposition 2.24. *Let $a, b \in L(G)$. If either a or b is a class function, then*

$$\langle a, b\rangle = \langle \overline{a}, \overline{b}\rangle.$$

Proof. Assume without loss of generality that a is a class function, meaning $a = \overline{a}$. Let $\{g_1, \dots, g_k\}$ be a complete set of representatives for the conjugacy classes of G. Then

$$\begin{aligned}
\langle a, b\rangle &= \langle \overline{a}, b\rangle \\
&= \frac{1}{|G|}\sum_{g\in G} \overline{a}(g)b(g)^* \\
&= \frac{1}{|G|}\sum_{i=1}^{k}\sum_{g\in C_{g_i}} \overline{a}(g)b(g)^* \\
&= \frac{1}{|G|}\sum_{i=1}^{k} \overline{a}(g_i)\sum_{g\in C_{g_i}} b(g)^* \\
&= \frac{1}{|G|}\sum_{i=1}^{k} \overline{a}(g_i)\left(|C_{g_i}|\overline{b}(g_i)^*\right) \\
&= \frac{1}{|G|}\sum_{i=1}^{k} |C_{g_i}|\overline{a}(g_i)\overline{b}(g_i)^* \\
&= \langle \overline{a}, \overline{b}\rangle.
\end{aligned}$$

□

Suppose $s\colon G \to \mathbb{C}$ is a statistic and $p\colon G \to [0,1]$ is a probability distribution. As in chapter 1, we can use p to construct a random walk on G, where at each step, we move from state a to state ag with probability $p(g)$. If the random walk starts from the identity, then after t steps, the probability of being at b is given by

$$p^{*t}(b) := (\underbrace{p * p * \cdots * p}_{t})(b).$$

Let $\mathbb{E}_p(s,t)$ denote the expected value of s after t steps of this random walk. By definition,

$$\mathbb{E}_p(s,t) = \sum_{g\in G} p^{*t}(g)s(g) = |G| \left\langle s, p^{*t} \right\rangle .$$

If either p^{*t} or s happens to be a class function, then by proposition 2.24,

$$\mathbb{E}_p(s,t) = |G|\langle \overline{s}, \overline{p^{*t}} \rangle. \tag{2.1}$$

Although this statement holds if either p^{*t} or s is a class function, Hultman restricts his attention to the case in which p is a class function (and consequently p^{*t} is a class function for all t). In this case, equation (2.1) substantially simplifies computation of $\mathbb{E}_p(s,t)$, since it's an inner product of class functions and can therefore be computed in the character basis.

Let $\{\chi_1, \ldots, \chi_k\}$ be the set of all irreducible characters of G. As mentioned earlier, this set forms a basis for the space of class functions on G. Hence, if p is a class function, there exist coefficients b_j such that

$$p = \sum_{j=1}^{k} b_j \chi_j.$$

Note that irreducible characters obey a very nice convolution formula.

Proposition 2.25.

$$\chi_j * \chi_l = \begin{cases} \frac{|G|}{f_j}\chi_j & l = j \\ 0 & l \neq j \end{cases}$$

where $f_j = \chi_j(e) = \deg \chi_j$.

This proposition yields the following expression for the t^{th} convolution power of p:

$$p^{*t} = \sum_{j=1}^{k} \left(\frac{|G|}{f_j}\right)^{t-1} b_j^t \chi_j.$$

So, decomposing p into characters immediately yields a decomposition of p^{*t} into characters.

For any statistic s on G, $\overline{s}$ is a class function, so there exist coefficients a_j such that

$$\overline{s} = \sum_{j=1}^{k} a_j \chi_j.$$

Since we have decomposed both $\overline{s}$ and p^{*t} into irreducible characters, we can compute their inner product in the character basis. We have

$$\mathbb{E}_p(s,t) = |G|\langle \overline{s}, p^{*t}\rangle = \sum_{j=1}^{k} \frac{|G|^t}{f_j^{t-1}} a_j (b_j^*)^t.$$

We have proven the following result.

Theorem 2.26. *Suppose $p\colon G \to [0,1]$ is a probability distribution and a class function, and $s\colon S_n \to \mathbb{C}$ is any statistic. If p and $\overline{s}$ have character decompositions*

$$\overline{s} = \sum_{j=1}^{k} a_j \chi_j \qquad \text{and} \qquad p = \sum_{j=1}^{k} b_j \chi_j,$$

then

$$\mathbb{E}_p(s,t) = |G|^t \sum_{j=1}^{k} \frac{a_j (b_j^*)^t}{f_j^{t-1}}$$

where f_j is the degree of χ_j.

This is a generalization of theorem 3.2 in Hultman (2014) to all finite groups. In light of this theorem, if p is a class function, then computing the expected value of some statistic s after t steps of the random walk governed by p comes down to decomposing p and $\overline{s}$ into characters. This sounds quite trivial, since we could just compute that decomposition by taking the inner products $\langle p, \chi_j \rangle$ and $\langle \overline{s}, \chi_j \rangle$ for each j. However, for the problems we care about this is often infeasible. For example, when studying random walks on S_n, we are looking for statements that hold for all n. It's not clear how to compute such an inner product on all symmetric groups simultaneously, so we need often need to find other ways to decompose these functions. Hultman (2014) and Gill (2013) do so for various permutation statistics using clever combinatorial arguments.

In the next chapter, we examine the problem of computing *variances* of permutation statistics on permutations sampled via random walks. As we will see, the methods we used to arrive at theorem 2.26 generalize to this problem somewhat, but we also run into some complications because of the complexity of the representation theory of S_n.

Chapter 3

Variances of permutation statistics, part I

At the end of the previous chapter, we extended the methods of Hultman (2014) to obtain a nice method for computing the expected value of a statistics after t steps of a random walk on a group G. It's natural to ask whether we can use similar techniques to find analogous formulas for variance, and potentially higher moments of s. In section 3.1, we briefly shift our focus toward cyclic groups instead of the symmetric group, which lets us build intuition for how we can approach the problem on S_n. In section 3.2 we attempt to lift the methods to S_n, and point out the limitations of this approach along the way. We end with a formula for the variance in the number of fixed points of a permutation sampled from a random walk.

3.1 Variances on cyclic groups

In section 1.1, we noted that the cyclic shuffle can be considered random walk on the group $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n under addition. It will be notationally convenient to write the groups we're working with multiplicatively, so we will instead consider the cyclic group C_n defined by

$$C_n \coloneqq \langle x \mid x^n = 1 \rangle .$$

The groups C_n and $\mathbb{Z}/n\mathbb{Z}$ are isomorphic, which we can see by identifying x^k in C_n with k in $\mathbb{Z}/n\mathbb{Z}$. So, analyzing variances of statistics on C_n will shed light on the cyclic shuffle.

C_n has n irreducible characters, which we will call χ_k for $k = 0, 1, \ldots, n-1$. They are given by

$$\chi_k(x^j) = \omega^{jk}$$

where $\omega = e^{2\pi i/n}$. Proposition 2.25 implies

$$\chi_j * \chi_k = \begin{cases} n\chi_j & j = k \\ 0 & \text{otherwise.} \end{cases}$$

The pointwise product of irreducible characters χ_j and χ_k is

$$\chi_j \chi_k = \chi_{j+k}.$$

This fact will prove crucial later. Recall that the characters are orthonormal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{n} \sum_{j=0}^{n-1} f(x^j) g(x^j)^*.$$

Also note that since C_n is abelian, every element of $L(C_n)$ is a class function, meaning it is a linear combination of irreducible characters.

Let $p\colon C_n \to [0,1]$ be a probability distribution and $s\colon C_n \to \mathbb{C}$ be a statistic. Define coefficients

$$a_k = \langle s, \chi_k \rangle \qquad \text{and} \qquad b_k = \langle p, \chi_k \rangle$$

so that

$$s = \sum_{k=0}^{n-1} a_k \chi_k \qquad \text{and} \qquad p = \sum_{k=0}^{n-1} b_k \chi_k,$$

as in theorem 2.26. Then, by our work in the previous section we know

$$p^{*t} = n^{t-1} \sum_{k=0}^{n-1} a_k^t \chi_k.$$

And, by theorem 2.26 we have

$$\mathbb{E}_p(s,t) = n^t \sum_{k=0}^{n-1} a_k (b_k^*)^t.$$

Let $\mathrm{Var}_p(s,t)$ denote the variance in s after t steps of the random walk governed by the probability distribution p. We will compute $\mathrm{Var}_p(s,t)$ using the formula

$$\mathrm{Var}_p(s,t) = \mathbb{E}_p(s^2,t) - \mathbb{E}_p(s,t)^2.$$

We already know how to compute the second term. To compute the first, we will need to decompose s^2 into characters. Using the formula for pointwise products of characters, we obtain

$$\begin{aligned} s^2 &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} a_j a_k \chi_j \chi_k \\ &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} a_j a_k \chi_{j+k} \\ &= \sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j a_{k-j}\right)\chi_k. \end{aligned}$$

So, the coefficient on χ_k in s^2 is $\sum_{j=0}^{n-1} a_j a_{k-j}$. We can now invoke theorem 2.26, which yields

$$\mathbb{E}_p(s^2,t) = n^t \sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j a_{k-j}\right)(b_k^*)^t = n^t \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} a_j a_{k-j}(b_k^*)^t. \tag{3.1}$$

Note that the formula for the coefficient on χ_k in s^2 looks very similar to a convolution over the group C_n. The only difference is that in this expression, the terms being convolved are lists of character coefficients, rather than coefficients in the group algebra. There's an intuitive explanation for this fact. The Fourier transform turns convolution in the group ring into pointwise multiplication of Fourier coefficients. So, we might expect it to turn a pointwise product in the group ring into a convolution of Fourier coefficients.

Example 3.1. Suppose $n = 6$. The cyclic shuffle corresponds to the random walk on C_6 given by the distribution p

g	1	x	x^2	x^3	x^4	x^5
$p(g)$	0	1/2	0	0	0	1/2

Here, we're identifying $x^k \in C_6$ with the permutation where card j is in the $(j+k)^{\text{th}}$ spot. As we noted in section 1.1, if we're shuffling cyclically and card 1 is in the m^{th} spot, then the permutation has $(m-1)(n-m+1)$ inversions. The permutation corresponding to x^k has card 1 in the $(k+1)^{\text{st}}$ spot, meaning it has $k(n-k)$ inversions. Inspired by this, we define the statistic $\textsc{Inv}\colon C_n \to \mathbb{N}$ by

$$\textsc{Inv}(x^k) := k(6-k)$$

where $k \in \{0, \ldots, 5\}$. The character coefficients $a_k = \langle s, \chi_k \rangle$ and $b_k = \langle p, \chi_k \rangle$ are listed in the table below.

k	0	1	2	3	4	5
a_k	35/6	−2	−2/3	−1/2	−2/3	−2
b_k	1/6	1/12	−1/12	−1/6	−1/12	1/12

Since all values b_k are real, we can replace every b_k^* with just b_k in our formulas. So, by theorem 2.26 the expected number of inversions after t steps of the cyclic shuffle is

$$\begin{aligned}
\mathbb{E}_p(\textsc{Inv}, t) &= n^t \sum_{k=0}^{n-1} a_k b_k^t \\
&= 6^t \Bigg(\left(\frac{1}{6}\right)^t \left(\frac{35}{6}\right) + \left(\frac{1}{12}\right)^t (-2) + \left(-\frac{1}{12}\right)^t \left(-\frac{2}{3}\right) \\
&\qquad + \left(-\frac{1}{6}\right)^t \left(-\frac{1}{2}\right) + \left(-\frac{1}{12}\right)^t \left(-\frac{2}{3}\right) + \left(\frac{1}{12}\right)^t (-2) \Bigg) \\
&= \frac{35 - 3(-1)^t}{6} - \frac{12 + 4(-1)^t}{3(2^t)}.
\end{aligned} \tag{3.2}$$

Now we'll compute the expected value of $\textsc{Inv}^2$. Using the convolution-like formula $\sum_{j=0}^{n-1} a_j a_{k-j}$, we can find the coefficients for the decomposition of s^2 into characters:

k	0	1	2	3	4	5
$\sum_{j=0}^{n-1} a_j a_{k-j}$	259/6	−20	−4/3	−1/2	−4/3	−20

Now equation (3.1) gives

$$\begin{aligned}
\mathbb{E}_p(\textsc{Inv}^2, t) &= n^t \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} a_j a_{k-j} b_k^t \\
&= 6^t \left(\left(\frac{1}{6}\right)^t \left(\frac{259}{6}\right) + \left(\frac{1}{12}\right)^t (-20) + \left(-\frac{1}{12}\right)^t \left(-\frac{4}{3}\right) \right. \\
&\qquad \left. + \left(-\frac{1}{6}\right)^t \left(-\frac{1}{2}\right) + \left(-\frac{1}{12}\right)^t \left(-\frac{4}{3}\right) + \left(\frac{1}{12}\right)^t (-20) \right) \\
&= \frac{259 - 3(-1)^t}{6} - \frac{120 + 8(-1)^t}{3(2^t)}.
\end{aligned}$$

So, we get the following formula for the variance.

$$\begin{aligned}
\mathrm{Var}_p(\textsc{Inv}^2, t) &= \mathbb{E}_p(\textsc{Inv}^2, t) - \mathbb{E}_p(\textsc{Inv}, t)^2 \\
&= \left(\frac{259 - 3(-1)^t}{6} - \frac{60 + 8(-1)^t}{3(2^t)} \right) \\
&\qquad - \left(\frac{35 - 3(-1)^t}{6} - \frac{12 + 4(-1)^t}{3(2^t)} \right)^2 \\
&= \frac{80 + 48(-1)^t}{9} + \frac{48 + 80(-1)^t}{9(2^t)} - \frac{160 + 96(-1)^t}{9(4^t)}. \qquad (3.3)
\end{aligned}$$

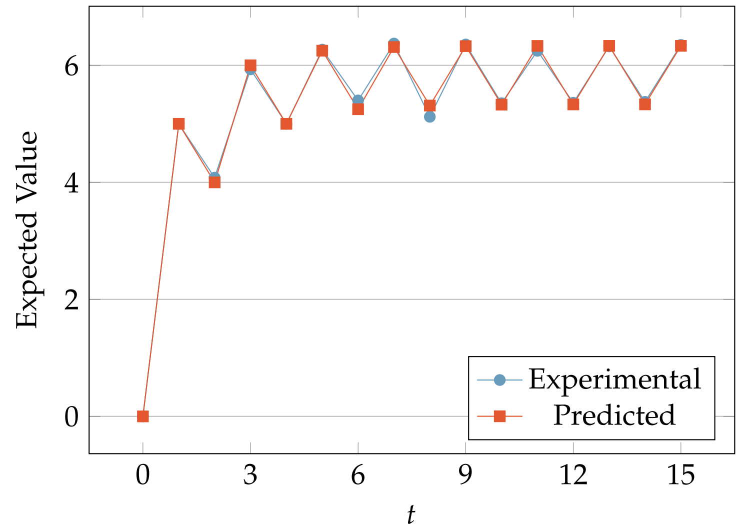
Equations 3.2 and 3.3 are plotted in figure 3.1.

3.2 Variances on the symmetric group

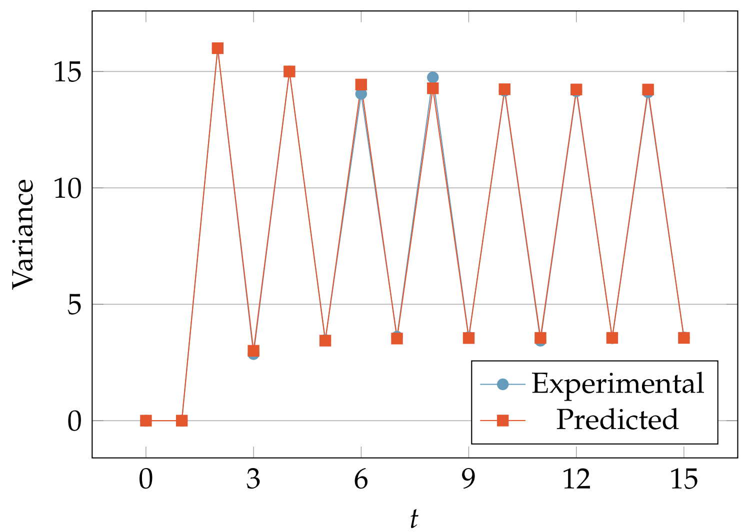
Now we'll employ the same approach for calculating $\mathrm{Var}_p(s, t)$ on S_n as we did on C_n. Assume the probability distribution p is a class function on S_n and s is any statistic on S_n. We will again employ the formula

$$\mathrm{Var}_p(s, t) = \mathbb{E}_p(s^2, t) - \mathbb{E}_p(s, t)^2.$$

As before, we know how to compute the second term using theorem 2.26. However, finding a decomposition for s^2 is a bit more subtle. To make the calculations as manageable as possible, we will assume the statistic s is also class function.



a. Expected number of inversions at step t of the cyclic shuffle on 6 cards.



b. Variance in the number of inversions at step t of the cyclic shuffle on 6 cards.

