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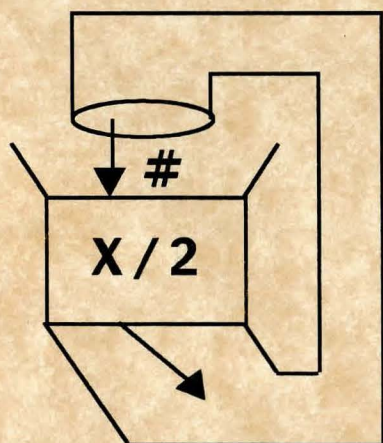
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Humanistic Mathematics Network Journal

Issue #18 November 1998



*"Do nondoing,
strive for nonstriving,
savor the flavorless,
regard the small as important,
make much of little,
repay enmity with virtue;
plan for difficulty when it is still easy,
do the great while it is still small.
The most difficult things in the world
must be done while they are easy;
the greatest things in the world
must be done while they are small."*

- from the Tao Te Ching by Lao Tzu

In this issue...

- Natural Math (p. 1)
- Humanistic teaching in elementary school (p. 10)
- Mathematical architecture (p. 35)

Invitation to Authors

Essays, book reviews, syllabi, poetry, and letters are welcomed. Your essay should have a title, your name and address, e-mail address, and a brief summary of content. In addition, your telephone number (not for publication) would be helpful.

If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text; these should be original photographs, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of your submission, double-spaced and preferably laser-printed, should be sent to:

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Essays and other communications may also be transmitted by electronic mail to the editor at AWHITE@HMC.EDU, or faxed to (909) 621-8366. The editor may be contacted at (909) 621-8867 if you have further questions.

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Cover

It's never too early: Many of the articles in this issue discuss humanistic mathematics with children in elementary grades, including Maria Droujkova's "Natural Math," John Eichinger's "Do the Great While It Is Still Small," and Kim Williams' "Architecture and Mathematics."

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Humanistic Mathematics Network Journal #18

November 1998

From the Editor

| | | | |
|---|-----------|--|-----------|
| Natural Math | 1 | Architecture and Mathematics: | 35 |
| <i>Maria Droujkova</i> | | An Introduction for Elementary and Middle School Children | |
| Do the Great While It Is Still Small: Humanistic Teaching in the Elementary School | 10 | <i>Kim Williams</i> | |
| <i>John Eichinger</i> | | Changing Ways of Thinking About Mathematics by Teaching Game Theory | 40 |
| The Legend of the Apple | 16 | <i>Ein-Ya Gura</i> | |
| <i>Raul A. Simon</i> | | Nine and One Third Circles of Rejoicing | 45 |
| Mathematics Found in Poetry | 17 | <i>Michael Capobianco</i> | |
| <i>Alexis Mann</i> | | Manifesto on Mathematical Education | 46 |
| Plastic Pebbles | 21 | <i>Saunders Mac Lane</i> | |
| <i>Virginie H. Mitchem</i> | | from <i>My Dance Is Mathematics: Poems From a Mathematics Teacher</i> | 47 |
| What Does Equality Mean?-- The Basque View | 22 | <i>JoAnne Grownney</i> | |
| <i>Marcia Ascher</i> | | In Future Issues... | |
| On Solving Equations, Negative Numbers, and Other Absurdities: Part II | 28 | | |
| <i>Ralph A. Raimi</i> | | | |

From Newsletter #1

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other and how they might come to better understand mathematics as a meaningful rather than arbitrary discipline were among the ideas of the first theme.

The second theme focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, discovery to verification, mathematics to science, truth to utility, and in general, mathematics to the culture within which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

- a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."
- b) An appreciation for the human dimensions that motivate discovery: competition, cooperation, the urge for holistic pictures.
- c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, and why it is investigated.
- d) A need for new teaching/learning formats that will help discourage our students from a view of knowledge as certain or to-be-received.
- e) The opportunity for students to think like mathematicians, including chances to work on tasks of low definition, generating new problems and participating in controversy over mathematical issues.
- f) Opportunities for faculty to do research on issues relating to teaching and be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures. . . the network might formally support writing, team-teaching, exchanges, conferences. . . .

Alvin White
August 3, 1987

From the Editor

Attempts to reform mathematics and science in the schools is not an activity for the timid. Reports in *Science* (16 October 1998, p. 387-9; 29 August 1997, p. 1192-5) are about people talking past each other.

Third graders in California will be taught about the periodic table, and sixth graders will learn about Earth's "lithospheric plates" under the standards approved by the state Board of Education. The presidents of the National Academy of Sciences and of the American Physical Society think that the standards focus too much on detailed knowledge and too little on concepts. Rote learning is substituted for understanding.

A high school chemistry teacher who helped draft the document thinks that it is perfect. "The average student with a caring teacher can get through this."

The president of the NAS complains that, "When you start teaching first and third graders about abstract things like atoms and molecules, what we actually do is not have kids understand anything...My hope is that the next governor takes care of this by commissioning a major overhaul of the standards."

According to *Science*, one major hindrance to the reform of mathematics in the schools is the vast number of teachers who took few math or science classes in college and have had no additional training. There are other pressures against reform. The 1992 California framework, based on the NCTM standards, called for teachers to question more and explain less, to group higher and lower ability students together, and to assign more projects and fewer workbook drills. By 1994 the radically new textbooks started appearing in classrooms.

The reaction was swift. Parent groups organized to fight what they called "fuzzy math" and "new New Math." They said the curriculum used untried methods and replaced basic skill drills, such as multiplication tables and long division with projects such as writing. In California, with support from Gov. Wilson, anti-reformist activists constitute the majority on the panels that are drafting both the new content and performance standards and the 1998 framework.

The new framework relies heavily on standards from Virginia and North Carolina. Shelley Ferguson, an elementary school teacher in San Diego who has been involved with the reform effort observes, "It's back to a laundry list of topics to know. Conceptual understanding and problem solving are pretty absent."

Whatever the outcome, reformers elsewhere say that the California math wars have taught them the importance of educating parents as well as teachers.

Natural Math

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ABSTRACT

The Natural Math project's main goal is to create mathematical curriculum around concepts of higher math (algebra, calculus and "post-calculus" subjects), presented in a way that makes them available with minimal prerequisites. In particular, the results of the project make it possible and desirable to teach higher math to very young children and math-anxious adults.

1. INTRODUCTION

Young people (ages 4 to 10) can learn higher mathematics. They can discover concepts for themselves, develop original algorithms, and take many elements of teaching into their own hands. Learning can be arranged in such a way that mathematics comes to students naturally, painlessly, and very fast.

I attempt to prove the above statements by presenting examples of lessons that are part of the Natural Math project. The main goal of the project is to create mathematics curriculum that is rich in concepts yet readily available to people of almost all ages and all levels of mathematical education. One of the methods to achieve that goal is to arrive at higher mathematical ideas through intuition, common knowledge and common language, adding rigor later.

Several rules are strictly followed in all experiments:

- Only volunteers can participate. Adults cannot volunteer children against their will. Every child can stop the program or start it again at any time.
- Students have to discover the key concepts for themselves. Teacher provides the environment that makes it possible.
- Students are never given any algorithms, but are led to generalize their concrete experiences into algorithms.
- Students have "veto rights" in choice of activities; mentors can ask them to lift the veto as a favor, but can't insist.

2. CHILDREN CAN DO IT.

Most lesson descriptions go much faster than the real lessons, because some examples and explanations are excluded.

1. Very young children can understand almost all arithmetic concepts.

By "arithmetic" I mean concepts that can be presented without algebraic generalizations, using only numbers. For example, the idea of multiplication is arithmetic, and the idea of derivative is not; the idea of negative numbers is arithmetic, and the idea of inverse function is not. This definition is intentionally fuzzy: I do not want to create the impression that children *cannot* do certain math before a certain age. We cannot prove general negative statements of that sort because any experiment necessarily uses particular teaching methods, and negative results only give information about the teaching methods used. If you have doubts, think about this: what if people used as much mathematics in everyday life as they use language? Wouldn't everybody learn a lot of math by the age of about three, together with his mother tongue(s)? (Now, that would be some experiment!)

Example 1.1. Genevieve (age 4) learns coordinates.

Genevieve could count to 10 when we started. We had some fun with zero (Montessori, p.329), when I asked: "Give me one marble, please... Now give me zero marbles! Jump two times... Now jump zero times," etc. I introduced negative numbers as something that comes before zero as we count. I cut out small pieces of paper, wrote numbers (-10 to 10) on them, and we hung them on a long cardboard in the appropriate order. I drew and cut out an animal (a dragon) that "lived on the number line," walking back and forth. Genevieve enjoyed questions of the sort: "If the dragon is at -2 and goes 3 steps to the left, where will it land?" I always followed by: "See, negative two minus three is negative five."

Then I pointed out that the dragon has wings and can fly, and demonstrated by moving the dragon next to a wall (e.g., two steps to the right, three steps down). We drew the movements on graphing paper with a coordinate system, and Genevieve was able to find coordinates of given points, and to find points when given coordinates, translating that into “steps” at first.