Figure 3.1 Values output by equation (3.2) (top) and equation (3.3) (bottom) plotted against experimental values from 1000 runs of the cyclic shuffle on 6 cards.

Recall that the irreducible representations of S_n are called *Specht modules*, and are indexed by integer partitions λ of n. We denote Specht modules by $(\rho^\lambda, S^\lambda)$, and the corresponding character by χ^λ.

Since s is a class function, there exist coefficients a_λ such that

$$s = \sum_{\lambda \vdash n} a_\lambda \chi^\lambda.$$

In the interest of computing s^2, we will need the following definition.

Definition 3.1. Let (π, V) and (ρ, W) be two representations of a finite group G. The *inner tensor product* of these representations is a representation $(\pi \otimes \rho, V \otimes W)$ given by

$$(\pi \otimes \rho)(g) = \pi(g) \otimes \rho(g).$$

We use the descriptor *inner* to distinguish from the outer tensor product, which is a method for constructing a representation of $G \times H$ given a representation of a group G and a representation of another group H.

Note that if V and W have characters χ_V and χ_W, respectively, then the inner tensor product has character

$$\chi_{V \otimes W}(g) = \operatorname{tr}(\pi(g) \otimes \rho(g)) = \operatorname{tr}(\pi(g)) \operatorname{tr}(\rho(g)) = \chi_V(g)\chi_W(g).$$

Hence, the character of the inner tensor product of two representations is the pointwise product of their characters. In general, the inner tensor product of two irreducible representations need not be irreducible. Thus, it's natural to ask how the inner product of two Specht modules S^λ and S^μ decomposes into a direct sum of Specht modules. More precisely, we know there exist coefficients $g^\nu_{\lambda,\mu}$ such that

$$S^\lambda \otimes S^\mu \cong \bigoplus_{\nu \vdash n} g^\nu_{\lambda,\mu} S^\nu$$

or equivalently

$$\chi^\lambda \chi^\mu \cong \sum_{\nu \vdash n} g^\nu_{\lambda,\mu} \chi^\nu.$$

These coefficients $g^\nu_{\lambda,\mu}$are called the *Kronecker coefficients* on the symmetric group. Although the existence of these coefficients is guaranteed by the representation theory of finite groups, computing them explicitly is quite difficult. For now, we will forge ahead assuming the coefficients are known.

Using these coefficients, we obtain the following decomposition of s^2:

$$\begin{aligned} s^2 &= \left(\sum_{\mu \vdash n} a_\mu \chi^\mu\right)\left(\sum_{\nu \vdash n} a_\nu \chi^\nu\right) \\ &= \sum_{\mu,\nu \vdash n} a_\mu a_\nu \chi^\mu \chi^\nu \\ &= \sum_{\mu,\nu \vdash n} a_\mu a_\nu \left(\sum_{\nu \vdash n} g^{\nu}_{\lambda,\mu} \chi^\lambda\right) \\ &= \sum_{\lambda,\mu,\nu \vdash n} g^{\lambda}_{\mu,\nu} a_\mu a_\nu \chi^\lambda. \end{aligned}$$

Hence, the coefficient on χ^λ in s^2 is $\sum_{\mu,\nu \vdash n} g^{\lambda}_{\mu,\nu} a_\mu a_\nu$. This is analogous to the convolution-like formula we observed in the previous section. However, due to the presence of Kronecker coefficients, it's considerably more difficult to work with than the previous one.

Given this formula for the coefficient on χ^λ in s^2, we can apply theorem 2.26 to obtain the following result.

Proposition 3.2. *Let* $s\colon S_n \to \mathbb{C}$ *be a statistic and* $p\colon S_n \to [0,1]$ *be a probability distribution. Suppose* s *and* p *are both class functions, and decompose into characters as*

$$s = \sum_{\lambda \vdash n} a_\lambda \chi^\lambda \qquad \text{and} \qquad p = \sum_{\lambda \vdash n} b_\lambda \chi^\lambda.$$

Then the expected value of s^2 *after* t *steps of the random walk defined by* p *is*

$$\mathbb{E}_p(s^2, t) = (n!)^t \sum_{\lambda,\mu,\nu \vdash n} \frac{g^{\lambda}_{\mu,\nu}}{f_\lambda^{t-1}} a_\mu a_\nu b_\lambda^t.$$

This can be used to obtain a formula for the variance of s, as we illustrate in the example below.

Example 3.2. Let s be the statistic Fix counting the number of fixed points in a permutation. This is a 1-local class function, and decomposes into characters as

$$\textsc{Fix} = \chi^{(n)} + \chi^{(n-1,1)}.$$

Hence, $a_{(n)} = a_{(n-1,1)} = 1$ and $a_\lambda = 0$ for all other λ.

Since this decomposition has so few nonzero terms, we only need a handful of Kronecker coefficients to apply proposition 3.2. Since the Specht module $S^{(n)}$ is the trivial representation,

$$S^{(n)} \otimes S^{\lambda} \cong S^{\lambda}$$

for any $\lambda \vdash n$. Thus,

$$g^{\lambda}_{(n),\nu} = \begin{cases} 1 & \lambda = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it is well known that for all $n \geq 4$,

$$S^{(n-1,1)} \otimes S^{(n-1,1)} \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,1^2)} \oplus S^{(n-2,2)}.$$

See, for example, Bowman et al. (2015). This implies

$$g^{\lambda}_{(n-1,1),(n-1,1)} = \begin{cases} 1 & \lambda_1 \geq n-2 \\ 0 & \text{otherwise} \end{cases}$$

Using these facts, proposition 3.2 gives the following equation.

$$\begin{aligned} \mathbb{E}_p(\mathrm{Fix}^2, t) &= (n!)^t \sum_{\lambda \vdash n} \left(\frac{g^{\lambda}_{(n),(n)} + g^{\lambda}_{(n-1,1),(n)} + g^{\lambda}_{(n),(n-1,1)} + g^{\lambda}_{(n-1,1),(n-1,1)}}{f_{\lambda}^{t-1}} \right) b^t_{\lambda} \\ &= \frac{(n!)^t}{f^{t-1}_{(n)}}(1+0+0+1)b^t_{(n)} + \frac{(n!)^t}{f^{t-1}_{(n-1,1)}}(0+1+1+1)b^t_{(n-1,1)} \\ &+ \frac{(n!)^t}{f^{t-1}_{(n-2,2)}}(0+0+0+1)b^t_{(n-2,2)} + \frac{(n!)^t}{f^{t-1}_{(n-2,1^2)}}(0+0+0+1)b^t_{(n-2,1^2)} \\ &= (n!)^t \Bigg(2b^t_{(n)} + \frac{3b^t_{(n-1,1)}}{(n-1)^{t-1}} + \frac{2^{t-1}b^t_{(n-2,2)}}{n^{t-1}(n-3)^{t-1}} \\ &\qquad + \frac{2^{t-1}b^t_{(n-2,1^2)}}{(n-1)^{t-1}(n-2)^{t-1}} \Bigg) \end{aligned} \tag{3.4}$$

Let's consider the random walk given by choosing transpositions uniformly at random. The relevant probability distribution is

$$p(\sigma) = \begin{cases} \frac{2}{n(n-1)} & \sigma \text{ is a transposition} \\ 0 & \text{otherwise.} \end{cases}$$

Say $p = \sum_{\lambda \vdash n} b_\lambda \chi^\lambda$. The coefficients b_λ are given by

$$b_\lambda = \langle p, \chi^\lambda \rangle = \frac{\chi_\lambda(\tau)}{n!},$$

where τ is any transposition. The *Murnaghan-Nakayama rule* is a combinatorial rule for evaluating characters of S_n (see Sagan (2001)). Using it we can compute the coefficients we need for this p (assuming $n \geq 4$). We find

$$b_{(n)} = \frac{1}{n!}, \qquad b_{(n-1,1)} = \frac{n-3}{n!},$$

$$b_{(n-2,2)} = \frac{(n-3)(n-4)}{2(n!)}, \qquad b_{(n-2,1^2)} = \frac{(n-2)(n-5)}{2(n!)}.$$

Plugging these into equation (3.4), we obtain the expression

$$\begin{aligned} \mathbb{E}_p(\textsc{Fix}^2, t) &= (n!)^t \left(\frac{2}{(n!)^t} + \frac{3(n-3)^t}{(n-1)^{t-1}(n!)^t} + \frac{2^{t-1}(n-3)^t(n-4)^t}{n^{t-1}(n-3)^{t-1}(2^t)(n!)^t} \right. \\ &\qquad \left. + \frac{2^{t-1}(n-2)^t(n-5)^t}{(n-1)^{t-1}(n-2)^{t-1}(2^t)(n!)^t} \right) \\ &= 2 + 3(n-1)\left(1 - \frac{2}{n-1}\right)^t + \frac{n(n-3)}{2}\left(1 - \frac{4}{n}\right)^t \\ &\qquad + \frac{(n-1)(n-2)}{2}\left(1 - \frac{4}{n-1}\right)^t. \end{aligned}$$

Using theorem 2.26, Hultman (2014) obtained the following formula for the expected value of $\textsc{Fix}$ after t steps of the random walk given by p:

$$\mathbb{E}_p(\textsc{Fix}, t) = 1 + (n-1)\left(1 - \frac{2}{n-1}\right)^t.$$

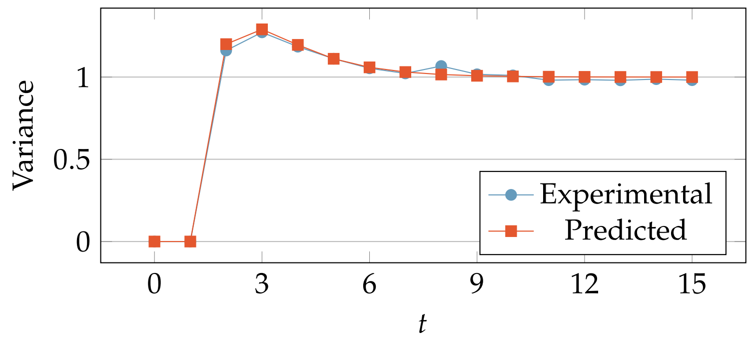
These two formulas allow us to compute the variance of Fix on permutations sampled via this random walk.

$$\begin{aligned}\mathrm{Var}_p(\textsc{Fix}, t) &= \mathbb{E}_p(\textsc{Fix}^2, t) - \mathbb{E}_p(\textsc{Fix}, t)^2 \\ &= 1 + (n-1)\left(1 - \frac{2}{n-1}\right)^t + \frac{n(n-3)}{2}\left(1 - \frac{4}{n}\right)^t \\ &\quad + \frac{(n-1)(n-2)}{2}\left(1 - \frac{4}{n-1}\right)^t - (n-1)^2\left(1 - \frac{2}{n-1}\right)^{2t}.\end{aligned} \tag{3.5}$$

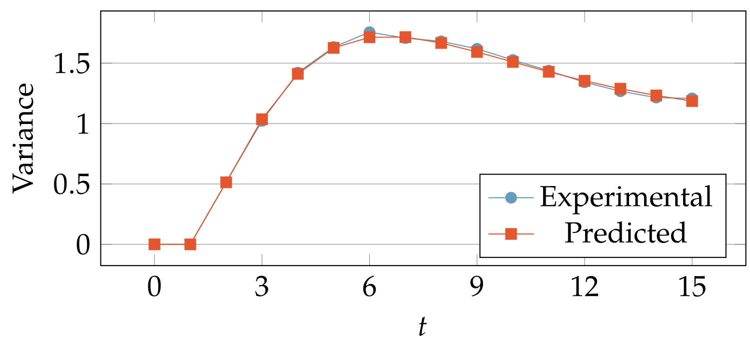
Hence, we have arrived at an exact formula for the variance in the number of fixed points in a product of t random transpositions from S_n, assuming $n \geq 4$. In figure 3.2, we plot the values output by this formula against values computed experimentally by simply running the random walk many times, recording the number of fixed points, and computing the variance of those data points.

As the previous example illustrates, proposition 3.2 can let us find formulas for variances of permutation statistics on permutations sampled from random walks. However, this approach is limited in a number of ways. Firstly, proposition 3.2 only holds for statistics that are class functions, but many statistics we care about are not (such as Inv and Des). Secondly, applying proposition 3.2 requires knowledge of Kronecker coefficients, which are notoriously difficult to characterize (see, for example, Pak et al. (2016)). For a 1-local statistic like Fix, we don't need very many Kronecker coefficients to apply the formula, so this doesn't pose a problem. However, the number of Kronecker coefficients needed becomes unmanageable if we were to examine statistics that aren't as low-frequency.

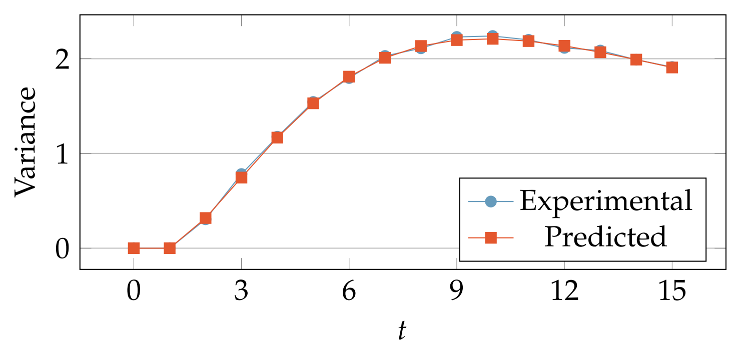
In the next section, we delve into the theory of the partition algebras. Although they may appear unrelated to the kinds of problems we're considering in this section, it will turn out that understanding partition algebras and their duality with the symmetric group will allow us to surpass some of the limitations we ran into in this section. Specifically, we will arrive at formulas analogous to the ones in example 3.2 for Inv, a non-class function.



a. Variance in the number of fixed points in the product of t random transpositions from S_5.



b. Variance in the number of fixed points in the product of t random transpositions from S_{10}.



c. Variance in the number of fixed points in the product of t random transpositions from S_{15}.

Figure 3.2 Values output by equation (3.5) plotted against experimental values from 10000 runs of the random walk, for $n = 5$ (top), $n = 10$ (middle) and $n = 15$ (bottom).

Chapter 4

Partition algebras

4.1 Motivation

As we have seen, computing variances of permutation statistics leads one to consider tensor squares of representations of S_n. Analogously, to gain information about the k^{th} moment of a statistic, we must examine k^{th} tensor powers of the relevant representations. Here we introduce the *partition algebras*, which are algebraic objects that arise naturally when studying tensor powers of certain representations of S_n. In particular, a symmetric group and its corresponding partition algebra exhibit *Schur-Weyl duality*, meaning their irreducible representations are very closely related to each other. In the following chapter, we will see several results about that take advantage of partition algebras and Schur-Weyl duality in order to conclude something about permutation statistics.

4.2 Definitions

Somewhat confusingly, the word "partition" in the name "partition algebra" does not refer to integer partitions as defined in definition 2.10, but instead to *set partitions*, a distinct but closely related combinatorial object.

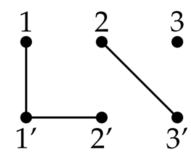
Definition 4.1. A *set partition* of a set U is a collection of nonempty disjoint sets $\{U_\alpha\}$ whose union is U. Individual sets in a set partition are called *blocks*.