Example 1.2. Cecali (age 9) studies percents.

I wrote down: “3%” and explained that this phrase in mathematical language means: “Three for every hundred.” Then we discussed questions of the sort: “What is 2% of 300?” At first, I had to translate it: “If we have 2 for every hundred, how many will we have for three hundreds?” After about 5 minutes, I did not have to translate anymore. Cecali did not know decimals yet, so we could not move much farther than, for example: “Find 7% of 250.” It never ceases to surprise me that problems that are “too simple” often mean trouble: Cecali could not find 15% of 100, and it took a while for her to figure out that “degenerate” proportion: 15 for every hundred, then how many for a hundred?. *It is a general tendency: if the example is so simple that it does not represent the concept anymore, young people almost always have difficulty with it.* See also Example 2.3, where Kirk could not understand the function $f(x) = x$.

Cecali promised to help me with taxes next year.

2. Children can easily understand “lower-level abstractions” that require few steps of reasoning, such as unknowns, more-less, etc. They are not always able to express their knowledge in mathematical (or any other) language, so it is quite a challenge to communicate with them.

Example 2.1. Genevieve (age 4) solves equations.

We started by playing with beads and a little basket. I put two beads in the basket without showing to Genevieve how many were there. I demonstratively added one more bead, and then shown the total of three and asked: “How many were there before?” The little girl thought for about half a minute (it’s a long time); she was very concentrated on the task, and finally figured it out.

One of the major difficulties in my project is that *most adults seem to be unable to tolerate the sight of a child quietly thinking for a long time.* Most people who observe my lessons have to be restrained from “helping” the

thinking child: they want to explain, to give hints, to reformulate the problem. I warn people that they should not interfere; many parents tell me that it is very hard for them. It is interesting that little children often are willing to think about a question much longer than adults. Often children say, “Please, don’t tell me the answer!” Many elementary math curricula concentrate on memorization, which means that most teacher’s questions have to be answered immediately. It follows that children do not have a chance to think. Sometimes I even have to convince those of my older students who go to school that it is OK to think before answering.

We played the same “guessing game” many times, and I started to move it onto paper by drawing and writing. I first called the unknown “something,” writing: $_ + 1 = 2$ and saying: “Something plus one is two, what is that something?” I mentioned the letter notation ($x + 1 = 2$), but it did not “sink in.” Unfortunately, Genevieve did not know how to read yet, but she was amused by the fact that she could “read mathematics:” she learned to recognize numbers up to 10, “+”, “-” and “=”. She proudly told her parents about her success.

Example 2.2. Jasmine (age 6) plays the “More-Less Game.”

The rules of the game are simple: the “host” takes a number, and the others have to guess it. If the guess is wrong, the “host” says if it is more or less than the number he has. “Host” put some marbles in a bag without showing how many (it gave the winner satisfaction of confirming the right guess by counting marbles). Jasmine had a lot of fun with the game. She learned the idea of “less and more” and figured out the beginning of the bisection method (the most efficient way to play is to find an interval containing the number and then bisect it repeatedly).

Example 2.3. Kirk (age 7) constructs “Function Machines”

“Function machine” is a machine that does something to numbers you put in it. I drew some mechanism for Kirk, and asked her to give me numbers to put in the machine until she could guess what the machine did. We had: in 1 out 2, in 5 out 6, in 2 out 3, and Kirk said: “The second number is one more.” I concluded: “The machine adds one.” Then we took turns creating “function machines,” and the drawings and the formulas got fancier and more interesting. This game is a big hit with every student to whom I show it. Kirk

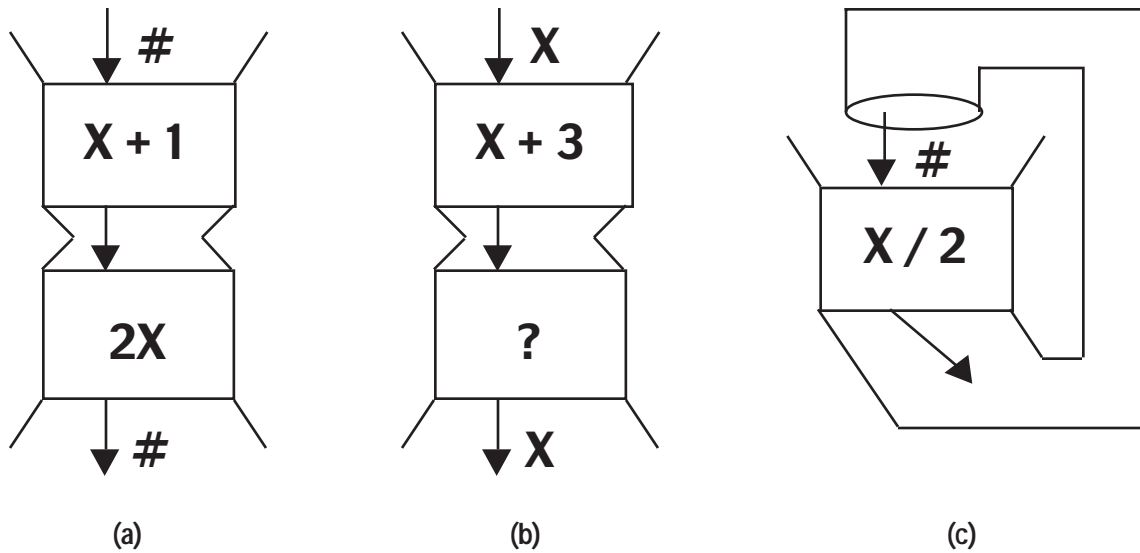


Figure 1: The Function Machines

Small cards are used to construct the "Function Machines." (a) Composition of functions (b) Inverse: what should be on the empty card is the machine always returns the same number we put in? (c) Iterations

was very puzzled when I constructed the machine that did: in 2 out 2, in 10 out 10 ($f(x)=x$). Compare to Example 1.2.

"Function Machines" can effectively lead even very young students to concepts such as composition of functions, inverse, iterations, and so on (Figure 1).

3. Children can develop "higher-level abstractions."

Sometimes the depth of their understanding is surprising, given very little knowledge.

Example 3.1. Emily (age7) investigates matrices.

This lesson was inspired by a book by D. Cohen (Cohen, pp. 5-8). Emily and I pretended to "go shopping." We chose what to buy and how much, and I wrote prices as follows:

books dolls \$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} = ?$$

Emily computed the total and wrote: "16". I told her that usually people call the thing we just did "multiplication of vectors," and spelled the word "vector." Then we had two days of shopping:

roses skirts dolls \$

$$\begin{array}{rcl} \text{day 1} & \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} & \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = 20 \\ \text{day 2} & \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} & \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = 9 \end{array}$$

I said that the traditional name for such an arrangement of numbers is "matrix," and spelled the word, explaining that we multiplied a matrix by a vector. Usually students quickly pick up appropriate "math language."

To motivate multiplication of two matrices, I suggested "comparison shopping" using names of local groceries:

item 1 item 2 item 3 S WD

$$\begin{array}{rcl} \text{day 1} & \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} & \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 5 \quad 5 \quad 9 \\ \text{day 2} & \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} & \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 22 \quad 15 \end{array}$$

Linear algebra students often have problems with the following: as a result, we are getting rows and columns that correspond to days and shops, but which is which? What does each entry in the answer matrix mean? To explore that, I asked: "Which shop is

cheaper?" After some confusion, Emily figured out that the columns were for shops and the rows were for days. Emily liked questions such as: "If you are a manager of Winn Dixie, will you use day 1 or day 2 in your advertising campaign? What if you are Shwegmann's manager? What to do if you are a customer?"

At first, Emily was using addition to find totals, e.g., $5 + 5 + 5$ instead of $3 * 5$. After a while, she seemed to remember some of these "facts." *She was doing multiplication faster by the minute, without going through a single drill on "times tables."* She and other children with whom we studied "shopping matrices" were able to figure out the rules for matrix multiplication without any explanations.

Next time, we coded every letter in the alphabet by its number, so "a" was "1", "b" was "2"... Emily chose words and coded them, for example, (4, 15, 7) for "dog". After Emily coded several words, I told her that it is very easy to guess what the code is. What about making a secret code? She asked me why people would want secret codes, so we talked about spies for a while. I suggested to divide every number by two (now "dog" was (2, 7 1/2, 3 1/2)). We sent each other secret messages and gave "keys" to them.

Message: (1 1/2, 1/2, 10). Key: multiply by 2

She decoded it quickly, and gave me a coded message of this sort with no difficulty. But it did not prepare me for her reaction to the next one I gave:

$$\text{Message: } \begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 2 & 5 \\ 3 & 6 \\ 12 & 1 \end{pmatrix} \quad \text{Key: multiply by } \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Emily: ???

Me: Just like our shopping! (pointing to the page with "shopping matrices")

Emily (immediately): *But we do not write like that.*

Me: Like what? What do you mean?

Emily: I said, we do not write like that!