Example 4.1. The set partitions of $\{a, b, c\}$ are

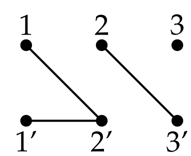
$$\{\{a, b, c\}\},\ \{\{a, b\}, \{c\}\},\ \{\{a, c\}, \{b\}\},\ \{\{b, c\}, \{a\}\},\ \{\{a\}, \{b\}, \{c\}\}.$$

We will use the term (k, k)-*set partitions* to refer to set partitions of the set $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$. These set partitions can be represented visually in the following way. Draw two rows of k vertices labeled 1 to k along the top and $1'$ to k' along the bottom, and draw edges between vertices so that two vertices are in the same connected component if and only if they are in the same block of the set partition.

For example, the (3,3)-set partition $\{\{1, 1', 2'\}, \{2, 3'\}, \{3\}\}$ may be represented by the diagram



Note that there may exist several different diagrams for the same set partition. In particular, we only care about whether a pair of vertices lies in the same connected component, so distinct diagrams with the same connected components are equivalent. An equivalent diagram to the one above is



With this, we are able to introduce partition algebras.

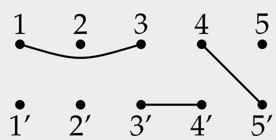
Definition 4.2. Let k be a positive integer, and let $n \in \mathbb{C}$. The *partition algebra* $P_k(n)$ is the $\mathbb{C}$-algebra of formal linear combinations of (k, k)-set partitions. The product of two (k, k) set partitions P_1 and P_2 is defined as follows.

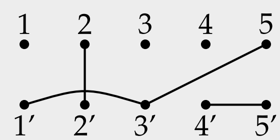
Take a diagram for P_1 and a diagram for P_2, and relabel all vertices in the P_2 diagram by adding a prime. Identify each single-primed vertex in P_1 with the corresponding single-primed vertex in P_2, yielding a graph with three rows of k vertices. Let $c(P_1, P_2)$ denote the number of connected components that are entirely in the middle row – that is, they consist only of single-primed vertices. The product of P_1 and P_2 is defined to be

$$P_1 P_2 = n^{c(P_1, P_2)} P_3,$$

where P_3 is the diagram obtained by deleting the middle row of the three row graph while preserving connections between the top and bottom rows, and then relabeling double-primed vertices to have single-primes.

Example 4.2. We will compute the product of

$P_1 =$ 

, and $P_2 =$ 

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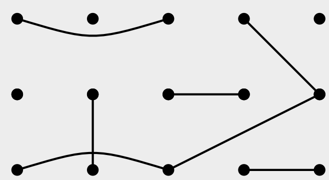
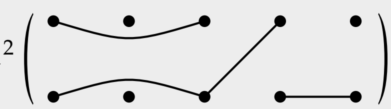
The stacked graph is shown below.



This has two components in the middle layer, meaning $c(P_1, P_2) = 2$. Hence,

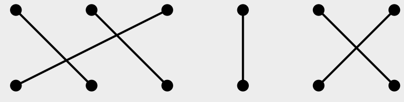
$P_1 P_2 = n^2$ 

A first observation about the this algebra is that it contains a copy of the symmetric group. Given a permutation $\omega \in S_k$, consider the (k, k)-set partition

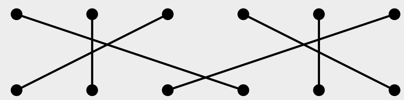
$$P_\omega = \{\{\omega(i), i'\} : i \in [k]\}.$$

As we illustrate in the following example, these set partitions behave like permutations with respect to multiplication.

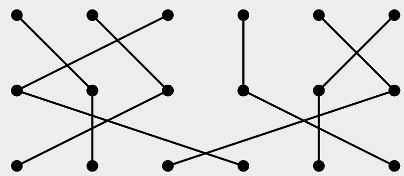
Example 4.3. Consider the permutations $\omega = (1\,3\,2)(5\,6)$, and $\pi = (1\,3\,6\,4)$ in S_6. The corresponding $(6, 6)$-set partitions are

$P_\omega =$ 

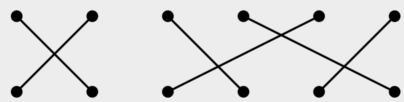
and

$$P_\pi =$$  .

Let's compute the product $P_\omega P_\pi$. The stacked diagram is



meaning

$$P_\omega P_\pi =$$  $$= P_{\omega\pi}.$$

Hence, multiplication in the partition algebra corresponds to composing the two permutations.

As such, the set

$$\text{span}\{P_\omega : \omega \in S_k\}$$

is a subalgebra of the partition algebra $P_k(n)$ isomorphic to $\mathbb{C}S_k$.

4.3 Schur-Weyl duality

The partition algebras arise naturally when studying tensor powers of representations of S_n, because they exhibit a relationship known as Schur-Weyl duality with $\mathbb{C}S_n$. In this section, we build up the key ideas of Schur-Weyl duality, and explain the emergence of the partition algebra in this setting.

For this chapter and the next, we let V denote an n-dimensional vector space with basis $\{v_1, \ldots, v_n\}$. The symmetric group S_n can act on this space by permuting basis vectors. Hence, we can define a representation map $\psi \colon S_n \to GL(V)$

$$\psi(\sigma)v_i := v_{\sigma(i)},$$

making V into an S_n-representation. We call this the *defining representation* of S_n. Extending linearly, we can consider ψ as a map from $\mathbb{C}S_n \to \text{End}_{\mathbb{C}} V$.

The group GL_n of invertible $n \times n$ complex matrices also has a natural action on V given by matrix multiplication. That is, we may define a representation $\Psi : GL_n \to GL(V)$ where if $A = (a_{ij}) \in GL_n$,

$$\Psi(A)v_i := \sum_{j=1}^{n} a_{ij}v_j.$$

Analogously to the case above, Ψ extends to a map from M_n — the set of all $n \times n$ complex matrices — to $\operatorname{End}_{\mathbb{C}} V$.

Notice that if A is a permutation matrix corresponding to the permutation σ, $\Psi(A)v_i = v_{\sigma(i)} = \psi(\sigma)v_i$. Hence, the S_n-representation ψ is just the restriction of the GL_n-representation Ψ to the subgroup $S_n \subset GL_n$ consisting of permutation matrices.

Now, consider the k^{th} tensor power of V, denoted $V^{\otimes k}$. Each of the representations defined above extends to a representation of $V^{\otimes k}$ (see definition 3.1), yielding the actions

$$\psi(\sigma)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_k)}$$

and

$$\Psi(A)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = Av_{i_1} \otimes Av_{i_2} \otimes \cdots \otimes Av_{i_k}.$$

We call these actions *diagonal*, since they act on each tensor component separately. We will often abbreviate the vector $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ as v_I, where I is the word $i_1 i_2 \dots i_k \in [n]^k$. We can then write the diagonal S_n-action more compactly as

$$\psi(\sigma)v_I = v_{\sigma(I)}$$

where $\sigma(I) = (\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k))$.

There exists another natural action of the symmetric group on $V^{\otimes k}$; S_k can act by permuting the tensor factors. Precisely, we may define a representation $\phi : S_n \to GL(V^{\otimes k})$ where

$$\phi(\omega)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{i_{\omega^{-1}(1)}} \otimes v_{i_{\omega^{-1}(2)}} \otimes \cdots \otimes v_{i_{\omega^{-1}(k)}}.$$

For example, if $k = 3$ and $\omega = (1\ 3\ 2)$, then

$$\phi(\omega)(v_2 \otimes v_3 \otimes v_5) = v_3 \otimes v_5 \otimes v_2.$$

With this action, we can consider $V^{\otimes k}$ as an S_k-representation (i.e. a $\mathbb{C}S_k$-module).

Notably, the S_k-action ϕ commutes with the GL_n-action Ψ on $V^{\otimes k}$. To see this, note that if $A \in GL_n$ and $\omega \in S_k$, we have

$$\begin{aligned}\phi(\omega)(\Psi(A)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})) &= \phi(\omega)(Av_{i_1} \otimes Av_{i_2} \otimes \cdots \otimes Av_{i_k}) \\ &= Av_{i_{\omega^{-1}(1)}} \otimes Av_{i_{\omega^{-1}(2)}} \otimes \cdots \otimes Av_{i_{\omega^{-1}(k)}} \\ &= \Psi(A)(v_{i_{\omega^{-1}(1)}} \otimes v_{i_{\omega^{-1}(2)}} \otimes \cdots \otimes v_{i_{\omega^{-1}(k)}}) \\ &= \Psi(A)(\phi(\omega)(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k})).\end{aligned}$$

In fact, we can say something even stronger: the *only* linear maps from $V^{\otimes k}$ to itself that commute with $\phi(\omega)$ for all $\omega \in S_k$ are those in the image of Ψ. Intuitively, this holds because if a map commutes with any permutation of the tensor factors, it must do the same thing to each factor, meaning it must be of the form $\Psi(A)$ for some $A \in M_n$

The following terminology helps us better characterize the relationship between these representations.

Definition 4.3. Let E be a $\mathbb{C}$-vector space and A be a subalgebra $\operatorname{End}_{\mathbb{C}} E$, the algebra of $\mathbb{C}$-linear maps from E to itself. The *centralizer* or *commutant* of A in E, denoted $\operatorname{End}_A E$, is the algebra of endomorphisms of E that commute with A, that is

$$\operatorname{End}_A E := \{b \in \operatorname{End} E : ba = ab \text{ for all } a \in A\}$$

By our reasoning above, we saw that $\Psi(M_n)$ is the centralizer of $\phi(\mathbb{C}S_k)$ in $\operatorname{End}_{\mathbb{C}}(V^{\otimes k})$. We may write this fact symbolically as

$$\operatorname{End}_{\phi(\mathbb{C}S_k)}(V^{\otimes k}) = \Psi(M_n).$$

The following theorem tells us that the relationship we have observed between these two algebras has extremely strong consequences in terms of representation theory.

Theorem 4.4 (Double Centralizer Theorem, see Etingof et al. (2011))**.** *Let A, B be two subalgebras of the algebra $\operatorname{End}_{\mathbb{C}} E$ of endomorphisms of a finite dimensional vector space E, such that A is semisimple and $B = \operatorname{End}_A E$. Then*:

(i) $A = \operatorname{End}_B E$ *(i.e., the centralizer of the centralizer of A is A).*

(ii) *B is semisimple.*

(iii) *As a representation of $A \otimes B$, E decomposes as*

$$E = \bigoplus_i V_i \otimes W_i,$$

where V_i are all the irreducible representations of A and W_i are all the irreducible representations of B.

Recalling that $\Psi(M_n) = \mathrm{End}_{\phi(\mathbb{C}S_k)}(V^{\otimes k})$, part (i) tells us that $\phi(\mathbb{C}S_k) = \mathrm{End}_{\Psi(M_n)}(V^{\otimes k})$. Hence, the algebras $\phi(\mathbb{C}S_k)$ and $\Psi(M_n)$ have a "dual" relationship with respect to their actions on $V^{\otimes k}$ – we say they are *mutual centralizers*. And, part (iii) tells us that as a representation of $\mathbb{C}S_k \otimes M_n$,

$$V^{\otimes k} = \bigoplus_\lambda S^\lambda \otimes L_\lambda$$

where the S^λ's are Specht modules (i.e. irreducible representations of S_k) and the L_λ's are irreducible representations of M_n. Thus, the mutual centralizer relationship between S_k and M_n sets up a pairing between their irreducible representations. This close relationship between their representation theories is known as *(classical) Schur-Weyl duality.*

Now, recall that S_n-representation ψ on $V^{\otimes k}$ sits inside of the GL_n-representation Ψ. We would like to find a similar statement to the one above with regard to this S_n-action. To start, we must first determine what the centralizer of the algebra $\psi(\mathbb{C}S_n)$ in $\mathrm{End}_\mathbb{C}(V^{\otimes k})$ is.

In what follows, we will denote $\psi(\sigma)v_I$ by simply $\sigma \cdot v_I$ to simplify notation. Suppose $T \in \mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$. Then T commutes with the action of S_n, so for any permutation $\sigma \in S_n$,

$$\sigma \cdot (Tv_I) = T(\sigma \cdot v_I) = Tv_{\sigma(I)}$$

which implies

$$Tv_I = \sigma^{-1} \cdot Tv_{\sigma(I)}.$$

Fixing the basis $\{v_I : I \in [n]^k\}$ for $V^{\otimes k}$ allows us to write T as an $n^k \times n^k$ matrix. Let $\langle \cdot, \cdot \rangle$ be an inner product on $V^{\otimes k}$ with respect to which this basis is orthonormal. Note that this inner product satisfies

$$\langle x, y \rangle = \langle \sigma \cdot x, \sigma \cdot y \rangle$$

for any $x, y \in V^{\otimes k}$ and $\sigma \in S_n$. Entries $T_{I,J}$ of the matrix T are indexed by pairs of k-element words $I, J \in [n]^k$, and are given by

$$T_{I,J} = \left\langle Tv_I, v_J \right\rangle.$$

It follows from the equation above that for any $\sigma \in S_n$,

$$T_{I,J} = \left\langle Tv_I, v_J \right\rangle = \left\langle \sigma^{-1} \cdot Tv_{\sigma(I)}, v_J \right\rangle = \left\langle \sigma\sigma^{-1} \cdot Tv_{\sigma(I)}, \sigma \cdot v_J \right\rangle$$
$$= \left\langle Tv_{\sigma(I)}, \sigma \cdot v_J \right\rangle = \left\langle Tv_{\sigma(I)}, v_{\sigma(J)} \right\rangle = T_{\sigma(I),\sigma(J)}. \quad (4.1)$$

Hence, in order to commute with the S_n-action, entries of T must be constant along S_n-orbits of tuples (I, J).

Example 4.4. Suppose $n = 4$ and $k = 2$, meaning $V^{\otimes 2}$ is a 16-dimensional vector space. Then $T \in \mathrm{End}_{S_n}(V^{\otimes 2})$ is a 16×16 matrix with entries indexed by pairs (I, J), where I and J are 2-long words on the set $\{1, 2, 3, 4\}$. Equation 4.1 implies that

$$T_{11,11} = T_{22,22} = T_{33,33} = T_{44,44},$$

since the transposition $\sigma = (1\ j)$ maps $(11, 11)$ to (jj, jj) for $j = 2, 3, 4$. Similarly,

$$T_{11,12} = T_{11,13} = T_{11,14} = T_{22,21} = T_{22,23} = \cdots$$

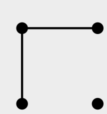
More generally, the S_n-action partitions the set of pairs of words (I, J) into orbits of the form

$$\mathrm{orb}(I, J) := \{(\sigma(I), \sigma(J)) : \sigma \in S_n\},$$

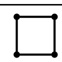
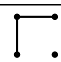
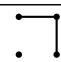
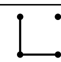
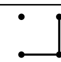
and $T_{I,J} = T_{I',J'}$ if (I, J) and (I', J') are in the same orbit.

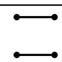
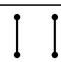
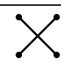
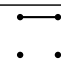
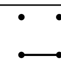
Given a pair (I, J), its orbit $\mathrm{orb}(I, J)$ consists of all pairs (I', J') with the exact same pattern of equalities and inequalities between individual entries. For example, $\mathrm{orb}(11, 12)$ consists of all pairs of words $(I, J) = (i_1 i_2, j_1 j_2)$ such that $i_1 = i_2 = j_1 \neq j_2$.

A pattern of equalities like this can be summarized by a $(2, 2)$-set partition, where the blocks of the set partition correspond to the sets of entries that are required to be equal to each other. Returning to our example from before, $\mathrm{orb}(11, 12)$ corresponds to the $(2, 2)$-set partition



since both entries of I and the first entry of J must be equal to each other, and the second entry of J must be different. Figure 4.1 shows all of the orbits and their corresponding $(2, 2)$-set partitions.

Orbit	orb(11, 11)	orb(11, 12)	orb(11, 21)	orb(12, 11)	orb(21, 11)
P					

Orbit	orb(11, 22)	orb(12, 12)	orb(12, 21)	orb(11, 23)	orb(12, 33)
P					

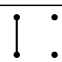
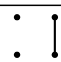
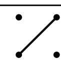
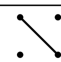
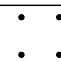
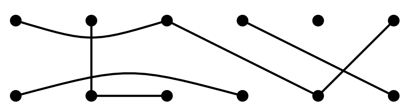
Orbit	orb(12, 13)	orb(12, 32)	orb(12, 23)	orb(12, 31)	orb(12, 34)
P					

Figure 4.1 The 15 orbits of pairs of 2-element words $(i_1 i_2, j_1 j_2)$ under the action of S_n, shown alongside the corresponding $(2, 2)$-set partition P.

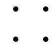
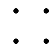
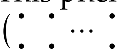
In the above example we assumed $k = 2$ and $n = 4$, but the same story holds for any n and k. For example, if $k = 6$ and $n = 5$, the orbit of $(121341, 522513)$ corresponds to the $(6, 6)$-set partition shown below.



Now is a good time to take a step back and examine what we've uncovered. We set out to understand the algebra $\text{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$. We noticed that the entries of any $T \in \text{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$ must be constant on certain orbits of pairs of words, and that those orbits are in one-to-one correspondence with (k, k)-set partitions. The matrix T is therefore fully described by its value on each orbit. Equivalently, we can describe T by specifying a value for each (k, k)-set partition. Thus, T can be identified with a formal linear combination of (k, k)-set partitions, i.e., an element of the partition algebra $P_k(n)$. Symbolically, we have the correspondence

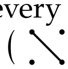
$$\sum_{\substack{(k,k)\text{-set} \\ \text{partitions } P}} t_P P \in P_k(n) \longleftrightarrow T \in \text{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$$

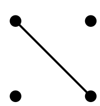
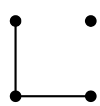
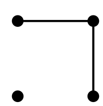
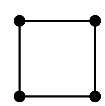
where T is the matrix that has value t_P on the orbit corresponding to P. This seems to imply a close connection between $\mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$ and $P_k(n)$.

However, this identification we have described has two caveats. Firstly, it is not always bijective. Consider the case where $k = 2$ and $n = 3$. The $(2, 2)$-set partition () would correspond to the orbit of pairs of words $(i_1 i_2, j_1 j_2)$ such that no two of i_1, i_2, j_1 and j_2 are equal. However, there are only three possible values for each of i_1, i_2, j_1, and j_2 (i.e. 1, 2, or 3), so it is impossible that all 4 are distinct. Hence any element $T \in \mathrm{End}_{\psi(\mathbb{C}S_n)}$ has no orbit of entries corresponding to (). This phenomenon occurs whenever $n < 2k$, since the (k, k)-set partition () as $2k$ distinct blocks, which is greater than the number of distinct values from 1 to n. Thus, the correspondence we described above is bijective if and only if $n \geq 2k$.

The second caveat is that the correspondence we have described does not respect the multiplicative structures of the two algebras in question. In other words, it is not an algebra homomorphism. However, following the techniques of Gaetz and Ryba (2021) we can tweak the definition of the correspondence slightly to fix this issue. To do so, we first define a partial order on the set of (k, k)-set partitions.

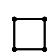
Definition 4.5. Let P and Q be (k, k)-set partitions. We write $P \leq Q$ if whenever two elements of $\{1, \ldots, k, 1', \ldots, k'\}$ are in the same block of Q, they are also in the same block of Q. If $P \leq Q$, we say that Q is a *coarsening* of P, or that P is a *refinement* of Q.

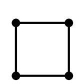
Equivalently, Q is a coarsening of P if every block in Q is a union of blocks in P. For example, the coarsenings of () are

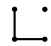
 ,  ,  , and  .

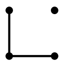
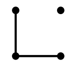
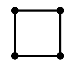
Definition 4.6 (See Gaetz and Ryba (2021))**.** Let P be a (k, k)-set partition. Define $x_P \in P_k(n)$ recursively by the equation.

$$P = \sum_{Q \geq P} x_Q.$$

For example, letting $P =$ () implies

$x_{\square} =$  .

Applying the equation again with $P = ($  $)$ yields

 $= x$ $+ x$, which implies x $=$  $-$ .

Continuing in this manner, we can compute x_P for any (k,k)-set partition P.

It isn't too hard to see that for any P,

$$x_P = P + (\text{some linear combination of other coarsenings of } P).$$

Based on this fact, an upper-triangularity argument shows that the set

$$\{x_P : P \text{ is a } (k,k)\text{-set partition}\}$$

is a basis for $P_k(n)$. If we tweak the map we discussed earlier to use this basis, we arrive at the following result.

Theorem 4.7 (see Halverson and Ram (2005))**.** *Consider the linear map $\Phi_{k,n}$: $P_k(n) \to \mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$ defined on the basis $\{x_P\}$ by*

$$\Phi_{k,n}(x_P) = T_P$$

where T_P is the matrix whose I,J entry is 1 if (J,I) is in the orbit corresponding to P, and 0 otherwise. Set $T_P = 0$ if no such orbit exists. Then $\Phi_{k,n}$ is a surjective algebra homomorphism.

When k and n are clear from context, we may denote $\Phi_{k,n}$ by just Φ. Note that the entry $(T_P)_{I,J}$ depends on the orbit of (J,I), not (I,J). This subtle detail is needed to make Φ a homomorphism instead of an anti-homomorphism. We can readily see that $\Phi_{k,n}$ has kernel

$$\ker \Phi_{k,n} = \mathrm{span}\{x_P : P \text{ has more than } n \text{ blocks}\}. \tag{4.2}$$

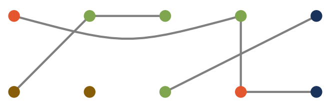
This is because the x_P's where P has more than n blocks are precisely the ones with $T_P = 0$. And, the set of T_P's where P has at most n blocks is a linearly independent set, so nothing in $\mathrm{span}\{x_P : P \text{ has at most } n \text{ blocks}\}$ is in $\ker \Phi_{k,n}$.

In particular, Equation 4.2 implies the following key fact.

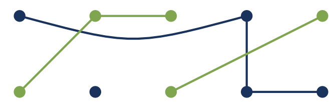
Corollary 4.8. *If $n \geq 2k$, $\Phi_{k,n} : P_k(n) \to \mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$ is an algebra isomorphism.*

Assuming $n \geq 2k$, we have finally successfully characterized the algebra $\mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k})$: it's the partition algebra $P_k(n)$! This implies that $\mathbb{C}S_k$ and $P_k(n)$ exhibit Schur-Weyl duality with respect to $V^{\otimes k}$. Before we explore the consequences of this fact, it will be helpful to understand how the partition algebras act on $V^{\otimes k}$. To do so, we introduce some terminology.

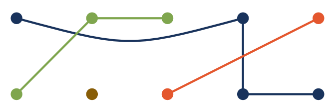
Definition 4.9. A *coloring* of a (k, k)-set partition is an assignment of colors 1 through n to the vertices in the set partition. We say a coloring is *good* if each block in the partition is monochromatic, and *perfect* if two vertices are the same color if and only if they are in the same block.



a. A coloring that is not good. $(C, C') = (13332, 44312)$.



b. A good coloring that isn't perfect. $(C, C') = (23323, 32332)$.



c. A perfect coloring. $(C, C') = (23321, 34122)$.

Figure 4.2 Three different colorings of the same $(5, 5)$-set partition by the set $\{1, 2, 3, 4\}$, where 1 = ■, 2 = ■, 3 = ■, and 4 = ■.

A coloring of a (k, k)-set partition can be described by a pair of $(C, C') = (c_1 c_2 \dots c_k, c'_1 c'_2 \dots c'_k)$ of words of length k on $[n]$; vertex l is given color c_l and vertex l' is given color c'_l.

Proposition 4.10. *Let P be a (k, k)-set partition and $I \in [n]^k$. Then*

$$\Phi_{k,n}(x_P)(v_I) = \sum_J v_J, \tag{4.3}$$

where the sum ranges over all $J \in [n]^k$ such that the coloring of P with $C = J$ and $C' = I$ is perfect. And,

$$\Phi_{k,n}(P)(v_I) = \sum_K v_K, \tag{4.4}$$

where the sum ranges over all $K \in [n]^k$ such that the coloring of P with $C = K$ and $C' = I$ is good.

Proof. By definition of $\Phi_{k,n}$,

$$\Phi_{k,n}(x_P)(v_I) = \sum_{J \in [n]^k} (T_P)_{I,J} v_J.$$

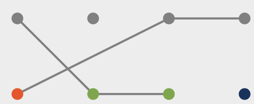
Recall that the entry $(T_P)_{I,J}$ is 1 if and only if (J, I) is in the orbit corresponding to P, and 0 otherwise. This is equivalent to the condition that the coloring of P with $(C, C') = (J, I)$ is perfect. Equation (4.3) follows.

For the second equation, first notice that

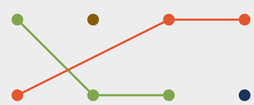
$$\Phi_{k,n}(P)(v_I) = \sum_{Q \geq P} \Phi_{k,n}(x_Q)(v_I).$$

Equation (4.4) now follows from equation (4.3) with the observation that a coloring of P is good if and only if it is a perfect coloring of some coarsening Q of P. □

Example 4.5. Let $n = 4$ and $k = 4$. Take $P =$ , and consider the word $I = 1332$. This tells us to color the bottom row of the diagram as



To compute $\Phi_{4,4}(x_P)(v_I)$, we must find all colorings of the top row such that the resulting coloring is perfect. The only such coloring is



meaning $\Phi_{4,4}(x_P)(v_I) = v_{3411}$.

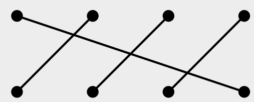
Similarly, to compute $\Phi_{4,4}(P)(v_I)$, we we must find all ways to color the top row that result in a good coloring. There are now 4 possibilities, shown below.

Hence $\Phi_{4,4}(P)(v_I) = v_{3111} + v_{3211} + v_{3311} + v_{3411}$.

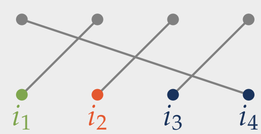
With this new description of Φ, it's easy to see that everything in image of Φ will commute with the diagonal action ψ of $\mathbb{C}S_n$ on $V^{\otimes k}$. Indeed, acting on (I, J) by some $\sigma \in S_n$ corresponds to relabeling the colors assigned to each vertex in a partition diagram, but this won't affect whether the coloring $(C, C') = (J, I)$ is good or perfect.

For set partitions P_ω corresponding to a permutation $\omega \in S_k$ as in example 4.3, the action of $\Phi_{k,n}(P_\omega)$ on a vector v_I reduces to the action of $\phi(\omega)$, which permutes the tensor factors (as defined on page 45). This is illustrated in the following example.

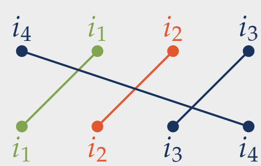
Example 4.6. Let $k = 4$ and suppose $\omega = (1\ 2\ 3\ 4) \in S_4$. Then P_ω is the set partition shown below.

$P_\omega =$ 

Let's compute $\Phi_{4,n}(P_\omega)(v_I)$ for some word $I = i_1 i_2 i_3 i_4 \in [n]^4$. In light of proposition 4.10, we can compute this by finding all ways of coloring the top row in the picture below so that the resulting coloring is good.



It's clear from the structure of P_ω that there always exactly only one good coloring for any I, and it is obtained by coloring each vertex in the top row the same color as its corresponding vertex in the bottom row:



The resulting coloring of the top row is given by permuting the entries of I according to ω; the color of the $\omega(l)^{\text{th}}$ entry is color of the l^{th} entry of the bottom row. Hence

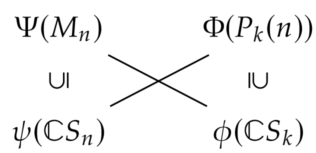
$$\Phi_{4,n}(P_\omega)(v_I) = v_{4123} = v_{i_{\omega^{-1}(1)}} \otimes v_{i_{\omega^{-1}(2)}} \otimes v_{i_{\omega^{-1}(3)}} \otimes v_{i_{\omega^{-1}(4)}} = \phi(\omega) v_I.$$

This same fact holds for general n and k for the exact same reason. Hence

$$\Phi(P_\omega)(v_I) = \phi(\omega) v_I. \tag{4.5}$$

Recall our observation earlier that the subalgebra span$\{P_\omega : \omega \in S_k\}$ of $P_k(n)$ is a copy of $\mathbb{C}S_k$. In this context, equation (4.5) tells us that *the representation ϕ of $\mathbb{C}S_k$ can be viewed as the restriction of the representation Φ of $P_k(n)$ to the subalgebra $\mathbb{C}S_k \subseteq P_k(n)$.*

The diagram below shows the relationships between $\mathbb{C}S_n$ and GL_n and their corresponding Schur-Weyl duality partners. As we have seen, restricting from a the algebra GL_n to the subalgebra $\mathbb{C}S_k$ means that the corresponding centralizer will be larger, since there are fewer things the maps are required to commute with. Goodman and Wallach (1998) calls this relationship a "seesaw pair," for self-evident reasons.



We end with one final consequence of the Schur-Weyl duality between $\mathbb{C}S_n$ and $P_k(n)$. Assume $n \geq 2k$. By corollary 4.8, we find

$$P_k(n) \cong \operatorname{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k}) \subseteq \operatorname{End}(V^{\otimes k}).$$

Hence $P_k(n)$ is the centralizer of $\psi(\mathbb{C}S_n)$ with respect to $V^{\otimes k}$. Applying theorem 4.4, we find that as a representation of $\mathbb{C}S_k \otimes P_k(n)$, $V^{\otimes k}$ has a decomposition of the form

$$V^{\otimes k} \cong \bigoplus_{\lambda} S^\lambda \otimes L_\lambda$$

where the S^λ's are Specht modules, and the L_λ's are irreducible representations of $P_k(n)$. Bowman et al. (2015) use this pairing between irreducible representations to turn the problem of computing Kronecker coefficients into a question about these L_λ's.

As we have seen, $P_k(n)$ emerges as the centralizer algebra of the image of $\mathbb{C}S_n$ in $\operatorname{End}_{\mathbb{C}}(V^{\otimes k})$. In the next chapter, we will take advantage of this relationship to prove results about permutation statistics on S_n. This will ultimately allow us to compute moments of permutation statistics sampled via certain random walks on S_n.

Chapter 5

Variances of permutation statistics, part II

As we saw at end of the previous section, $\mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes k}) \cong \Phi_{k,n}(P_k(n))$, meaning every endomorphism of $V^{\otimes k}$ that commutes with everything in $\mathbb{C}S_n$ is of the form $\Phi_{k,n}(x)$ for some element x of the partition algebra $P_k(n)$. Thus, if we have some endomorphism f of $V^{\otimes k}$ that commutes with $\psi(g)$ for all $g \in S_n$, then f is in the image of Φ. Hence there exist coefficients $a_P \in \mathbb{C}$ such that

$$f = \Phi_{k,n}\left(\sum_P a_P P\right) \tag{5.1}$$

where the sum P ranges over all (k, k)-set partitions.

Recall theorem 2.21:

Theorem 2.21 (Gaetz and Ryba (2021))**.** *Let $\phi \in S_k$ be a fixed permutation thought of as a classical k-pattern. For a permutation $\sigma \in S_n$, let m_i denote the number of i-cycles in σ. Then as a function on all symmetric groups at once, $\overline{\mathrm{Pat}_\phi^d}$ is described by a single polynomial of degree at most dk in the variables $n, m_1, \ldots, m_{dk}$, where n has degree 1 and each m_i has degree i.*

We now have the tools necessary to understand the proof of this result. In particular, our observation above is one of a few key ingredients to the argument. Via some clever manipulations, Gaetz and Ryba are able to encode $\overline{\mathrm{Pat}_\phi^d}$ into an endomorphism of $V^{\otimes k}$ which commutes with ψ. As in equation (5.1), this endomorphism decomposes into a linear combination of (k, k)-set partitions. After proving some technical lemmas characterizing how (k, k)-set partitions act on $V^{\otimes k}$, this yields the desired result about the original statistic.