Only then I noticed that she was waving her hand *up and down*. She instantly understood that the answer will be a *column* vector (something most college students would notice much later) and was trying to explain that it is improper in English

to write from top to bottom, not from left to right. English is not my native language; she was trying to teach me as she does occasionally (I always appreciate it and thank her properly). It is amazing that she understood such an abstract fact in a few seconds. Consider steps she had to take:

- 1) recall the process of matrix multiplication
- 2) understand, *without performing any operations*, that she will get a column vector
- 3) connect it with orientation of writing, something most people don't even notice
- 4) find a way to explain it (having very limited math vocabulary).

After I wrote transposes, Emily agreed to decode the message (her name), which proves that her column vs. row consideration was not a guess.

Emily's Matrix Poem

*A little fun a little bit
And then we coded the alphabet
And on the coach was a pet
Matrix matrix matrix*

*A lot of fun a lot of it
We coded words and wrote it
and on the table was a pet
Matrix matrix matrix*

*A little fun a lot of it
We played with coordinate
And on the ceiling was a pet
Matrix matrix matrix*

*A lot of fun a little bit
We played a game and stitched it
And on the chair was a pet
Matrix matrix matrix*

*And on the bed was a pet
Matrix matrix matrix
Matrix!*

Emily was so excited about matrices that she wrote a poem, reproduced above, which I used to talk about combinatorics.

4. Children can construct and use algorithms for solving problems.

That also requires some *classification* of the problems (to figure out what algorithm to use), which is an abstraction in itself. Sometimes it is desirable to lead children to some particular algorithm; in this case,

restrictions can be used to make other methods “illegal.” Restrictions can be presented as “the rules of the game” (see Example 4.1 below), which saves a lot of explanations by referring to the “game culture” that is a common frame of reference (cf., Davis, pp. 107-140) for many children. The “game frame,” with its notions of fairness, fixed rules, sharing, competing, and taking turns, is a very powerful teaching tool especially suited for mathematics.

Example 4.1. Aidar (age 7) solves equations by “reversing.”

After a “hands-on” introduction to equations (see Example 2.1) I explained that it is a tradition in mathematics to write letters, especially “X”, instead of - or “something,” and demonstrated: $_ + 1 = 4$ is the same as $X + 1 = 4$
 $X = 3$

Some education psychologists argue that young children can not “reverse,” i.e., cannot see that subtraction is the inverse of addition. Sure enough, the idea that: $X = 4 - 1$ was not something that naturally occurred to my young students. They had no need for “reversing,” being perfectly able to find the solution by guessing. My task was to create that need. Since my students were not fluent in large numbers and fractions, it was impossible to present examples that are usually used to motivate high school students who only want to guess, such as $5X = 7$ or $X + 1997 = 2870$. I tried to ask (about $X + 1 = 4$): “What are you doing with 4 and 1 to get 3?” which produced a lot of puzzled looks and answers of the sort: “You find what you should add to 1 to get 4.” After many more futile attempts (allow me to spare the description) I finally found a game that led students to “reversing.”

The game uses a calculator (TI-82, for example) where all numbers and operations show and stay on the screen. Here is my talk with Aidar (translated from Russian):

Me: Do you remember how to solve equations? Try this one: $X + 2 = 5$

Aidar: X is 3

Me: Here is a new game with this calculator (some explanations were required about the “enter” key). We want to get the answer of the equation on the screen, but we can’t just type it in. The rule of the game is to use any buttons except the number that is the answer, so you can’t use “3” here. Try it!

Aidar typed the following:

5+2 “clear” (he has noticed it does not work before he pressed “enter” to find the answer)

2 “clear”

5-2 “enter”

We took turns solving equations with the calculator. Aidar did not have to use “clear” button anymore. It worked in a similar way with all my young students: after understanding the rules of the game, they figured out the “reversing” method of solving equations, without any hints from me. I used marbles to demonstrate the method again: “There are some marbles hidden in my hand. I add two more, and now we have five. To find out how many were there before, we can just take away the two marbles I added, in order to undo what I did.”

This example makes one wonder what does the phrase: “Children are not able to understand concept X until age Y” mean? Very often, it means that before age Y children do not have any *experiences* that require understanding of concept X. Hence, if a teacher provides such experiences, students might understand concepts early.

Example 4.2. Emily (age 7) adds large numbers.

Emily wanted to learn how to operate with large (3-4 digit) numbers. She understood the idea of place value and the fact that it is convenient to add tens to tens, hundreds to hundreds, etc. However, she wanted to do operations from right to left and did not know what to do after the following step: $32 + 81 = ?$

$$\begin{array}{r} 32 \\ + 81 \\ \hline 113 \end{array}$$

The problem was that I could not understand what exactly she was doing (I only saw that she was writing from left to right). I had to iterate my suggestions, by trial and error getting closer and closer to her yet unknown to me technique (numerical method of pedagogy). She refused to use approaches that were too far from her own. And I am happy she did, because she invented something original. One wonders how many inventions are lost because inventors do not defend them strongly enough. Finally, we figured out how to make her algorithm work. She added easily, if slowly, and had no problems with 3- and 4-digit ex-

amples:

$$\begin{array}{r} 1\ 7\ 3 \\ +\ 2\ 6\ 9 \\ \hline 3\ 13\ 12 \\ 4\ 3\ 4\ 2 \\ 4\ 4\ 2 \end{array}$$

Emily added from left to right (first $1+2$, then $7+6$, then $3+9$), then “carried,” looking at the next number while considering numbers from left to right. If I had to teach “long addition” by “telling them the rule,” I would use Emily’s algorithm rather than conventional algorithm, because it is more straightforward. Emily and I started to do “long multiplication” on the same day the lesson above happened.

5. Children can understand some topics from almost all branches of mathematics, including “postcalculus.”

Presentation has to be adjusted, of course. See examples from other sections, plus:

Example 5.1 Kirk (age 7) plays with cyclic groups.

Young people love cyclic groups. There is something fascinating in being able to do arithmetic with finite amount of numbers.

With Kirk, we started by looking at the ordinary clock. Numbers never grow past 12, so if it is one o’clock now, 15 hours later it will be 4, not 16. I pointed to the wall clock and wrote: $15 + 1 = 4$. With much laughter, Kirk solved several problems of this sort. Then we talked about other planets, where the day can be longer or shorter than 12 hours. Kirk chose the 3-hour-long day and drew the “space alien clock” with numbers 1, 2, and 3. Then I asked: “If it’s two o’clock on that planet, what time will it be 14 hours later?” I expected Kirk to count hours around the clock (2,3,1,2,3,1 ...); however, she immediately said: “One o’clock.” She solved it faster than I did (I used the fastest way, i.e., remainders). She did other problems as fast, so it was not a lucky guess (however, she was much slower next week when we returned to the topic). Judging by the time of her response, she was using a very efficient method to solve these problems (cf., Woods, Resnick and Groen), without being told about any methods whatsoever!

I explained that mathematicians invented a way to write about alien clocks without confusion: they write

$2+14= 1 \pmod{3}$ (I read it aloud). Kirk had no problems with notation. We explored alien clocks of different modulo, using examples with positive and negative numbers, such as:

$$2-4=3\pmod{5}$$

Kirk developed a vague notion of remainders when she noticed patterns in numbers:

$$4=1\pmod{3}$$

$$5=2\pmod{3}$$

$$6=0\pmod{3}$$

$$7=1\pmod{3} \text{ again, and so on.}$$

Cyclic groups is now one of Kirk’s favorite topics in mathematics. The topic can be used to talk about division and multiplication (“What time will it be on the planet 13 hours after midnight?” or “How many hours are there in 7 days?”) and is an effective way to introduce remainders.

Example 5.2 Emily (age 8) starts to understand linear independence of vectors.

We played a game: we take a piece of graphing paper and draw some “obstacle course” on it, made of any objects, e.g., trees, lakes, castles... The objective of the game is for one of us to guide the other through that maze from the start to some treasure at the finish. The difficulty is that we can only use commands in numbers (I told a science fiction story to go with it, about commands received through a very primitive radio that could translate only numbers), so one has to give coordinates of vectors to guide, for example (1,0) for one step to the right. We played the game for a while to make sure that all relevant terminology and concepts were exposed (such as “vector addition,” “vector coordinates,” etc.).

When the game started to get too easy for Emily, I suggested the new rules: now we guide the broken robot that can make only two kinds of steps, but as many of them as we want, in positive and negative directions (Picture 3). Emily was guiding the robot at first, but when time came to make steps in negative direction, she did not know what to do. I asked her if I could guide the robot, and she said, relieved and reluctant at the same time: “We guided each other before, but now we guide the robot, so it is OK for you to do that.”

During any lessons students are very vulnerable: just by being there they admit to the whole world that they do not know something, can't do something, and need a special person, the teacher, to help them. Too easily it can be turned into a threatening situation. The teacher has to exercise extreme caution and sensitivity not to hurt students' feelings. It is especially true if students are unable to do some task or if they make a mistake. *Students sometimes feel hurt by the gentlest offer of help.* So, I try to provide excuses for them to ask for help, or play along when they invent excuses as Emily did. Sometimes I would just recommend students to think some more, of course. Let me mention here that if the teacher has to administer any sort of formal tests that affect students' life in any way, the situation almost always gets ugly, sensitivity or not.