In this chapter, we explain the key ingredients of that argument. Their final result asserts that $\overline{\mathrm{PAT}^d_\phi}$ is a polynomial, but they don't attempt to compute any examples of such polynomials. However, their argument roughly outlines a method to arrive at that polynomial. By carefully stepping through the parts of their argument and computing along the way, we are able to find the polynomial corresponding to the statistic $\overline{\mathrm{INV}^2} = \overline{\mathrm{PAT}^2_{(2-1)}}$. This will yield the following new result.

Proposition 2.22 (S.). *Let $\sigma \in S_n$ be a permutation, and let m_i denote the number of i-cycles in σ. Then*

$$\begin{aligned}\overline{\mathrm{INV}^2}(\sigma) = \frac{1}{720}(&5m_1^4 + 20m_1^3 n - 14m_1^3 - 12m_1^2 m_2 + 50m_1^2 n^2 - 90m_1^2 n\\ &- 25m_1^2 - 24m_1 m_2 n + 12m_1 m_2 - 24m_1 m_3 + 60m_1 n^3 - 126m_1 n^2\\ &+ 94m_1 n + 98m_1 + 60m_2^2 - 20m_2 n^2 + 108m_2 n - 124m_2\\ &- 24m_3 n - 48m_3 - 24m_4 + 45n^4 - 130n^3 + 111n^2 - 98n).\end{aligned}$$

At the end of this chapter, we use this result to analyze random walks on S_n. We conclude with a formula for the variance of the number of inversions in permutations sampled via a random walks.

5.1 Polynomiality argument

At a macroscopic level, Gaetz and Ryba's argument that $\overline{\mathrm{PAT}^d_\phi}$ is a polynomial goes as follows. Firstly, they show the following proposition, which shows that the action of any (k, k)-set partition can be described by a polynomial of the desired form. They then show that $\overline{\mathrm{PAT}^d_\phi}$ can be decomposed into (k, k)-set partitions in a nice way, which follows the logic that led us to equation (5.1). In this section, we focus on the first part of this argument, which explains why the actions of (k, k)-set partitions are characterized by polynomials.

Proposition 5.1 (2.7 in Gaetz and Ryba (2021)). *Let P be a fixed (k, k)-set partition. For a permutation $\sigma \in S_n$, let m_i denote the number of i-cycles in σ. Then*

$$\mathrm{tr}_{V^{\otimes k}}(\Phi_{k,n}(P)\psi(\sigma))$$

is a polynomial of degree at most k in the variables n, $m_1, m_2, \ldots m_k$, where n has degree 1 and m_i has degree i.

Proof. Note that

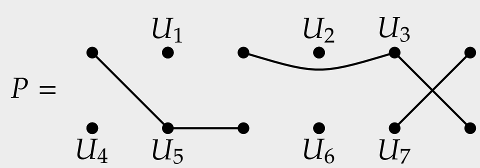
$$\begin{aligned}\mathrm{tr}_{V^{\otimes k}}(\Phi_{k,n}(P)\psi(\sigma)) &= \sum_{I\in[n]^k} \langle v_I, \Phi_{k,n}(P)(\psi(\sigma)v_I)\rangle \\ &= \sum_{I\in[n]^k} \left\langle v_I, \Phi_{k,n}(P)(v_{\sigma(I)})\right\rangle .\end{aligned}$$

By proposition 4.10, the coefficient of v_I in $\Phi_{k,n}(P)(v_{\sigma(I)})$ is 1 if the coloring of P by $(C, C') = (I, \sigma(I))$ is good, and 0 otherwise. Hence,

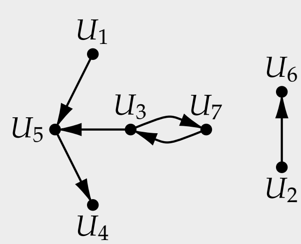
$$\begin{aligned}&\mathrm{tr}_{V^{\otimes k}}(\Phi_{k,n}(P)\psi(\sigma)) \\ &\qquad = \#\{I \in [n]^k : \text{coloring of } P \text{ by } (C, C') = (I, \sigma(I)) \text{ is good}\}.\end{aligned}$$

In order to count such colorings, we can transform the question into a question about graphs. Let $U_1, U_2, \ldots, U_m$ be the blocks in the set partition P. Construct a directed multigraph $G(P)$ with a vertex corresponding to each block U_i, and a directed edge from U_i to U_j if there is some integer $l \in [k]$ such that $l \in U_i$ and $l' \in U_j$.

Example 5.1. Consider the following $(6, 6)$-set partition with the blocks labeled U_1 through U_7.



Then $G(P)$ is the graph shown below.

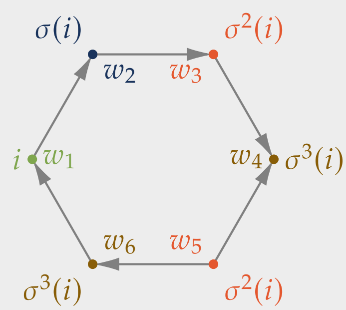


Note that if the coloring of P by $(C, C') = (I, \sigma(I))$ is good, then each block U_i is monochromatic, so we can think of assigning a color to each block rather than to each element of $\{1, \dots, k, 1', \dots, k'\}$ individually. Note that if $l \in U_i$ and l' in U_j, then the color assigned to U_j is σ of the color assigned to U_i. Equivalently, the color assigned to U_i is σ^{-1} of the color assigned to U_j. In other words, if $U_i \to U_j$ in the graph $G(P)$, then the color of either U_i or U_j determines the color of the other. Hence, the color of a single block in $G(P)$ determines the colors of all other blocks in its connected component.

Next, we determine how many ways there are to color a single connected component. We will call a sequence of vertices $(w_1, \dots, w_l)$ in $G(P)$ a *weak cycle* if $w_1 = w_k$, no other vertices are repeated, and for each $1 \le i \le k-1$, either $w_i \longrightarrow w_{i+1}$ or $w_i \longleftarrow w_{i+1}$. In other words, a weak cycle is a cycle in $G(P)$ where one is allowed to traverse any edge regardless of its direction. For a weak cycle $W = (w_1, \dots, w_l)$, define the *length* $L(W)$ to be the number of edges traversed with their orientation (i.e. $w_i \longrightarrow w_{i+1}$) minus the number of edges traversed against their orientation (i.e. $w_i \longleftarrow w_{i+1}$).

Note that $w_i \longrightarrow w_{i+1}$ implies the color assigned to w_{i+1} is σ of the color assigned to w_i, whereas $w_i \longleftarrow w_{i+1}$ implies the color assigned to w_{i+1} is σ^{-1} of the color assigned to w_i. If $i \in [n]$ is the color assigned to w_1, then following the entire weak cycle implies $\sigma^{L(W)}(i) = (i)$.

Example 5.2. A weak cycle with length 4. If w_1 is assigned color i, following the cycle around yields the condition $\sigma^4(i) = i$.



For a connected component C in $G(P)$, let $L_{\text{gcd}}(C)$ be the (positive) greatest common divisor of $L(W)$ for all weak cycles W in C. Let $L_{\text{gcd}}(C) = 0$ if C has no weak cycles. By our reasoning above, a vertex in C can be assigned any color i such that $\sigma^{L_{\text{gcd}}}(i) = i$. The color of any one vertex in C determines the colors of all other vertices in C. Hence, the number of ways to assign a color to the connected component C is the number of $i \in [n]$ such that $\sigma^{L_{\text{gcd}}}(i) = i$, which is

$$\sum_{d|L_{\text{gcd}}} dm_d.$$

Notice that if $L_{\text{gcd}} = 0$ then $p_{L_{\text{gcd}}}$ reduces to

$$\sum_d dm_d = n,$$

which is consistent with the fact that if $L_{\text{gcd}} = 0$, then that there are no restrictions on the color assigned to a vertex in that connected component. Inspired by this, we define

$$p_l(m_1, m_2, \dots, m_n) = \sum_{d|l} dm_d.$$

The total number of colorings of $G(P)$ is the product of $p(L_{\text{gcd}}(C))$ over all connected C components in G, so

$$\text{tr}_{V^{\otimes k}}(\Phi_{k,n}(P)\psi(\sigma)) = \prod_C p_{L_{\text{gcd}}(C)}(m_1, m_2, \dots, m_n), \tag{5.2}$$

where the product ranges over all connected components C in $G(P)$. This is clearly a polynomial in the variables n and $m_1, \dots, m_n$. We now need to prove that this polynomial is of the claimed form.

Note that the graph G has at exactly k edges, since each $l \in [k]$ contributes one edge. Hence $L_{\text{gcd}} \leq k$ for each connected component. This shows that if $i > k$, m_i does not appear in the polynomial. Moreover, if we consider m_i to be of degree i, then $\deg p_l = l$. This tells us that

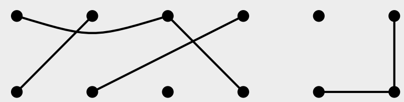
$$\deg \prod_C p_{L_{\text{gcd}}(C)}(m_1, m_2, \dots) = \sum_C L_{\text{gcd}}(C) \leq k,$$

since $L_{\text{gcd}}(C)$ is less than or equal to the number of edges in C, and there are k edges in total. □

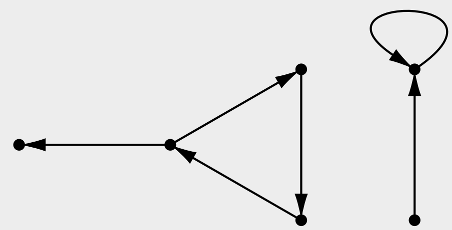
Although we don't have a closed form for computing the polynomial corresponding to a given partition P, equation (5.2) does give us a procedure for computing it. That is, we construct the graph $G(P)$, compute $L_{\text{gcd}}(C)$ of each connected component, compute the corresponding polynomial $p_{L_{\text{gcd}}}(C)$, and take the product of all of these polynomials over all connected components.

Example 5.3. We will compute the polynomial corresponding to the $(6, 6)$-partition set partition

$P =$



The corresponding graph $G(P)$ is



Call the left connected component C_1 and the right C_2. Note that C_1 has one weak cycle of length 3 and C_2 has one weak cycle of length 1, meaning $L_{\text{gcd}}(C_1) = 3$ and $L_{\text{gcd}}(C_2) = 1$. Thus for any permutation $\sigma \in S_n$, if m_i denotes the number of i-cycles in σ then

$$\begin{aligned} \text{tr}_{V^{\otimes k}}(\Phi_{n,6}(P)\psi(\sigma)) &= p_3(m_1, m_2, \ldots, m_6)p_1(m_1, m_2, \ldots, m_6) \\ &= (m_1 + 3m_3)(m_1) = m_1^2 + 3m_1m_3. \end{aligned}$$

5.2 Permutation statistic polynomials

As we saw in the previous section, traces of elements of the partition algebra can be described by polynomials in n and $m_1, \ldots, m_k$. We wish to show the same is true of moments of pattern statistics. In this section, we describe how to decompose these moments of statistics into traces of (k, k)-set partitions. We begin by extending the notion of a classical pattern (definition 2.12) to non-bijective functions.

Definition 5.2. Let $I = i_1 i_2 \ldots i_k \in [n]^k$ be a word of length k and let $\alpha : [k] \to [k]$. We say I is *α-sorted* if for any $1 \le a, b \le k$, $i_a < i_b$ if and only if $\alpha(a) < \alpha(b)$.

Just as we do with permutations, it will be helpful to denote these generalized patterns α in one-line notation. We will represent the function

$\alpha\colon [k] \to [k]$ by the word $\alpha_1\alpha_2\ldots\alpha_k$, where α_i denotes $\alpha(i)$. For example, if $k = 4$ and α is the function defined by $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 1$, and $\alpha(4) = 3$, we denote α by simply 2113. As an example, the words 2113, 3115, and 2116 are α-sorted, but 1112 and 4125 are not. Notice that if α happens to be bijective, then this definition coincides with the notion of an occurrence of the classical pattern α.

Definition 5.3 (Gaetz and Ryba (2021))**.** Let $\alpha\colon [k] \to [k]$. Define the linear map $E_\alpha\colon \mathbb{C} \to V^{\otimes k}$ by

$$E_\alpha(1) = \sum_{\substack{I\in[n]^k,\\ I \text{ is } \alpha\text{-sorted}}} v_I$$

It follows that $E_\alpha^\top\colon V^{\otimes k} \to \mathbb{C}$ is given by

$$E_\alpha^\top(v_I) = \begin{cases} 1 & I \text{ is } \alpha\text{-sorted,} \\ 0 & \text{otherwise.} \end{cases}$$

These linear maps allow us to reinterpret a pattern statistic as the trace of a linear operator.

Proposition 5.4 (3.2 in Gaetz and Ryba (2021))**.** *For any pattern $\phi \in S_k$ and any permutation $\sigma \in S_n$,*

$$\textsc{Pat}_\phi(\sigma) = \mathrm{tr}_{\mathbb{C}}(E_\phi^\top \psi(\sigma) E_{\mathrm{id}}),$$

where $\mathrm{id}\colon [k] \to [k]$ *is the identity map.*

Before presenting a proof, we should discuss what the above equation means. As defined in the previous section, $\psi(\sigma)$ is a linear operator from $V^{\otimes k}$ to itself given by the diagonal action of S_n on $V^{\otimes k}$. Hence the product $E_\phi^\top \psi(\sigma) E_{\mathrm{id}}$ is a linear map from $\mathbb{C}$ to $\mathbb{C}$. The only linear maps from $\mathbb{C}$ to itself are those that multiply every complex number by some constant, and we write $\mathrm{tr}_{\mathbb{C}}$ to extract that constant.

Proof. Note that $\mathrm{tr}_{\mathbb{C}}(E_\phi^\top \psi(\sigma) E_{\mathrm{id}}) = E_\phi^\top \psi(\sigma) E_{\mathrm{id}}(1)$. By definition, $E_{\mathrm{id}}(1)$ is the sum of v_I for all words $I = (i_1, i_2, \ldots, i_k)$ with $i_1 < i_2 < \cdots < i_k$. A word I of this form is an occurrence of the pattern ϕ in σ if and only if $\sigma(I)$ is ϕ-sorted. That is to say, $E_\phi^\top(v_{\sigma(I)})$ is 1 if I is an occurrence of ϕ in σ and 0 otherwise. Note that $E_\phi^\top(v_{\sigma(I)}) = E_\phi^\top \psi(\sigma)(v_I)$. Therefore, the total number of occurrences of ϕ in σ is given by summing $E_\phi^\top \psi(\sigma)(v_I)$ over all increasing words I, which is precisely $E_\phi^\top \psi(\sigma) E_{\mathrm{id}}(1) = \mathrm{tr}_{\mathbb{C}}(E_\phi^\top \psi(\sigma) E_{\mathrm{id}})$. □

As a consequence of this proposition, $\textsc{Inv}(\sigma) = \mathrm{tr}_{\mathbb{C}}(E_{21}^{\top}\psi(\sigma)E_{12})$. The following proposition characterizes the way these E_α maps interact with tensor products.

Proposition 5.5 (3.3 in Gaetz and Ryba (2021))**.** *Given $\alpha\colon [k_1] \to [k_1]$ and $\beta\colon [k_2] \to [k_2]$,*

$$E_\alpha \otimes E_\beta = \sum_\gamma E_\gamma$$

where the sum is over all $\gamma\colon [k_1+k_2] \to [k_1+k_2]$ such that $\gamma_1\gamma_2\dots\gamma_{k_1}$ is α-sorted, $\gamma_{k_1+1}\gamma_{k_1+2}\dots\gamma_{k_1+k-2}$ is β-sorted, and the image of γ is of the form $\{1,\dots,r\}$ for some r.

Example 5.4. This proposition gives us the following decompositions, assuming $n \geq 4$:

$$\begin{aligned} E_{12} \otimes E_{12} &= E_{1212} + E_{1213} + E_{1223} + E_{1234} + E_{1312} + E_{1323} \\ &\quad + E_{1324} + E_{1423} + E_{2312} + E_{2313} + E_{2314} + E_{2413} + E_{3412}, \\ E_{21} \otimes E_{21} &= E_{2121} + E_{2131} + E_{2132} + E_{2143} + E_{3121} + E_{3132} \\ &\quad + E_{3142} + E_{4132} + E_{3221} + E_{3231} + E_{3241} + E_{4231} + E_{4321}. \end{aligned}$$

Applying proposition 5.5 repeatedly yields the following useful corollary.