Next, I drew two vectors, $(-1, 1)$ and $(-2, 2)$ and asked: "If the robot can only go in the steps like that, will we be able to get him to the treasure?" Emily saw immediately that the robot would only go along a straight line. I told her that mathematicians say that vectors like that are "not independent." After practicing with couples of vectors (in the last few examples Emily could see independence by coordinates, without having to draw vectors), we were ready to play the next "maze game." I drew two independent vectors and added the third, not parallel to each of the two. I asked: "Do we need the third vector, or is two of them enough?" And after Emily said that two is enough, I explained that three are not independent again, because all

"steps" of the third kind can be achieved with the first two vectors.

6. Many elements of teaching can be successfully managed by even the youngest students.

These elements include, but are not limited to, planning and designing parts of curriculum (Example 6.1), and evaluation (Example 6.2).

One of the simplest and most efficient tools I use to arrange "self-teaching" is taking turns with students in all activities. In most cases, students move to more advanced topics much faster than most teachers would move designing curriculum; in other cases, students want to stay longer on topics that interest them and/or are difficult to them. One is reminded of nutrition studies showing that toddlers, given full freedom, choose the diet that is best for them (I do not have the reference; the study results were published in periodicals).

Example 6.1 Kirk (age7) writes her own math textbook.

Once Kirk started our lesson by throwing her math workbook into the recycling bin. Her parents and I liked her workbooks, but Kirk, being a very independent person, did not accept the idea of doing regular exercises. She asked me if I would write a better book for her, and I suggested that she can write her own math text, and promised to help. We meet once a week and discuss the book's

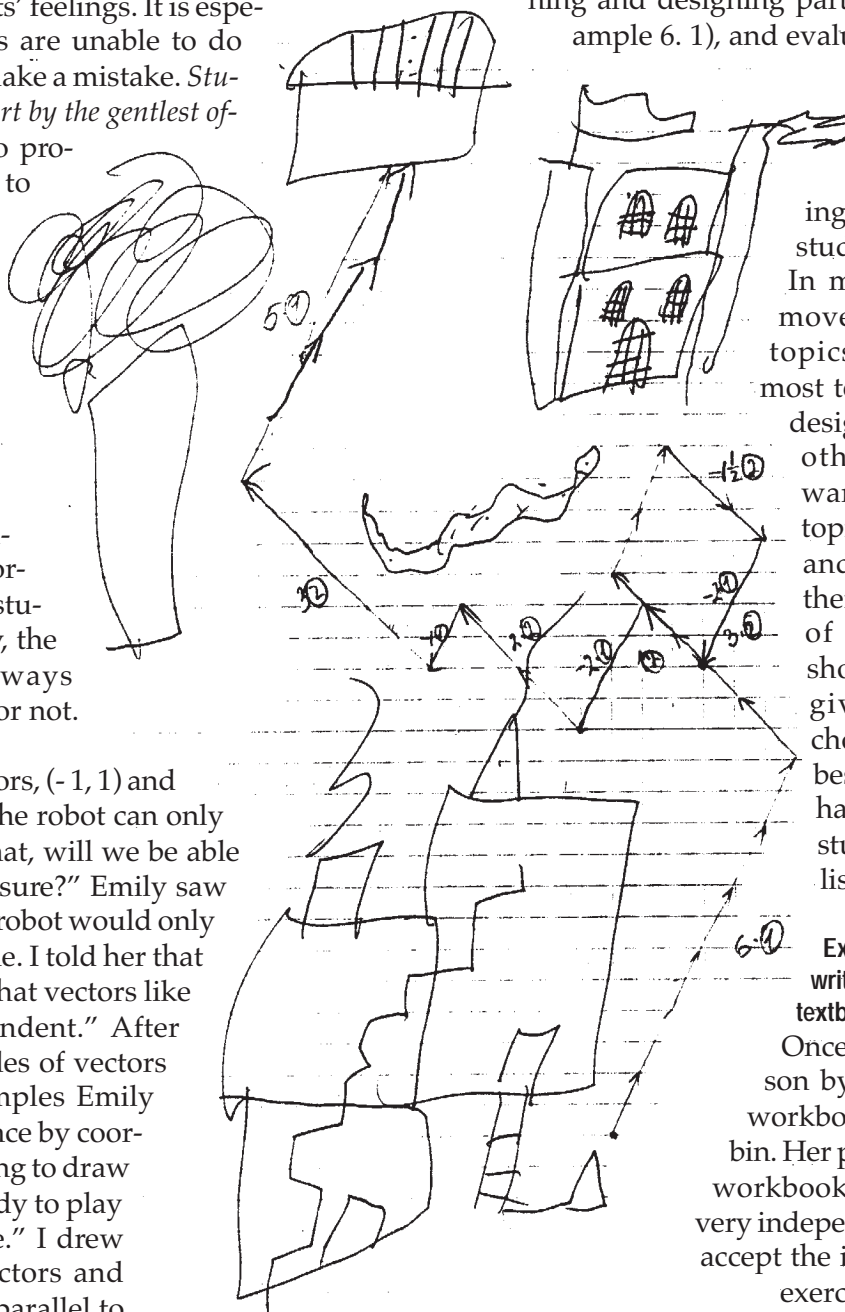


Figure 2: The Maze and the Broken Robot
Our robot could only make two kinds of steps, $(1, 2)$ and $(-2, 2)$. Yet he reached the treasure, a bookshelf (Emily loves to read). The game can be used to discuss linear combinations and different coordinate systems.

progress, and we exchange e-mail. Kirk invents math games, creates exercises, and writes stories to accompany them. I show her some math she does not know yet, and she transforms and internalizes it. Here is her story to accompany a series of simple arithmetic exercises she put in the book; she also draws pictures for every page:

“Kirk’s mom told her she may have an Oreo, but she is too small to reach the shelf. Every time Kirk solves something, she grows. Help her to get the cookies.”

An example of interesting problems Kirk raises:

“One human year is about seven dog years. How old is my dog in his years, if he is eight in our years?” It led to an interesting discussion of proportions (and biology: we talked about other creatures’ lives).

Most of Kirk’s math education is planned by her parents (she is homeschooled). However, the little part of her learning that her book constitutes seems to be important to her, and, of course, much fun.

Example 6.2 Aidar (age 7) creates assessment.

Aidar initiated a “game of school”: one of us would give the other exercises and then evaluate them. I was playing the student first, and Aidar graded me (A+). I noticed that he was nervous about the grades. Then he was playing the student:

Me (returning his paper with solved exercises): Very good!

Aidar (looking for the grade and not finding it): Where is my grade?

Me: I don’t like grades.

Aidar: Oh... (thinking for a while, sadly looking at the paper with huge A+ he wrote as a “gift” for me, then getting an idea) But you can make tiny little marks next to every problem that is solved right!

Aidar has found a way to make softer (and more informative) evaluations. We followed his suggestion, of course.

In summary, children can do a lot of quality mathematics before the age of 10. It does not even take much time (none of my students spend more than two hours a week on the lessons). The next section tries to answer the question: “What is it for?”

3. WHY TEACH HIGHER MATH TO YOUNG PEOPLE?

1) “Bird’s eye view.” There is more understanding of “what mathematics is about.” My young students would not think, as many people of their age (and, unfortunately, too many adults): “Math is addition, subtraction, and some multiplication.”

2) Fun. Young students like higher math more than primitive arithmetic (if rigor is not pushed too far). It leaves more place for creativity. The best time for learning is when you like what you learn. None of my linear algebra students at Tulane University were inspired enough by the subject to write a poem (as Emily in Example 3.1). Children often enjoy (or don’t mind) learning material that is boring and/or difficult for adults (consider languages, for example).

3) New frames of reference. Giving students even a vague notion of new concepts creates frames of reference in their minds. It builds a base for further teaching. It helps to break the vicious circle of “you can’t start to learn it unless you understand it, but you can’t understand it until you learn it.” Students are initiated into richer mathematical culture, and it is done gently, without the undue strain people experience in college when they have to learn too much at once. They have more time to get used to that culture and to absorb it.

4) Advanced problem solving methods. Experiences in higher mathematics lead students from purely arithmetical methods to algebraic methods involving various symbolic operations, abstractions, etc. This differs from many programs for “gifted” where children are given the most difficult problems they can possibly solve using elementary methods they already know. As a child, I often felt “cheated” having to solve problems with great difficulties using primitive methods, only to learn later that those tricky problems turn into elementary exercises in more advanced methods. I am sure everybody can supply his own examples. My big personal sore was in elementary physics of movement (velocity, acceleration, etc.) where problems were hard to solve without using derivatives, but were just standard exercises with derivatives. About half a year I wasted because of that seems to be more than enough time to learn necessary calculus.

5) Meaningful exercise. Higher mathematics is “arith-

metic-intense,” presenting excellent review opportunities. Many people do not seem to learn subject # n until they use it for subject # $(n+1)$ (compare to the usual lament of students taking, for example, a differential equations course: “If I took calculus now, I would get an “A” so easily!”)

6) Future science education. Many people who are successful in mathematics say that they are very comfortable with the parts of mathematics they learned early (Danger! People who are *forced* to learn things they are not ready for may not benefit). For example, Professor Vladimir Arnol’d says (Lui): “Many Russian families have the tradition of giving hundreds of such problems [very nonstandard old merchant problems] to their children, and mine was no exception. Very young children start thinking about such problems even before they have any knowledge of numbers. Children five to six years old like them very much and are able to solve them ... The feeling of discovery that I had then [as a child] was exactly the same as in all the subsequent much more serious problems ...” Anecdotal evidence shows that *gentle* exposure to higher mathematics at an early age is very helpful for future understanding. In practice, “gentle” means “less rigorous, very intuitive, with no formal tests.”

7) Time saving. Learning arithmetic through higher math saves time, for obvious reasons (returning to concepts several times makes the periods “in between” work for understanding; review is imbedded in learning of new things; students see arithmetic as a mere tool for higher math, which reduces fear considerably). In practice it means that people who are interested in mathematics and science can start doing graduate-level research several years earlier than usual. Those who are not interested in math receive *an opportunity to spend considerably less time with the subject, reaching the same or better results*, and doing work that can even change their attitude towards math.

8) Application to remedial education. Methods of teaching that work well for young people can be successfully used with older people having learning problems. After all, young students and “lower track” students face the same challenge of *limited prerequisites*. Of course, the situation is usually worse for people who learned math in a wrong way: they require *rehabilitation* more than anything. My experience shows that the methods I use with children do a good job rehabilitating math-anxious people.

Clearly, more research is needed in the area of teaching higher mathematics to people with limited prerequisites. I hope that more mathematicians and education specialists will address the exciting pedagogical problem of making mathematics more available. It can and should result, in particular, in very different curricula for young people, whose learning potential is too often grossly underestimated. I also hope that more people will realize that students themselves hold in their hands solutions to many pedagogical problems, and that students need some freedom in learning, as well as a lot of unobtrusive help, in order to solve these problems.

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Do the Great While It Is Still Small: Humanistic Teaching in the Elementary School

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"Do nondoing,
strive for nonstriving,
savor the flavorless,
regard the small as important,
make much of little,
repay enmity with virtue;
plan for difficulty when it is still easy,
do the great while it is still small.
The most difficult things in the world
must be done while they are easy;
the greatest things in the world
must be done while they are small."

- from the *Tao Te Ching* by Lao Tzu, translated by Thomas Cleary (1991, p. 48)

These words from the *Tao Te Ching* offer practical guidance regarding the teaching of humanistic mathematics and science. If mathematics and science are indeed critical human endeavors, inextricably tied to culture and social interaction, and therefore integral to a full understanding of the human condition, then as humanistic disciplines they must become integral aspects of school curriculum. "Do the great while it is still small," suggests that educators begin teaching math and science as humanistic disciplines at the earliest possible point, that is, in elementary schools. We cannot afford to wait until students are in college to present math and science in a humanistic context. Research has shown us that student interest in these subjects is highest when in the elementary schools, and that by the time they are in junior high school many able students have lost much of their interest in science and mathematics (Yager & Penick, 1986). Appropriate instruction and learning opportunities can be provided for students as young as pre-Kindergarten, thereby allowing youngsters to grow up in a world of exciting, useful, and challenging math-and-science-related experiences. Adults raised under these conditions will be more likely to understand the fascinating and subtle aspects of science and math as human enterprises, leading to greater math/science

interest, achievement, and appreciation.

The *Tao* reminds us too that, "The most difficult things in the world must be done while they are easy." Elementary students typically exhibit strong interest in suitably presented science and mathematics, thus offering a perfect opportunity for them to learn these subjects in a humanistic context. What I propose to do in this article is to make a case for presenting humanistic math and science to elementary school children, and then to introduce a theoretical, yet practical, framework for teaching these subjects in K-6 classrooms.

I refer to the subjects of math and science jointly in this article because, especially in pre-college educational settings the two can be, or should be, intimately linked. In the elementary classroom science and mathematics reinforce one another, each discipline drawing upon the techniques and tools of the other to offer students an experience and an awareness that is greater than the sum of the parts. Problem solving skills, whether in or out of school, will be strengthened when students can draw freely from the strategies of math and/or science as necessary. When these subjects are offered jointly students are able to make natural connections between math and science, enhancing comprehension and appreciation of both.

Imagine standing before a room full of thirty-two ten year olds. Now imagine that you are responsible for their understanding of mathematics and science, as well as other vital subjects including literacy, social studies, communication, collaboration/cooperation, and the arts. That is, imagine that you are an elementary school teacher. Next, consider how you will instruct these young students in this cornucopia of disciplines. Don't forget that those young people are potentially future graduate students in math and science, future workers at math/science related occupations, and perhaps even future teachers. Also, don't forget that as an elementary school teacher your aca-

demic background in math and science is probably marginal at best. Will you rely on the traditional textbook methodologies, or will you try the riskier and more demanding approach of action-oriented and individualized instruction? Imagine a curriculum that would motivate these students to reach high levels of achievement in math and science, while simultaneously encouraging their curiosity and personal interest. Imagine a curriculum that would lay the groundwork for a deep understanding of math and science as humanistic efforts. A great deal of educational research indicates that such goals are within our reach, and that they may be reached by our elementary-level students, working with motivated and inspiring teachers. (Drew, 1996; Myers & Fouts, 1992; Vargas-Gomez & Yager, 1987; Yager & Penick, 1986).

What should be the basic structure of a curriculum rich in humanistic aspects of math and science? A review of research regarding effective elementary pedagogy (Bruner, 1977; Dewey, 1926; Freire, 1970; Maslow, 1971; Rogers, 1983; Vygotsky, 1978; Yager & Lutz, 1995) suggests that a curriculum aimed at teaching the humanistic, aesthetic, and pragmatic aspects of science and mathematics should center on four theoretical and functionally interconnected components. Such a curriculum could apply to the teaching of math, of science, or of math/science. A humanistic curriculum would include *interactive/collaborative*, *holistic/relevant*, *interdisciplinary*, and *problem-based* components, each of which will now be considered at greater length.

INTERACTIVE/COLLABORATIVE COMPONENT

Elementary students, as fundamentally concrete thinkers, require a personal and interpersonal experience of humanistic math/science if we wish to offer them a deep and practical understanding of these subjects. Students must be actively involved in their explorations of scientific/mathematic phenomena. Engaging lessons that encourage personal involvement and provide opportunities for meaningful understanding are most satisfying and therefore optimally motivational for students. In action, this component will utilize what is known as “hands-on” or “minds-on” classroom activities, the former referring to ex-

plorations involving objects and materials actually manipulated by students (e.g., directing students to separate a large pile of various leaves into two piles, based on observable characteristics, then to construct a bar graph based on the piles), and the latter referring to activities that promote the use of higher order thinking skills, but not necessarily involving the use of materials by students (e.g., an inquiry demonstration presented by the teacher). These concepts are well described in the national standards now set for science and mathematics teaching (AAAS, 1993; NCTM, 1989; NRC, 1996) since they form the basis for the pedagogy described in those documents. These techniques are particularly crucial for marginalized, at-risk, and underachieving students.



*What sorts of experiences do they, the students, encounter in their lives?
What do they believe?
What do they want?
Who are they?
Who do they want to become?*

Further, to be fully effective, interactive studies must be undertaken in a collaborative manner. Methods involving cooperative group work are essential to learning about science and math as humanistic endeavors.

Not only is learning dependent on socialization (Vygotsky, 1978), but the basis of humanistic math and science lies in fostering an awareness of the interpersonal aspects of those disciplines. They cannot be taught in a social vacuum, i.e., simply reading about humanistic math and science is antithetical to developing authentic and functional comprehension and appreciation in these areas. A deeper understanding may be cultivated by actual problem solving in social settings and augmenting those experiences with media such as texts, videos, and computer-based learning.

HOLISTIC/RELEVANT COMPONENT

Closely associated with the Interactive/Collaborative Component is the need to present lessons that are relevant to the students themselves. Student-centered instruction focuses on student interests, student questions, student ideas, and student-generated projects. Humanistic math/science remains oxymoronic in a traditional classroom where teacher-centeredness is the rule. Memorization and retention of facts are not enough; a deeper understanding is required, which can only be accomplished through a process of scaffolding student learning from the familiar to the un-

familiar. The entire life of the child then becomes important to the humanistic educator. What sorts of experiences do they, the students, encounter in their lives? What do they believe? What do they want? Who are they? Who do they want to become? What do they like? What do they dislike? By taking a holistic view of the child, as opposed to limiting the curriculum to the cognitive dimension alone, the teacher may find numerous opportunities to creatively attach humanistic math and science to the child's daily experiences. Learning, founded on students' actual lives, can then build up and out in an ever-widening spiral.