Corollary 5.6. *Let $\alpha : [k] \to [k]$ and $d \geq 1$. Then*

$$E_\alpha^{\otimes d} = \sum_\gamma E_\gamma$$

where the sum is over all $\gamma : [dk] \to [dk]$ such that the words

$$\gamma_1\gamma_2\cdots\gamma_k, \quad \gamma_{k+1}\gamma_{k+2}\cdots\gamma_{2k}, \quad \cdots \quad , \gamma_{(d-1)k+1}\gamma_{(d-1)k+2}\cdots\gamma_{dk},$$

are all α-sorted, and the image of γ is of the form $\{1,\dots,r\}$ for some r.

Given $\alpha\colon [k] \to [k]$ and $d \geq 1$, let $\Gamma(\alpha, d)$ denote the set of all $\gamma\colon [dk] \to [dk]$ that meet the conditions of corollary 5.6. That is,

$$\Gamma(\alpha,d) := \left\{\gamma\colon [dk] \to [dk] \;\middle|\; \begin{array}{c} \gamma_{jk+1}\gamma_{jk+1}\cdots\gamma_{(j+1)k} \text{ is } \alpha\text{-sorted for all } 0 \leq j \leq d-1, \\ \operatorname{im}\gamma = \{1,\dots,r\} \text{ for some } r. \end{array}\right\}.$$

This notation allows us to write

$$E_\alpha^{\otimes d} = \sum_{\gamma\in\Gamma(\alpha,d)} E_\gamma.$$

We need one more technical result before we are able to prove the the main theorem of Gaetz and Ryba (2021).

Theorem 5.7 (3.4 in Gaetz and Ryba (2021))**.** *Given* $\alpha, \beta\colon [k] \to [k]$*, the map* $T_{\alpha,\beta}\colon V^{\otimes k} \to V^{\otimes k}$ *given by*

$$T_{\alpha,\beta} = \frac{1}{n!}\sum_{g\in S_n} \psi(g)^{-1}E_\alpha E_\beta^T \psi(g)$$

may be written in the form

$$T_{\alpha,\beta} = \Phi_{k,n}\left(\sum_P a_P P\right)$$

where the sum ranges over all (k,k)*-set partitions, and each* $a_P \in \mathbb{C}$ *does not depend on* n*.*

As the following proof illustrates, the partition algebra shows up here for the exact same reason that it did in equation (5.1).

Proof. Since $T_{\alpha,\beta}$ is defined by averaging over S_n, it commutes with the S_n-action given by ψ:

$$\begin{aligned}
T_{\alpha,\beta}(\psi(\phi)v_I) &= \frac{1}{n!}\sum_{g\in S_n} \psi(g)^{-1}E_\alpha E_\beta^T \psi(g)\psi(\phi)v_I \\
&= \frac{1}{n!}\sum_{g\in S_n} \psi(g)^{-1}E_\alpha E_\beta^T \psi(g\phi)v_I \\
&= \frac{1}{n!}\sum_{h\in S_n} \psi(h\phi^{-1})^{-1}E_\alpha E_\beta^T \psi(h)v_I \qquad \text{(reindexing by } h = g\phi) \\
&= \frac{1}{n!}\sum_{h\in S_n} \psi(\phi)\psi(h)^{-1}E_\alpha E_\beta^T \psi(h)v_I \\
&= \psi(\phi)T_{\alpha,\beta}(v_I).
\end{aligned}$$

Hence, $T_{\alpha,\beta} \in \mathrm{End}_{\psi(\mathbb{C}S_n)}(V^{\otimes}) = \Phi_{k,n}(P_k(n))$. Therefore we can write $T_{\alpha,\beta}$ in the form

$$T_{\alpha,\beta} = \Phi_{k,n}\left(\sum_P a_P(n) P\right).$$

It remains to be shown that these coefficients $a_P(n)$ actually don't depend on n. This will be easier to show in the $\{x_P\}$ basis, so let $b_P(n)$ be coefficients such that

$$\sum_P b_P(n)x_P = \sum_P a_P(n)P.$$

Let P be any (k,k)-set partition and let $I, J \in [n]^k$ be words of length k such that the coloring of P by $(C, C') = (J, I)$ is perfect. Then by proposition 4.10,

$$\left\langle v_J, \Phi_{k,n}(x_Q)v_I \right\rangle = \begin{cases} 1 & Q = P, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$b_P(n) = \left\langle v_I, \Phi_{k,n}\left(\sum_P b_P(n)x_P\right) v_J \right\rangle = \left\langle v_J, T_{\alpha,\beta} v_I \right\rangle.$$

By definition of $T_{\alpha,\beta}$,

$$\begin{aligned}
\left\langle v_J, T_{\alpha,\beta} v_I \right\rangle &= \frac{1}{n!}\sum_{g\in S_n} \left\langle v_J, \psi(g)^{-1} E_\alpha E_\beta^\top \psi(g) v_I \right\rangle \\
&= \frac{1}{n!}\sum_{g\in S_n} \left\langle \psi(g) v_J, E_\alpha E_\beta^\top \psi(g) v_I \right\rangle \\
&= \frac{1}{n!}\sum_{g\in S_n} \left\langle E_\alpha^\top v_{g(J)}, E_\beta^\top v_{g(I)} \right\rangle.
\end{aligned}$$

Note that $\left\langle E_\alpha^\top v_{g(J)}, E_\beta^\top v_{g(I)} \right\rangle$ is 1 if $g(I)$ is β-sorted and $g(J)$ is α-sorted, and 0 otherwise. Hence,

$$b_P(n) = \frac{1}{n!}\#\{g \in S_n : g(I) \text{ is } \beta\text{-sorted and } g(J) \text{ is } \alpha\text{-sorted}\}.$$

Finally, let L denote the set of elements of $[n]$ that appear in either I or J. The condition on g in the above equation only depends on what g does to elements of L, so it simplifies to

$$b_P(n) = b_P = \frac{1}{|L|!}\#\{g \in S_L : g(I) \text{ is } \beta\text{-sorted and } g(J) \text{ is } \alpha\text{-sorted}\},$$

where S_L denotes the set of permutations of the set L. This expression does not depend on n, which completes the proof. □

Given $\alpha, \beta \colon [k] \to [k]$, let $a_P^{\alpha,\beta}$ and $b_P^{\alpha,\beta}$ be the coefficients satisfying

$$T_{\alpha,\beta} = \Phi_{k,n}\left(\sum_P a_P^{\alpha,\beta} P\right) = \Phi_{k,n}\left(\sum_P b_P^{\alpha,\beta} x_P\right).$$

In the proof above, we found the following explicit formula for computing the coefficient $b_P^{\alpha,\beta}$:

$$b_P^{\alpha,\beta} = \frac{1}{|L|!} \#\left\{g \in S_L : g(I) \text{ is } \beta\text{-sorted and } g(J) \text{ is } \alpha\text{-sorted}\right\}. \tag{5.3}$$

Note that

$$\sum_P b_P^{\alpha,\beta} x_P = \sum_P a_P^{\alpha,\beta} P = \sum_P a_P^{\alpha,\beta}\left(\sum_{Q \geq P} x_Q\right) = \sum_Q \left(\sum_{P \leq Q} a_P^{\alpha,\beta}\right) x_Q,$$

and hence

$$b_P^{\alpha,\beta} = \sum_{Q \leq P} a_Q^{\alpha,\beta}. \tag{5.4}$$

Since we can compute the coefficients $b_P^{\alpha,\beta}$ using equation (5.3), this formula allows us to compute the coefficients $a_P^{\alpha,\beta}$ recursively.

We are finally able to prove the main theorem of Gaetz and Ryba (2021).

Proof of theorem 2.21. Let $\sigma \in S_n$ and let m_i denote the number of i-cycles in σ. Then

$$\begin{aligned}
\overline{\mathrm{P{\scriptsize AT}}_\phi^d}(\sigma) &= \frac{1}{n!}\sum_{g \in S_n} \mathrm{P{\scriptsize AT}}_\phi(g\sigma g^{-1})^d \\
&= \frac{1}{n!}\sum_{g \in S_n} \mathrm{tr}_{\mathbb{C}}(E_\phi^\top \psi(g\sigma g^{-1}) E_{\mathrm{id}})^d && \text{(by proposition 5.4)} \\
&= \frac{1}{n!}\sum_{g \in S_n} \mathrm{tr}_{\mathbb{C}}(E_\phi^\top \psi(g)\psi(\sigma)\psi(g)^{-1} E_{\mathrm{id}})^d \\
&= \frac{1}{n!}\sum_{g \in S_n} \mathrm{tr}_{V^{\otimes k}}(\psi(g)^{-1} E_{\mathrm{id}} E_\phi^\top \psi(g)\psi(\sigma))^d \\
&= \frac{1}{n!}\sum_{g \in S_n} \mathrm{tr}_{V^{\otimes dk}}(\psi(g)^{-1} E_{\mathrm{id}}^{\otimes d} (E_\phi^{\otimes d})^\top \psi(g)\psi(\sigma)).
\end{aligned}$$

By corollary 5.6, we have

$$E_{\mathrm{id}}^{\otimes d} = \sum_{\alpha \in \Gamma(\mathrm{id},d)} E_\alpha, \qquad \text{and} \qquad E_\phi^{\otimes d} = \sum_{\beta \in \Gamma(\phi,d)} E_\beta.$$

Hence, the expression expands further as

$$\begin{aligned}
\overline{\mathrm{P{\scriptsize AT}}_\phi^d}(\sigma) &= \frac{1}{n!} \sum_{g \in S_n} \mathrm{tr}_{V^{\otimes dk}}(\psi(g)^{-1} E_{\mathrm{id}}^{\otimes d} (E_\phi^{\otimes d})^\top \psi(g)\psi(\sigma)) \\
&= \frac{1}{n!} \sum_{g \in S_n} \sum_{\alpha \in \Gamma(\mathrm{id},d)} \sum_{\beta \in \Gamma(\phi,d)} \mathrm{tr}_{V^{\otimes dk}}(\psi(g)^{-1} E_\alpha E_\beta^\top \psi(g)\psi(\sigma)) \\
&= \frac{1}{n!} \sum_{\alpha \in \Gamma(\mathrm{id},d)} \sum_{\beta \in \Gamma(\phi,d)} \mathrm{tr}_{V^{\otimes dk}} \left(\left(\sum_{g \in S_n} \psi(g)^{-1} E_\alpha E_\beta^\top \psi(g) \right) \psi(\sigma) \right) \\
&= \sum_{\alpha \in \Gamma(\mathrm{id},d)} \sum_{\beta \in \Gamma(\phi,d)} \mathrm{tr}_{V^{\otimes dk}}(T_{\alpha,\beta}\psi(\sigma)).
\end{aligned}$$

Recall that by theorem 5.7,

$$T_{\alpha,\beta} = \Phi_{dk,n}\left(\sum_P a_P^{\alpha,\beta} P \right).$$

This yields

$$\begin{aligned}
\overline{\mathrm{P{\scriptsize AT}}_\phi^d}(\sigma) &= \sum_{\alpha \in \Gamma(\mathrm{id},d)} \sum_{\beta \in \Gamma(\phi,d)} \mathrm{tr}_{V^{\otimes dk}} \left(\Phi_{dk,n}\left(\sum_P a_P^{\alpha,\beta} P \right) \psi(\sigma) \right) \\
&= \sum_{\alpha \in \Gamma(\mathrm{id},d)} \sum_{\beta \in \Gamma(\phi,d)} \sum_P a_P^{\alpha,\beta} \, \mathrm{tr}_{V^{\otimes dk}}(\Phi_{dk,n}(P)\psi(\sigma)). \qquad (5.5)
\end{aligned}$$

By proposition 5.1, $\mathrm{tr}_{V^{\otimes k}}(\Phi_{dk,n}(P)\psi(\sigma))$ is a polynomial of degree at most dk in n, $m_1, \ldots, m_{dk}$, where m_i has degree i. Since $\overline{\mathrm{P{\scriptsize AT}}_\phi^d}(\sigma)$ is a linear combination of these polynomials, it itself is a polynomial of this form. □

Equation (5.5) actually gives an explicit formula for the polynomial describing $\overline{\mathrm{P{\scriptsize AT}}_\phi^d}$. Recall that equations 5.3 and 5.4 allow us to compute the coefficients $a_P^{\alpha,\beta}$, and the polynomials describing $\mathrm{tr}_{V^{\otimes dk}}(\Phi_{dk,n}(P)\psi(\sigma))$ can be computed from equation (5.2) as in example 5.1

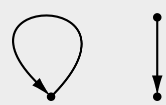
Example 5.5. We will use this formula to compute $\overline{\textsc{Pat}^1_{(2-1)}} = \overline{\textsc{Inv}}$. We have

$$\Gamma(12, 1) = \{12\} \qquad \text{and} \qquad \Gamma(21, 1) = \{21\},$$

so equation (5.5) simplifies to just

$$\overline{\textsc{Inv}}(\sigma) = \sum_P a_P^{12,21} \operatorname{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$$

where the sum ranges over all $(2,2)$-set partitions. For each P, $\operatorname{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$ is some polynomial in n, m_1 and m_2, where m_1 and m_2 are the number of 1- and 2-cycles in σ, respectively. We can compute these polynomials using equation (5.2). For example, $P = (\,|\;\;\colon)$ has corresponding graph $G(P)$ that looks like



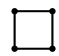
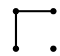
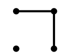
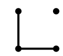
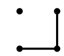
which yields the polynomial $\operatorname{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma)) = nm_1$.

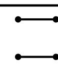
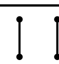
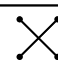
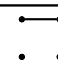
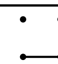
We can compute the coefficients $b_P^{12,21}$ via equation (5.3). For example, consider $P = (\,|\;\;\colon)$ as above. The words $I = 12$ and $J = 13$ make the coloring of P by $(C, C') = (I, J)$ perfect, so we find that

$$b_P^{12,21} = \frac{1}{6}\#\{g \in S_3 : g(12) \text{ is 21-sorted and } g(13) \text{ is 12-sorted}\}.$$

The only permutation $g \in S_3$ that meets these conditions is 213, so $b_P^{12,21} = 1/6$.

Once we have computed the values $b_P^{12,21}$ for all P, we can compute the coefficients $a_P^{12,21}$ using equation (5.4). Figure 5.1 lists the values of $\operatorname{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$, $b_P^{12,21}$ and $a_P^{12,21}$ for all $(2,2)$ set partitions P. Using

P					
$\mathrm{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$	m_1	m_1	m_1	m_1	m_1
$b_P^{12,21}$	0	0	0	0	0
$a_P^{12,21}$	0	0	0	0	0

P					
$\mathrm{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$	n	m_1^2	m_1+2m_2	n	n
$b_P^{12,21}$	0	0	1/2	0	0
$a_P^{12,21}$	1/4	−1/12	1/12	−1/4	−1/4

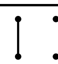
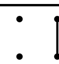
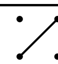
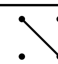
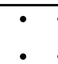
P					
$\mathrm{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$	$m_1 n$	$m_1 n$	n	n	n^2
$b_P^{12,21}$	1/6	1/6	1/3	1/3	1/4
$a_P^{12,21}$	−1/12	−1/12	1/12	1/12	1/4

Figure 5.1 Values of $\mathrm{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))$, $a_P^{12,21}$ and $b_P^{12,21}$ for all $(2,2)$-set partitions P.

these values, we find

$$\begin{aligned}\overline{\mathrm{Inv}}(\sigma) &= \sum_P a_P^{12,21}\,\mathrm{tr}_{V^{\otimes 2}}(\Phi_{2,n}(P)\psi(\sigma))\\ &= \frac{1}{4}(n) - \frac{1}{12}(m_1^2) + \frac{1}{12}(m_1+2m_2) - \frac{1}{4}(n) - \frac{1}{4}(n) - \frac{1}{12}(m_1 n)\\ &\qquad - \frac{1}{12}(m_1 n) + \frac{1}{12}(n) + \frac{1}{12}(n) + \frac{1}{4}(n^2)\\ &= \frac{3n^2 - n - m_1^2 - 2m_1 n + 2m_2}{12},\end{aligned}$$

which agrees with theorem 2.19.

We can also apply this formula to compute $\overline{\mathrm{Inv}^2} = \overline{\mathrm{Pat}^2_{(2-1)}}$. As we saw in example 5.4, $E_{12} \otimes E_{12}$ and $E_{21} \otimes E_{21}$ each decompose into a sum of 13 terms of the form E_γ where $\gamma\colon [4] \to [4]$. In other words, $|\Gamma(12,2)| = |\Gamma(21,2)| = 13$. Moreover, there are 4140 distinct $(4,4)$-set partitions. Hence, there are $13 \cdot 13 \cdot 4140 = 699660$ terms to compute in the triple sum in equation (5.5). Thus, evaluating the formula by hand is impractical, but is very feasible to do on a computer. Running the calculation in Mathematica, I arrived at the following new result.

Proposition 2.22 (S.). *Let $\sigma \in S_n$ be a permutation, and let m_i denote the number of i-cycles in σ. Then*

$$\begin{aligned}\overline{\textsc{Inv}^2}(\sigma) = \frac{1}{720}(&5m_1^4 + 20m_1^3 n - 14m_1^3 - 12m_1^2 m_2 + 50m_1^2 n^2 - 90m_1^2 n \\ &- 25m_1^2 - 24m_1 m_2 n + 12m_1 m_2 - 24m_1 m_3 + 60m_1 n^3 - 126m_1 n^2 \\ &+ 94m_1 n + 98m_1 + 60m_2^2 - 20m_2 n^2 + 108m_2 n - 124m_2 \\ &- 24m_3 n - 48m_3 - 24m_4 + 45n^4 - 130n^3 + 111n^2 - 98n).\end{aligned}$$

5.3 Character polynomials

Since $\overline{\textsc{Inv}^2}$ is a class function, it can be written as a linear combination of irreducible characters of S_n. In this section, we compute that decomposition.

Note that since Inv is 2-local, corollary 2.18 implies $\textsc{Inv}^2$ is 4-local. Using this fact, it follows that for any $\lambda \vdash n$ with $\lambda_1 < n-4$,

$$\begin{aligned}\left\langle \overline{\textsc{Inv}^2}, \chi^\lambda \right\rangle &= \left\langle \textsc{Inv}^2, \chi^\lambda \right\rangle && \text{(by proposition 2.24)} \\ &= \sum_{\sigma \in S_n} \textsc{Inv}^2(\sigma)\chi^\lambda(\sigma) \\ &= \operatorname{tr}\left(\sum_{\sigma \in S_n} \textsc{Inv}^2(\sigma)\rho^\lambda(\sigma)\right) \\ &= \operatorname{tr}\left(\rho^\lambda(\textsc{Inv}^2)\right) \\ &= \operatorname{tr}(0) && \text{(by theorem 2.23)} \\ &= 0.\end{aligned}$$

Hence, the only characters that appear in the decomposition of $\overline{\textsc{Inv}^2}$ are those corresponding to partitions $\lambda \vdash n$ with $\lambda_1 \geq n-4$.

Since irreducible characters on S_n are class functions, they only depend on the cycle structure – *i.e.* the number of 1-cycles, 2-cycles, ... – of a given permutation. Hence if m_i denotes the number of i-cycles in σ, then

$$\chi^\lambda(\sigma) = X^\lambda(m_1, m_2, \dots, m_n)$$

for some function X^λ.

The Murnaghan-Nakayama rule is a combinatorial rule for evaluating irreducible characters of S_n at specific permutations (see Sagan (2001)). Using it, we can compute examples of these functions X^λ.

$$\begin{aligned}
\chi^{(n)}(\sigma) &= 1\\
\chi^{(n-1,1)}(\sigma) &= m_1 - 1\\
\chi^{(n-2,2)}(\sigma) &= m_2 + \frac{1}{2}m_1^2 - \frac{3}{2}m_1\\
\chi^{(n-2,1,1)}(\sigma) &= -m_2 + \frac{1}{2}m_1^2 - \frac{3}{2}m_1 + 1\\
\chi^{(n-3,3)}(\sigma) &= m_3 + m_1m_2 + \frac{1}{6}m_1^3 - m_2 - m_1^2 + \frac{5}{6}m_1\\
&\vdots
\end{aligned}$$

Surprisingly, these functions turn out to be polynomials in variables $m_1, \dots m_n$. Moreover, these polynomials have a form that looks very similar to the polynomials that theorem 2.21 asserts the existence of. The following theorem will make this connection more precise.

Given a partition λ and an integer $n \geq |\lambda| + \lambda_1$, let $\lambda_{[n]}$ denote the partition $(n - |\lambda|, \lambda_1, \lambda_2, \dots) \vdash n$. For example, if $\lambda = (2, 1, 1)$, then $\lambda_{[n]} = (n-4, 2, 1, 1)$. Using this notation, the following theorem characterizes character polynomials.

Theorem 5.8 (see Gaetz and Pierson (2023)). *Let $\sigma \in S_n$ and let m_i denote the number of i-cycles in σ. Let $\lambda = (\lambda_1, \lambda_2, \dots)$. Then for any $n \geq |\lambda| + \lambda_1$ character $\chi^{\lambda_{[n]}}$ is a polynomial in $m_1, \dots, m_{|\lambda|}$ of degree at most $|\lambda|$ where m_i has degree i.*

Note that the constraints on the degrees of these polynomials (i.e. that m_i is considered to be degree i) are precisely the same as in theorem 2.21. The only difference is that the polynomials described by theorem 2.21 may depend on n, while these character polynomials do not.

We compute these character polynomials for all partitions λ with $\lambda_1 \geq n - 4$, since we know these are the only characters that can appear in the

λ	Polynomial for χ^λ
(n)	1
$(n-1,1)$	$m_1 - 1$
$(n-2,2)$	$m_2 - \frac{1}{2}m_1^2 - \frac{3}{2}m_1$
$(n-2,1,1)$	$-m_2 + \frac{1}{2}m_1^2 - \frac{3}{2}m_1 + 1$
$(n-3,3)$	$m_3 + m_1m_2 - m_2 + \frac{m_1^3}{6} - m_1^2 - \frac{5}{6}m_1$
$(n-3,2,1)$	$-m_3 + \frac{1}{3}m_1^3 - 2m_1^2 + \frac{8}{3}m_1$
$(n-3,1,1,1)$	$m_3 - m_1m_2 + m_2 + \frac{1}{6}m_1^3 - m_1^2 + \frac{11}{6}m_1 - 1$
$(n-4,4)$	$m_4 + m_1m_3 - m_3 + \frac{1}{2}m_2^2 + \frac{1}{2}m_1^2m_2 - \frac{3}{2}m_1m_2 + \frac{m_2}{2} + \frac{1}{24}m_1^4 - \frac{5}{12}m_1^3 + \frac{23}{24}m_1^2 - \frac{7}{12}m_1$
$(n-4,3,1)$	$-m_4 - \frac{1}{2}m_2^2 + \frac{1}{2}m_1^2m_2 - \frac{3}{2}m_1m_2 + \frac{3}{2}m_2 + \frac{1}{8}m_1^4 - \frac{5}{4}m_1^3 + \frac{27}{8}m_1^2 - \frac{9}{4}m_1$
$(n-4,2,2)$	$-m_1m_3 + m_3 + m_2^2 - 2m_2 + \frac{1}{12}m_1^4 - \frac{5}{6}m_1^3 + \frac{29}{12}m_1^2 - \frac{5}{3}m_1$
$(n-4,2,1,1)$	$m_4 - \frac{1}{2}m_2^2 - \frac{1}{2}m_1^2m_2 + \frac{3}{2}m_1m_2 + \frac{1}{2}m_2 + \frac{1}{8}m_1^4 - \frac{5}{4}m_1^3 + \frac{31}{8}m_1^2 - \frac{15}{4}m_1$
$(n-4,1,1,1,1)$	$-m_4 + m_1m_3 - m_3 + \frac{1}{2}m_2^2 + \frac{1}{2}m_1^2m_2 + \frac{3}{2}m_1m_2 - \frac{3}{2}m_2 + \frac{1}{24}m_1^4 - \frac{5}{12}m_1^3 + \frac{35}{24}m_1^2 - \frac{25}{12}m_1 + 1$

Table 5.1 Character polynomials for all partitions $\lambda = (\lambda_1, \lambda_2 \ldots) \vdash n$ such that $\lambda_1 \geq n-4$

decomposition of $\overline{\text{Inv}^2}$. These polynomials are listed in table 5.1. With all of these polynomials in hand, we can readily find the character decomposition of $\overline{\text{Inv}^2}$ using the polynomial in proposition 2.22. This yields the following decomposition.

Corollary 5.9. *If $\sigma \in S_n$ and $n \geq 6$,*

$$\begin{aligned}
\overline{\mathrm{Inv}^2}(\sigma) = &\left(\frac{n^4}{16} - \frac{7n^3}{72} + \frac{5n^2}{48} - \frac{5n}{72}\right)\chi^{(n)}(\sigma) + \left(-\frac{n^3}{12} + \frac{n^2}{30} + \frac{11n}{60} + \frac{1}{15}\right)\chi^{(n-1,1)}(\sigma) \\
&+ \left(\frac{n^2}{18} + \frac{n}{10} + \frac{2}{45}\right)\chi^{(n-2,2)}(\sigma) + \left(-\frac{n^2}{12} + \frac{3n}{20} + \frac{1}{15}\right)\chi^{(n-2,1,1)}(\sigma) \\
&+ \left(\frac{n}{15} + \frac{2}{15}\right)\chi^{(n-3,2,1)}(\sigma) + \left(\frac{n}{30} + \frac{1}{30}\right)\chi^{(n-3,1,1,1)}(\sigma) \\
&+ \frac{1}{15}\chi^{(n-4,2,2)}(\sigma) + \frac{1}{30}\chi^{(n-4,1,1,1,1)}(\sigma).
\end{aligned}$$

Note that since the character polynomials don't depend on n, all of the dependence of $\overline{\mathrm{Inv}^2}$ on n is contained the coefficients. The condition that $n \geq 6$ is necessary so that all of the characters in this decomposition are defined. For instance, in order for the the character $\chi^{(n-4,2,2)}$ to exist, we need $(n-4,2,2)$ to be an integer partition, which imposes the restriction that $n \geq 6$. With this character decomposition, we can now analyze random walks on S_n.

5.4 Random walk results

Using the character decomposition of $\overline{\mathrm{Inv}^2}$ given in corollary 5.9, we can invoke theorem 2.26 to compute the expected value of Inv^2 sampled via a random walk. In particular, if p is the uniform distribution on transpositions, we find the following formula (assuming $n \geq 6$).

$$\begin{aligned}\mathbb{E}_p(\textsc{Inv}^2, t) &= \frac{n^4}{16} - \frac{7n^3}{72} + \frac{5n^2}{48} - \frac{5n}{72} \\ &+ \frac{1}{60}(n-1)\left(-5n^3 + 2n^2 + 11n + 4\right)\left(1 - \frac{2}{n-1}\right)^t \\ &+ \frac{1}{120}(n-1)(n-2)\left(-5n^2 + 9n + 4\right)\left(1 - \frac{4}{n-1}\right)^t \\ &+ \frac{1}{180}n(n-3)\left(5n^2 + 9n + 4\right)\left(1 - \frac{4}{n}\right)^t \\ &+ \frac{1}{180}(n+1)(n-1)(n-2)(n-3)\left(1 - \frac{6}{n-1}\right)^t \\ &+ \frac{1}{45}n(n-4)(n-2)(n+2)\left(1 - \frac{6}{n}\right)^t \\ &+ \frac{1}{720}(n-1)(n-2)(n-3)(n-4)\left(1 - \frac{8}{n-1}\right)^t \\ &+ \frac{1}{180}n(n-1)(n-4)(n-5)\left(1 - \frac{4(2n-3)}{(n-1)n}\right)^t.\end{aligned}$$

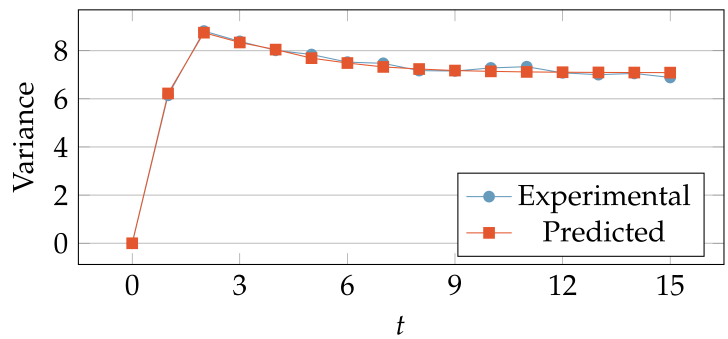
Using theorem 2.26, we can obtain the following formula for the expected number of inversions (see Hultman (2014)).

$$\begin{aligned}\mathbb{E}_p(\textsc{Inv}, t) = \frac{n(n-1)}{4} &- \frac{2(n+1)(n-1)}{12}\left(1 - \frac{2}{n-1}\right)^t \\ &- \frac{(n-1)(n-2)}{12}\left(1 - \frac{4}{n-1}\right)^t.\end{aligned}$$

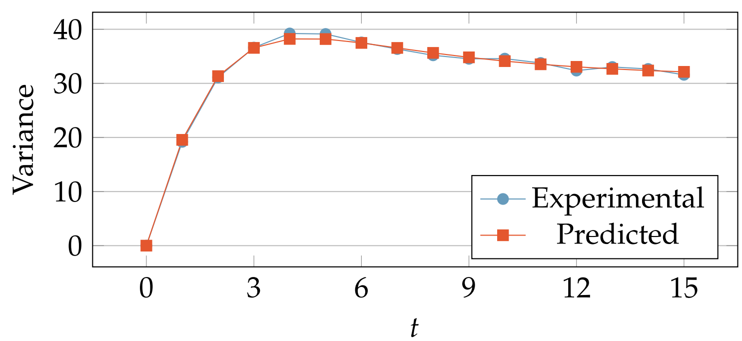
This allows us to find the following formula for the variance in the number of inversions in the product of t random transpositions.

$$\begin{aligned}\mathrm{Var}_p(\textsc{Inv}^2, t) &= \mathbb{E}_p(\textsc{Inv}^2, t) - \mathbb{E}_p(\textsc{Inv}, t)^2 \qquad (5.6)\\
&= \frac{1}{72}\left(2n^3+3n^2-5n\right) + \frac{1}{30}\left(n^3+2n^2-n-2\right)\left(1-\frac{2}{n-1}\right)^t\\
&\quad + \frac{1}{30}\left(n^3-2n^2-n+2\right)\left(1-\frac{4}{n-1}\right)^t\\
&\quad + \frac{1}{180}\left(5n^4-6n^3-23n^2-12n\right)\left(1-\frac{4}{n}\right)^t\\
&\quad + \frac{1}{180}\left(n^4-5n^3+5n^2+5n-6\right)\left(1-\frac{6}{n-1}\right)^t\\
&\quad + \frac{1}{45}\left(n^4-4n^3-4n^2+16n\right)\left(1-\frac{6}{n}\right)^t\\
&\quad + \frac{1}{720}\left(n^4-10n^3+35n^2-50n+24\right)\left(1-\frac{8}{n-1}\right)^t\\
&\quad + \frac{1}{180}\left(n^4-10n^3+29n^2-20n\right)\left(1-\frac{4(2n-3)}{(n-1)n}\right)^t\\
&\quad - \frac{1}{36}\left(n^4-3n^3+n^2+3n-2\right)\left(1-\frac{2(3n-7)}{(n-1)^2}\right)^t\\
&\quad - \frac{1}{36}\left(n^4-2n^2+1\right)\left(1-\frac{2}{n-1}\right)^{2t}\\
&\quad - \frac{1}{144}\left(n^4-6n^3+13n^2-12n+4\right)\left(1-\frac{4}{n-1}\right)^{2t}.\end{aligned}$$

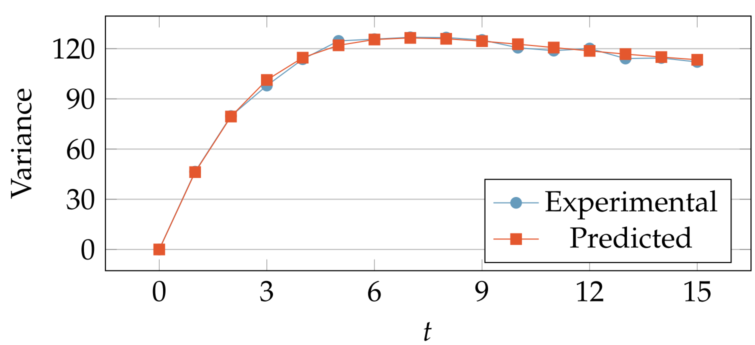
Figure 5.2 shows values output by this formula for various values of n and t alongside experimental data obtained by manually running the random walk 10000 times. In all of the plots, the two lines coincide almost exactly, indicating that equation (5.6) correctly computes the variance in the number of transpositions in the product of t random transpositions. The small discrepancies between the values in these plots are due to noise in the experimental data.



a. Variance in the number of inversions in the product of t random transpositions from S_6.



b. Variance in the number of inversions in the product of t random transpositions from S_{10}.



c. Variance in the number of inversions in the product of t random transpositions from S_{15}.