The humanistic curriculum must also be holistic in the sense that it involves the entire child. Caring (Noddings, 1993), respecting, and empathizing (Rogers, 1983) are values that support students as unique thinking and feeling individuals in the process of growing and understanding the world. An ethic of care and compassion, openly and appropriately expressed, encourages their exploration of the unknown, both inside and outside the classroom. The teacher's style of interacting with students, in fact, has been shown to be a critical variable associated with student success in science and science-related classes (Ebenezer & Zoller, 1993; Eichinger, 1992, 1997; Myers & Fouts, 1992).

INTERDISCIPLINARY COMPONENT

Mathematics and science do not happen in a vacuum. They are composed of meaningful acts performed by real people in the courses of their lives. Just as I have recommended the blending of math and science throughout this article, these two subjects (traditionally treated as discrete entities in school) can also be combined effectively with other school disciplines. Interdisciplinary combinations not only promote the presentation of the subjects in a holistic and relevant context (as recommended above), but also provide opportunities for imaginative and personal connections between students and subject matter, which serve to further enhance understanding and motivation.

Examples of interdisciplinary strategies involving humanistic mathematics and/or science abound, combining art and mathematics (Hall, 1995; Reiner, 1994; Williams, 1995), art and science (Eichinger, 1996a; Kohl & Potter, 1993), art, mathematics, and science (Eichinger, 1997), music and mathematics

(Huylebrouck, 1996; Kitts, 1996), chemistry and the dramatic arts (Budzinsky, 1995), literature and mathematics (Bernard, 1994; Growney, 1994; Lew, 1996), literature, art, and mathematics (Swetz, 1996), and history and mathematics (Priestley, 1996). Although not all of the aforementioned studies were written with elementary school teaching in mind, any of them could be modified to accommodate students in grades K-6. A servicable procedure for integrating units of study in elementary math and science was proposed by Francis and Underhill (1996). Examples of appropriately integrated math and science curriculum at the elementary school level include those by Curran-Everett (1997), who explores the properties of the Möbius Band, Scarnati (1996), who teaches observation techniques through the description and assembly of Lego shapes, and Eichinger (1996b), who challenges students to learn about thermodynamics through experimentation, data collection, and interpretation.

Other aspects of humanistic instruction that are often overlooked in traditional elementary settings, such as technological applications and the development of a critical social consciousness, are readily accessible through an interdisciplinary approach. The Science/Technology/Society movement (STS) is defined by Yager and Lutz (1995) as "the teaching and teaming of science in the context of human experience, including the technological applications of science" (p.30). STS instruction is therefore intimately tied to practical applications of mathematics and leads students to a deeply relevant understanding of the place of these subjects in their lives. STS techniques are empowering for students since, as stated by Yager and Lutz, "The richness of STS comes from contributions of the individual students, their creative ideas, and the central role they play in planning and carrying out the STS investigations" (p.35). Hurd (1994) called for a science/technology curriculum "that relates science to human affairs, the quality of life, and social progress" (p. 109), and whose "ultimate purposes are to have students who can take part in helping to plan the science/technology aspects of our sociocultural future" (p. 109). In this sense, notions of critical social consciousness, human rights, and social action may be forwarded in the elementary classroom through interdisciplinary humanistic instruction that includes authentic reflection and dialogue based upon real-world issues. In this way, the humanistic mathemat-

ics and science curriculum will “help students explore their personal and group identities relative to the social structures in which they live, others who live within the same social structures, the inequities that exist there, and students’ roles in suffering from or benefiting from them” (Jennings & Eichinger, 1996, p. 12).

Another reason for encouraging an interdisciplinary facet to the humanistic curriculum is that it will provide teachers with more time to teach science and mathematics in a very busy curricular day. The accumulation of academic responsibilities, headed by the need to teach reading, writing, and mathematical calculation, leaves teachers too little time to explore other subjects in depth, especially if those subjects are taught in isolation from one another. In addition to providing opportunities for making meaningful connections to the other disciplines, the proposed interdisciplinary curriculum will create more space for teaching math/science in a humanistic context. Tie math/science into reading and writing. Connect it also with social studies, art, and physical education. Blend these subjects in new and innovative ways.

PROBLEM-BASED COMPONENT

The last of the four interrelated components refers to the importance of grounding the humanistic curriculum in meaningful, challenging problems and opportunities for authentic inquiry. Gone are the days when rote memorization of facts and algorithms suffice for a math/science education. An essential feature of the current standards in science and mathematics is a call for deeper, more active, and more relevant study of these subjects at all grades for all students. As stated by the National Research Council in the *National Science Education Standards* (1996), “Learning science [and/or math] is something students do, not something that is done to them” (p. 20). Posing realistic, interesting, and challenging problems for students or groups of students to solve is a mainstay of the current movement toward curriculum reform in math and science. The problem-solving instructional format has been associated with increases in student achievement and motivation at all school levels. Perhaps most

importantly, students will understand and appreciate the value of math and science as humanistic endeavors only if they have used it to solve problems of interest *to them*. Through problem solving, students learn not only to effectively confront challenges in the classroom, but also to confidently face future choices and tasks presented by “real life,” including those related to occupation, citizenship, leisure, and interpersonal relations.

Wheatley (1991) proposed a problem-centered model of mathematical and scientific learning designed to promote students’ construction of subject matter knowledge in the classroom. That model is composed

of three elements: 1) students are challenged with a task, 2) work is done in small groups, and 3) after working on the problem the groups convene to discuss their solutions. Group presentations are made to the class, not to the teacher, whose role is that of non-

judgmental and encouraging facilitator. The implications of the problem solving approach have been discussed by various authors, including Meier, Hovde, and Meier (1996) who stress the importance of “real life” and interdisciplinary applications, and Lipson (1995), who reported on student reactions to this sort of instruction.

A clear advantage of the problem-centered approach, as opposed to traditional, memory-based methods of instruction, is that it encourages the inclusion of more complex thinking skills. Critical thinking skills (e.g., analysis, synthesis, application, evaluation), metacognition (i.e., reflective thinking), and process-thinking skills (e.g., observing, predicting, inferring, questioning, experimenting, and communicating) are all a part of effective problem solving, and are also critical to an understanding of humanistic science and math.

A challenge presented by problem-centered instruction is that of assessment. Techniques of assessment and evaluation must be aligned with instruction, i.e., they must be congruent with the knowledge constructed by problem solving, rather than with traditional memory-centered pedagogy (i.e., testing for

“
...students will understand and appreciate the value of math and science as humanistic endeavors only if they have used it to solve problems of interest to them.”

simple recall of facts and concepts). Problem-based learning necessitates assessment strategies that involve observation of actual student performance and solutions/products, and that note whether students can *apply and use* information.

THE PRACTICE OF TEACHING HUMANISTIC MATHEMATICS/SCIENCE.

What does the humanistic curriculum look like in practice? Do programs exist that incorporate aspects of interaction/collaboration and holism/relevance within an interdisciplinary and problem-based instructional format? The good news is yes, there are some appropriate programs in existence. The bad news is that there are not enough such programs nor are they necessarily in wide enough use. Teachers, pressed for time to teach so many subjects in a school day, are likely to “overlook” subjects with which they are least familiar, and few are very familiar with dynamic and student-centered science and mathematics. Research shows us that teachers who are familiar with aspects of the humanistic style outlined above are more comfortable with the content and pedagogy of such a curriculum, and are therefore more likely to teach in a humanistic manner (Eichinger & Anderson, 1996).

Many classroom teachers employ their own uniquely designed humanistic curricula, but appropriate, larger scale programs do exist. Examples of instructional programs that tend to approach math and science in the four-pronged manner noted above can be found in the inquiry-based science/math curricula of Pasadena, CA, and Mesa, AZ, elementary schools. A number of packaged programs in math and/or science also offer options that approach an effective humanistic curriculum. Such programs include Project AIMS (Activities Integrating Mathematics and Science), Math Their Way, Full Option Science System (FOSS), Math Renaissance, GEMS (Great Explorations in Math and Science), and Mathland, among others. The professional journals *Teaching Children Mathematics* (formerly *The Arithmetic Teacher*), *School Science and Mathematics*, and *Science and Children* are also useful resources for the humanist elementary school teacher. Any curriculum package or program can be misused, however, and the best way to reach the greatest number of students is to be sure that the teachers themselves

understand and appreciate the human aspect of math and science. Excellent instructional programs require excellent teachers, since, in the end, it is largely the teacher’s expertise that determines the quality of the classroom experience.

Teachers, functioning as decision-making professionals and not merely as classroom “technicians,” must be encouraged and supported in their pursuit of more effective humanistic instructional strategies. Viewing a popular movie such as *The Lost World: Jurassic Park* might stimulate a teacher to ask some interesting questions of her or his students. Just how big was *Tyrannosaurus rex*? Could we draw one in chalk on the playground asphalt? What color might its skin have been? What makes you think so? Color in the skin with more chalk. Now let’s estimate the volume of *T. rex* - how can we do that? How many ways can we think of to estimate its surface area, and which method is likely to be the most accurate? Could we build a scale model of *T. rex*? How big would a human be in comparison? How far do you think *T. rex* could jump, and how could you decide? Could it climb? Swim? What makes you think so? Can you find any evidence for your answers? What other questions do you have regarding *T. rex*? How could you find those answers? What resources are available to tell you more about *T. rex*? These sorts of investigations are based on the children’s own interests, and combine math and science as tools to help young students discover what *they* want to know. Thus, math and science may be seen as relevant and useful *in their daily lives*.