Figure 5.2 Values output by equation (5.6) plotted against experimental values from 10000 runs of the random walk, for $n = 6$ (top), $n = 10$ (middle) and $n = 15$ (bottom)

Chapter 6

Conclusion

6.1 Next directions

Broadly, our goal has been to compute moments of some statistic $s\colon S_n \to \mathbb{C}$ on permutations sampled via a random walk given by some probability distribution $p\colon S_n \to [0,1]$. The approach we employed in chapter 3 is only possible if both s and p are class functions. In chapter 5, we demonstrated that by using the theory of the partition algebra, one can compute moments of non class function statistics s, but this approach still requires that p be a class function. It would be nice to be able to drop this restriction as well, since it would allow us to analyze a larger class of random walks.

For example, Bousquet-Mélou (2009) computed the expected number of inversions in a product of t random *adjacent* transpositions. In other words, they computed the first moment of $\textsc{Inv}$ on permutations sampled via the random walk governed by the non-class function distribution

$$p(\sigma) = \begin{cases} 1/(n-1) & \sigma \text{ is an adjacent transposition,} \\ 0 & \text{otherwise.} \end{cases}$$

This calculation is considerably more involved than the analogous ones done by Hultman (2014) when considering class functions, and doesn't use many of the same tools. As such it's not clear how to extend the methods of this thesis to consider problems of this sort.

Secondly, the methods we used to compute the polynomial for $\overline{\textsc{Inv}^2}$ in proposition 2.22 can't extend very far. Recall equation (5.5), which was the basis for that computation:

$$\overline{\text{Pat}^d_\sigma}(\pi) = \sum_{\alpha\in\Gamma(\text{id},d)} \sum_{\beta\in\Gamma(\sigma,d)} \sum_P a_P^{\alpha,\beta} \operatorname{tr}_{V^{\otimes dk}}(\Phi_{dk,n}(P)\psi(\pi)). \tag{5.5}$$

Let's consider using this equation to compute the polynomial for $\overline{\text{Inv}^3} = \overline{\text{Pat}^3_{(2-1)}}$. The sets $\Gamma(12,3)$ and $\Gamma(21,3)$ each have 409 elements. And, the innermost sum ranges over all $(6,6)$-set partitions, of which there are 4213597. Hence, computing this polynomial in this way would require us to compute

$$409 \cdot 409 \cdot 4213597 = 704854719757$$

different terms. Even with computational tools, this becomes infeasible quickly. Clearly, in order to compute polynomials corresponding to higher moments or larger patterns, we need more efficient ways of computing these polynomials.

One potential method to compute these polynomials more effectively is some form of multivariate polynomial interpolation. Theorem 2.21 gives quite strong restrictions on what terms can appear in the polynomial $\overline{\text{Pat}^d_\sigma}$. If we could compute the value of $\overline{\text{Pat}^d_\sigma}$ at enough points, we could potentially figure out the coefficients on each of those terms by just solving a linear system. It seems as though this approach may be more computationally efficient than the one using equation (5.5), but I haven't had time to investigate it this year.

6.2 Open questions

We conclude with a list of questions to motivate future work. Firstly, we could continue to work toward the question that motivated this thesis: characterizing distributions of permutation statistics like Inv.

Question 6.1. *What does the distribution of* Inv *look like for a product of t random transpositions? In particular, what are its k^{th} moments for $k \geq 3$? What about other classical pattern statistics?*

Of course, this question is very open ended, as there might not be a nice simple answer to what this distribution "looks like." Still, working toward this question has unearthed numerous interesting insights and connections, so it seems fruitful to continue to push in this direction.

Stepping back a bit, recall theorem 2.20:

Theorem 2.20 (Fulman (1998)). *Let $\sigma \in S_n$ be a permutation, and let m_i be the number of i-cycles in σ. Then*

$$\overline{\mathrm{DES}}(\sigma) = \frac{n^2 - n + 2m_2 - m_1^2 + m_1}{2n}.$$

Hence, the statistic $\overline{\mathrm{DES}}$ is described by a simple rational function, and this function closely resembles the kinds of polynomials that arise for classical pattern statistics. This suggests that it might be possible to generalize theorem 2.21 to characterize moments of vincular pattern statistics. However, this generalization must be somewhat subtle, since theorem 2.20 demonstrates that vincular pattern statistics may not be described by polynomials, due to the n in the denominator.

Question 6.2. *Can theorem 2.21 be extended to vincular pattern statistics like* DES *in a natural way? In particular, is* $\overline{\mathrm{DES}^k}$ *always a rational function whose numerator is a polynomial in $n, m_1, \ldots, m_{2k}$ and whose denominator is a polynomial in n?*

I believe the key steps toward proving such a result would be finding suitable generalizations of proposition 5.4, proposition 5.5 and theorem 5.7 to the case of vincular pattern statistics.

In section 5.3, we wrote the statistic $\overline{\mathrm{INV}^2}$ as a linear combination of irreducible characters of S_n, where the coefficients are polynomials in n. The same can be done in general; there exist polynomials $a^{\lambda}_{\sigma,d}(n)$ such that for $n \geq 2dk$,

$$\overline{\mathrm{PAT}^d_\sigma} = \sum_{|\lambda| \leq dk} a^{\lambda}_{\sigma,d}(n) \chi^{\lambda[n]}.$$

See Gaetz and Ryba (2021) for a complete proof that such a decomposition always exists. The following question about these polynomials was posed by Gaetz and Pierson (2023).

Question 6.3. *If $\mathrm{id}_k : [k] \to [k]$ is the identity map, is $a^{\lambda}_{\mathrm{id}_k,1}(n)$ a nonnegative integer for all $n \geq 2k$ and λ with $|\lambda| \leq k$? In other words, does $\overline{\mathrm{PAT}_{\mathrm{id}_k}}$ always decompose into a nonnegative integer combination of irreducible characters of S_n?*

If this were the case, it would mean that $\overline{\mathrm{PAT}_{\mathrm{id}_k}}$ is the character of some (reducible) representation of S_n. An elegant way to prove the conjecture would be to construct an S_n-representation whose character is $\mathrm{PAT}_{\mathrm{id}_k}$.

In their paper, Gaetz and Pierson verified that the conjecture holds for the partitions $\lambda = \emptyset, (1), (2)$, and $(1, 1)$. With the methods of chapter 5, we can compute the polynomials $a^{\lambda}_{\mathrm{id}_k}(n)$ directly when k is small enough, which could shed light on the conjecture.

Appendix A

Fourier transforms

λ	$\rho^\lambda(\textsc{Fix})$	$\rho^\lambda(\textsc{Exc})$
(5)	$\begin{bmatrix}120\end{bmatrix}$	$\begin{bmatrix}240\end{bmatrix}$
$(4,1)$	$\begin{bmatrix}30&0&0&0\\0&30&0&0\\0&0&30&0\\0&0&0&30\end{bmatrix}$	$\begin{bmatrix}-15&-5\sqrt{3}&-\frac{5\sqrt{6}}{2}&-\frac{3\sqrt{10}}{2}\\5\sqrt{3}&-15&-\frac{15\sqrt{2}}{2}&-\frac{3\sqrt{30}}{2}\\\frac{5\sqrt{6}}{2}&\frac{15\sqrt{2}}{2}&-15&-3\sqrt{15}\\\frac{3\sqrt{10}}{2}&\frac{3\sqrt{30}}{2}&3\sqrt{15}&-15\end{bmatrix}$
$(3,2)$	$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$
$(3,1,1)$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$
$(2,2,1)$	$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$
$(2,1,1,1)$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$
$(1,1,1,1,1)$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$

Table A.1 Fourier transforms of the statistics Fɪx and Exc on S_5

λ	$\rho^{\lambda}(\mathrm{Des})$
(5)	$\begin{bmatrix} 240 \end{bmatrix}$
$(4,1)$	$\begin{bmatrix} -3 & -\sqrt{3} & -\frac{\sqrt{6}}{2} & -\frac{3\sqrt{10}}{2} \\ -3\sqrt{3} & -3 & -\frac{3}{\sqrt{2}} & -\frac{3\sqrt{30}}{2} \\ -3\sqrt{6} & -3\sqrt{2} & -3 & -3\sqrt{15} \\ -3\sqrt{10} & -\sqrt{30} & -\sqrt{15} & -15 \end{bmatrix}$
$(3,2)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(3,1,1)$	$\begin{bmatrix} -\frac{20}{3} & -\frac{5}{3\sqrt{2}} & -\frac{5\sqrt{6}}{2} & -\frac{\sqrt{5}}{30} & -\frac{\sqrt{10}}{6} & -\frac{8\sqrt{5}}{3} \\ -\frac{10\sqrt{2}}{3} & -\frac{5}{6} & -\frac{5\sqrt{3}}{2} & -\frac{\sqrt{15}}{6} & -\frac{\sqrt{5}}{6} & -\frac{4\sqrt{10}}{3} \\ -\frac{10\sqrt{6}}{3} & -\frac{5\sqrt{3}}{6} & -\frac{15}{2} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{15}}{6} & -\frac{4\sqrt{30}}{3} \\ -\frac{2\sqrt{30}}{3} & -\frac{\sqrt{15}}{6} & -\frac{3\sqrt{5}}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{4\sqrt{6}}{3} \\ -2\sqrt{10} & -\frac{\sqrt{5}}{2} & -\frac{3\sqrt{15}}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -4\sqrt{2} \\ -4\sqrt{5} & -\frac{\sqrt{10}}{2} & -\frac{3\sqrt{30}}{2} & -\frac{\sqrt{6}}{2} & -\frac{1}{\sqrt{2}} & -8 \end{bmatrix}$
$(2,2,1)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(2,1,1,1)$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$(1,1,1,1,1)$	$\begin{bmatrix} 0 \end{bmatrix}$

Table A.2 Fourier transform of the statistic Des on S_5

λ	$\rho^{\lambda}(\textsc{Inv})$
(5)	$\begin{bmatrix} 600 \end{bmatrix}$
(4,1)	$\begin{bmatrix} -6 & -6\sqrt{3} & -6\sqrt{6} & -6\sqrt{10} \\ -6\sqrt{3} & -18 & -18\sqrt{2} & -6\sqrt{30} \\ -6\sqrt{6} & -18\sqrt{2} & -36 & -12\sqrt{15} \\ -6\sqrt{10} & -6\sqrt{30} & -12\sqrt{15} & -60 \end{bmatrix}$
(3,2)	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
(3,1,1)	$\begin{bmatrix} -\frac{10}{3} & -\frac{5\sqrt{2}}{3} & -\frac{5\sqrt{6}}{3} & -\frac{\sqrt{30}}{3} & -\sqrt{10} & -2\sqrt{5} \\ -\frac{5\sqrt{2}}{3} & -\frac{5}{3} & -\frac{5\sqrt{3}}{3} & -\frac{\sqrt{15}}{3} & -\sqrt{5} & -\sqrt{10} \\ -\frac{5\sqrt{6}}{3} & -\frac{5\sqrt{3}}{3} & -5 & -\sqrt{5} & -\sqrt{15} & -\sqrt{30} \\ -\frac{\sqrt{30}}{3} & -\frac{\sqrt{15}}{3} & -\sqrt{5} & -1 & -\sqrt{3} & -\sqrt{6} \\ -\sqrt{10} & -\sqrt{5} & -\sqrt{15} & -\sqrt{3} & -3 & -3\sqrt{2} \\ -2\sqrt{5} & -\sqrt{10} & -\sqrt{30} & -\sqrt{6} & -3\sqrt{2} & -6 \end{bmatrix}$
(2,2,1)	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
(2,1,1,1)	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
(1,1,1,1,1)	$\begin{bmatrix} 0 \end{bmatrix}$

Table A.3 Fourier transform of the statistic Inv on S_5

λ	$\rho^\lambda(\textsc{Cyc}_2)$	$\rho^\lambda(\textsc{Cyc}_3)$
(5)	$\begin{bmatrix}60\end{bmatrix}$	$\begin{bmatrix}40\end{bmatrix}$
$(4,1)$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$
$(3,2)$	$\begin{bmatrix}12&0&0&0&0\\0&12&0&0&0\\0&0&12&0&0\\0&0&0&12&0\\0&0&0&0&12\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$
$(3,1,1)$	$\begin{bmatrix}-10&0&0&0&0&0\\0&-10&0&0&0&0\\0&0&-10&0&0&0\\0&0&0&-10&0&0\\0&0&0&0&-10&0\\0&0&0&0&0&-10\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$
$(2,2,1)$	$\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}-8&0&0&0&0\\0&-8&0&0&0\\0&0&-8&0&0\\0&0&0&-8&0\\0&0&0&0&-8\end{bmatrix}$
$(2,1,1,1)$	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}10&0&0&0\\0&10&0&0\\0&0&10&0\\0&0&0&10\end{bmatrix}$
$(1,1,1,1,1)$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0\end{bmatrix}$

Table A.4 Fourier transforms of the statistics $\textsc{Cyc}_2$ and $\textsc{Cyc}_3$ on S_5

λ	$\rho^{\lambda}(\text{PAT}_{(2-3-1)})$
(5)	$\begin{bmatrix} 200 \end{bmatrix}$
$(4,1)$	$\begin{bmatrix} -9 & -9\sqrt{3} & -9\sqrt{6} & -9\sqrt{10} \\ -3\sqrt{3} & -13 & -17\sqrt{2} & -7\sqrt{30} \\ 3\sqrt{6} & \sqrt{2} & -14 & -10\sqrt{15} \\ 9\sqrt{10} & 5\sqrt{30} & 2\sqrt{15} & -30 \end{bmatrix}$
$(3,2)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & -\sqrt{3} & -2\sqrt{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & -1 & -2\sqrt{2} \\ 0 & -2\sqrt{6} & 0 & -2\sqrt{2} & -8 \end{bmatrix}$
$(3,1,1)$	$\begin{bmatrix} \frac{10}{3} & \frac{5\sqrt{2}}{3} & \frac{5\sqrt{6}}{3} & \frac{\sqrt{30}}{3} & \sqrt{10} & 2\sqrt{5} \\ \frac{5\sqrt{2}}{3} & \frac{5}{3} & \frac{5\sqrt{3}}{3} & \frac{\sqrt{15}}{3} & \sqrt{5} & \sqrt{10} \\ \frac{10\sqrt{6}}{3} & \frac{10}{\sqrt{3}} & \frac{15}{2} & 2\sqrt{5} & \frac{3\sqrt{15}}{2} & \sqrt{30} \\ \frac{\sqrt{30}}{3} & \frac{\sqrt{15}}{3} & \sqrt{5} & 1 & \sqrt{3} & \sqrt{6} \\ 2\sqrt{10} & 2\sqrt{5} & \frac{3\sqrt{15}}{2} & 2\sqrt{3} & \frac{9}{2} & 3\sqrt{2} \\ 6\sqrt{5} & 3\sqrt{10} & 2\sqrt{30} & 3\sqrt{6} & 6\sqrt{2} & 6 \end{bmatrix}$
$(2,2,1)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & -1 \end{bmatrix}$
$(2,1,1,1)$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$(1,1,1,1,1)$	$\begin{bmatrix} 0 \end{bmatrix}$

Table A.5 Fourier transform of the statistic $\text{PAT}_{(2-3-1)}$ on S_5

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