Mathematics and science are not just topics in a book; they are interrelated elements of our everyday experiences as human beings. They can be living, exciting, and inspiring subjects when studied in a humanistic and relevant setting. What I envision is a time when children nationwide (dare I hope, worldwide?) will find a deeper connection to mathematics and science as humanistic pursuits. They may, for example, view broadcast images of math/science in action such as the travels of the Mars Rover Sojourner, exclaiming with enthusiasm and joy, “That looks like what we did in school!” To accomplish this goal, we can’t afford to wait until these students enter college. We must act on the knowledge that “...the greatest things in the world must be done while they are small.”

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The Legend of the Apple

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Slowly darkens the English countryside;
pale and distant, the moon sails the sky,
announcing to the green and sleepy farms
the coming of a new warm summer night.
Silent and brooding, the young scholar sits
close to the door of his ancestral manor,
and in the melancholy, timeless peace
surrounding him, his mind leisurely wanders
into half-closed domains of time and space.

Behold. One of the savory red fruits
noisily falls down from an apple tree,
compelled by its own sweet maturity.
The truth-searcher, lifting his idle gaze,
beholds both fallen fruit and silv'ry disk,
and the sharp edge of cruel inner lightning
pierces in silence the young scholar's brain:
Is it then possible that star and fruit
obey one law, both cases being one?
(Why does satellite not fall to earth,
but instead, far into the past and future,
once and again follow dutiful ellipse?)

Before dawn comes, will Isaac Newton find
the law of gravitation in his mind.

Mathematics Found in Poetry

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Mathematics is found everywhere, whether it be building a house or planting a tree. Yet, you may not know that mathematics is also found in poetry, ranging anywhere from the amount of lines within a stanza to internal rhythm patterns.

Like everyday life, poetry is surrounded by patterns and rhythm. Biologists say that even before you were born, you could feel the rhythm of your mother's walking pattern and of her heartbeat. From the human heartbeat many sounds have derived, all based on the soft then loud stresses of the heart: lub-DUBB, soft, hard. Due to the theory that we speak in the pattern of heartbeats, we can measure and count the hard and soft stresses. The most common one is iambic, from the Greek. This pattern is soft hard, soft hard, soft hard.¹ We use this pattern often, adopting it as the basic unit of speech, such as found in William Shakespeare's quote, *To be or not to be, that is the question*. Patterns within the syllables are also important. Usually the writer expresses him/herself in hidden ways of repetition, such as in ancient Greece, where they used the length of syllables to make patterns. We also use syllables to express important ideas. You would not want a soft stress on an important word. To account for both syllables and patterns we use the method of Syllable-stress. This measures how much of both are in each line. To make a poem where one thought is understood and remembered, some poets use repetition, either of words or phrases. The repeated lines usually come in the idea of waves. Once you reach the end of a line or wave, a new wave starts with the same type of rhythm, bringing you to the end of the next line. Poetry can also be measured in musical notes, which dictate to the reader how long one word should be held. Music is found in poetry a lot. It is the measure of chronometric time. Once you start dividing the full unit of time you come out with different patterns; one beat equals a whole note, half a beat equals a half note and so on until you reach sixteenth notes.²

Poetry can also be compared to geometry, for in some

ways they are alike. What makes a geometric figure is the consistency of its elements such as angles or line segments which stay proportional as it goes through different transformations. In other words, one reason a triangle is considered a geometric figure is because if it were flipped and expanded to a hundred times its original size, its angles and lines would still be in the same proportions as it was originally. Within poetry and other well-written works, proportion is a very important. One example of this in poetry is in a fourteen line poem, where the line stanzas are divided into the proportion 6:8:14.

Fibonacci numbers play a big part in understanding and analyzing both literature and biology. For instance, if you look at a sunflower it seems to be an unorganized mess of petals and untamed yellows, but if you look closely enough, you see that there is an order to its growth. When you look you find that there are twenty-one florets spiraling in the clockwise direction and thirty-four going in the counterclockwise direction. These two numbers are part of the "golden number" sequence, also known as Fibonacci numbers.

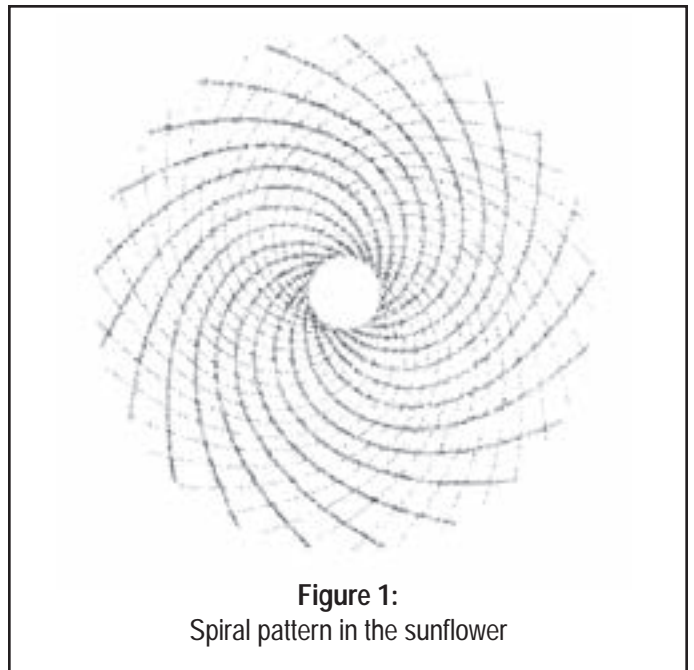


Figure 1:
Spiral pattern in the sunflower

In poetry you can find parts of this “golden sequence.” In the Aeneid, for instance, one part of book five has lines grouped in numbers of five, eight and thirteen. Another place where Fibonacci numbers might be found are in the number of syllables found within poem’s lines.³

In order for figures to be classified in geometry they have to have certain characteristics.⁴ This is also true in poetry. A square is listed under quadrilaterals because it has four sides. In poetry a fourteen line poem is classified as a sonnet, yet there are more detailed descriptions for each. Not only does a square have four sides, but they all have to be equal. If this were not true it would not be a square, but instead a rectangle. It is the same for a sonnet. This type of poem has to have some sort of ordered rhyme scheme. Rhyme scheme is the art of placing a letter at the end of each line according to what sound is found there. For instance, if your first line is, *come out and play* you would place an A at the end of the line signifying *play* as your first sound, then if you found anything which rhymed with *play* later in the poem, you repeat the letter A to show the same sound has been repeated. This lets you trace the repetition of sound throughout the poem. Such set examples of rhyme schemes can be found in any well written sonnet, yet some of the best examples are Shakespearean sonnets. Some patterns found within his poems are abab cdcd efef gg, abba abba cde cde, and abba abba cde edc. In Sir Philip Sidney’s poem, “With How Sad Steps, O Moon,” the rhyme pattern is abba abba cdcd ee.

With How Sad Steps, O Moon

*With how sad steps, O moon, thou climb’st the skies,
How silently, and with how wan a face.
What, may it be that even in heavenly place
That busy archer his sharp arrows tries?
Sure, if that long-with-love-acquainted eyes
Can judge of love, thou feel’st a lover’s case;
I read it in thy looks; thy languisht grace,
To me that feel the like, thy state describes.*

*Then even of fellowship, O moon, tell me
Is constant love deemed there but want of wit?
Are beauties there as proud as here they be?
Do they above love to be loved, and yet
Those lovers scorn whom that love doth possess?
Do they call virtue there ungratefulness?*

Besides the mechanics of creating a poem, there is also the need for creative language and situations. Without the understanding of why the scene is taking place, no poet can create a good poem. Within geometry you also have to know the reason and theory behind accepted facts, because without that you could not fully understand geometry. In order to accomplish this you use the facts which you know are true. Then, using deductive reasoning you combine statements which are accepted as true and piece them together, coming up with a logical conclusion, for if they say p is true then it can’t be not p. For example, you know that the vertices of a triangle have a total of one hundred and eighty degrees. Although you could prove this mathematically, most people accept it as a fact. In poetry you do the same thing. You are given a situation that the poet tells you is true and then from clues found within the poem, you come up with an ending which the writer hoped you would discover. For example: *Darkness creeps over the sleeping mountains*. You know that darkness means night and that sleeping also has to do with night, so it is most likely nighttime. Because it is nighttime, then it can’t be day. In Shakespeare’s “Sonnet 130,” he describes his mistress whom he loves dearly, yet he confides in the reader that she is truly not pretty. He goes through telling about how her lips are not red like coral and her eyes are not like the morning sun. Although in this society beauty is sometimes considered the measurement of love, this poem defies that and instead says the opposite. In this case her ugliness only enhances her beauty.

Sonnet 130

*My mistress’ eyes are nothing like the sun;
Coral is far more red than her lips’ red;
If snow be white, why then her breasts are dun;
If hairs be wires, black wires grow on her head.
I have seen roses damasked, red and white,
But no such roses see I in her cheeks;
And in some perfumes is there more delight
Than in the breath that from my mistress reeks.
I love to hear her speak, yet well I know
That music hath a far more pleasing sound.
I grant I never saw a goddess go;
My mistress when she walks treads on the ground:
And yet, by heaven! I think my love as rare
As any she, belied with false compare.⁵*

Within deductive reasoning there are certain elements

which must be present: terms, axioms and theorems. Terms within geometry can be categorized as undefined or defined. Undefined terms are points, lines and planes. You know they exist but they have no dimensions or mass. In poetry undefined terms are abstract feelings or events, ones which you know exist, but have no color, smell, taste or sound. Because undefined terms are important, people have come up with symbols which help them visualize what they are talking about. A point is represented by a dot and a plane is represented by a flat surface which extends indefinitely, yet can be represented by any flat surface. In poetry anger might be associated with red or depression with black. Although undefined terms are shapeless, they are very important to the plane they are describing, as are the abstract thoughts which help illustrate concrete images. One poem which shows the use of intangible objects to create a feeling is "Fantasy."

Fantasy

*The night's sweet breath,
breathes desires past my ear,
whispering songs of fantasies,
which I keep contained under lock and key.
And only in the darkness do I let my mind wander
forgetting about reality,
letting my soul run free.*

—Alexis Mann

In this poem there are almost no concrete images, yet the poet gets her point across. Although you can not touch, smell or taste the desires, you know they are there. She lets you experience the speaker's desires by relating them through feelings which every person has.

Axioms are another idea that makes up geometry. They are statements which are assumed to be true and therefore go without being proven, such as Euclid's fifth axiom or parallel postulate. This states that there is a point not on a given line and only one line can be drawn through the point parallel to the given line. For years scientists have been trying to prove this axiom right or wrong, yet none have been able to do either. Thus, it is still classified as an axiom. In poetry axioms can be described as a situation made up of unproved facts. This is because the author is telling you the situation is real so there is no reason for the

reader to investigate further, such as in *the little girl walks down the dirt New Hampshire road*. The writer is telling you the little girl is in New Hampshire, so there is no reason to use other facts to prove this is true.

The next and final part is theorems. Unlike axioms, theorems can be proven using deductive reasoning. To do this, though, you need references to other proven theorems or additional information. One example is how to find the congruency of two triangles. Based on other knowledge, you know that if all three corresponding angles and corresponding sides are the same, then the triangles must be congruent. So, if you know that two sides and one angle of triangle ABC are congruent to the same two sides and one angle in triangle DEF, then you have just proved their congruency, using side angle side (SAS). Theorems can be proven in poetry. To prove an idea or situation, you use facts found in earlier stanzas or information you have collected in everyday life, then relate it back to the line you are reading. This helps you to understand what is truly going on within the poem. This also helps by making sure you do not become confused due to the metaphors or symbols the poet might be using.⁶ Such an example can be found in "My Papa's Waltz" by Theodore Roethke.

My Papa's Waltz

*The whiskey on your breath
Could make a small boy dizzy;
But I hung on like death:
Such waltzing was not easy.*

*We romped until the pans
Slid from the kitchen shelf;
My mother's countenance
Could not unfrown itself.*

*The hand that held my wrist
Was battered on one knuckle;
At every step you missed
My right ear scraped a buckle.*

*You beat time on my head
With a palm caked hard by dirt,
Then waltzed me off to bed
Still clinging to your shirt.*

The writer here wants you to think of the father's

drunkenness as a waltz, but you know by past knowledge that this dance is no dance at all but instead a father beating his son. If you read this poem literally without relating it back to prior knowledge, you would have never understood the poet's attempt to hide this horrible event within a beautiful dance. Therefore, using deductive reasoning, you have proven that the father is not actually dancing.

In my final comparison between mathematics and poetry I will look at direct and indirect proofs. In mathematics, proofs are arguments which establish a statement's truth. A mathematical proof has a certain defined structure, which can be divided into steps. First is the initial step or hypotheses, which are assertions that are considered true without having to be proven. In poetry, the hypothesis is the structure of the situation, whether it be true or false.⁷ For example, if you are writing a poem about a cat and in the first line you say it lives in a house and drinks water, then throughout the poem you must make sure your facts stay consistent. If in stanza one she lives in a house and in stanza four she lives in an apartment, then that original statement is now false because she no longer lives in a house. To make the first statement true you must make the cat still live in a house even in stanza four. Indirect proofs can also be formed if there is a contradiction found, or if you assume the conclusion is false. This is found in poetry. When you read a poem, and as you get toward the end, a simple fact switches the meaning of the poem, such as in William Shakespeare's "Sonnet 33."

Sonnet 33

*Full many a glorious morning have I seen
Flatter the mountain tops with sovereign eye,
Kissing with golden face the meadows green,
Gilding pale streams with heavenly alchemy;
Anon permit the basest clouds to ride
With ugly rack on his celestial face,
And from the forlorn world his visage hide,
Stealing unseen to west with this disgrace.
Even so my sun one early morn did shine
With all-triumphant splendor on my brow;
But out alack he was but one hour mine,
The region cloud hath masked him from me now,
Yet him for this my love no whit disdaineth;
Suns of the world may stain when heaven's sun
staineth.*

This poem at first shows a dark lifeless morning. Shakespeare then goes through describing it and how terrible it is, yet towards the end the sun comes out for a instant. This is the turning point of the poem when the whole meaning switches from the feeling of depression and hopelessness to the possibility of happiness.

Before I this started researching this paper, I was not sure how much mathematics really related to writing and poetry, yet as I searched I started to realize poetry's connection with mathematics. I hope you have learned a little something while reading my report; I know I have learned a great deal.

*Mistress of mine, time and
Again you have wooed me with your
Theorems and proofs,
Held me captive with your abstract beauty, and
Enchanted me with your dance.
Mistress of mine, time and again I have been
Awed by the
Transcendent melodies you weave and the
Infinite tapestries you spin from only a sparse
Collection of symbols and signs.
Mistress of mine, it has been a long and glorious romance⁸
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Continued on page 34

Plastic Pebbles

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A seagull
Measures the height of winter surf,
Sun and wind seeping through his feathers
While beneath him waves gather rocks from the
 shoreline,
Sift, sort, grind, and leave them at low tide,
Glinting pebbles, glistening sand.

People count, sort, and tally the pebbles
As the tide rises and falls to the rhythm of the moon.
We read the moon's language
Measuring days, nights, months, years
According to the passage of sun and moon;
Our gaze soars from stars into the depths of space.

And we leave those pebbles on the beach and build
 machines to describe our universe.
Knowledge multiplies; accumulated thought patterns
Illuminate the darkness of abstraction.
The tide takes bottles from our shoreline,
Grinds them smooth, then tosses them back,
Muted green and brown
While a foghorn sounds through the mist
Barely audible above the wave roar.

Children gather rocks and bottle-pebbles from the
 beach to sort, count, tally, and weigh.
Parents feed facts into computers
Until waves of words and figures
Innundate our world behind the shoreline.

When thinkers left their counting pebbles by the sea
They built their theories on the supposition "If..."
They built, bound only by imagination and logic.

Still we create new theories from the depths of our
 insatiable minds,
Framing deductions, mathematical reason—
Concise amongst verbosity.
Our machines produce, computers test new ventures
Inspired by wind, sun, and space.

One day,
Two men taught and a computer performed.
The computer performed and the men learned.
Together they solved a problem,
Adding new dimension to our thought.
Together they built a proof mathematicians had
 sought alone for a hundred years.*

But what of limits?
What if applications clash with oceans
Or distortions destroy?

Today's tides pluck plastic bottles from the shoreline
And cannot toss them back ground smooth and
 glistening wet.
Instead toss them bent but indestructable onto rocks
Or gather them in eddies and currents to be carried
 through oceans
To contaminate distant beaches.

We will learn with our machines, produce, judge,
 explain, and solve
In unimagined ways
While seagulls watch,
Sun and wind filtering through their feathers;
Waves grind rocks and bottle-pebbles green and
 brown
To glistening sand;
Waves will silence the foghorn,
And what of the plastic?

* In 1976 two graph theorists, Kenneth Appel and Wolfgang Haken, proved that four colors suffice to color any map drawn on a plane so that no two adjacent countries are the same color. This is the first documented mathematical proof including computations compiled by computers (1,200 hours, 3 computers, used both as a research tool and in final computations). University of Illinois, July 1976. While solving this problem, the mathematicians learned from the computations carried out by the computers, and likewise, the computers' calculations were modified based on what the researchers deduced from the earlier calculations.

What Does Equality Mean?--The Basque View

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INTRODUCTION

In general, surrounding or underlying the concept of equality there is an idea of sameness. But same in what sense? In mathematics, an equivalence relation is defined as one which is reflexive, symmetric, and transitive,¹ but there are many such relations and so, depending on the case at hand, this is specified further. In geometry, for example, there is a distinction between numerical equality of, say, areas of triangles, and equality (or congruence) of both sizes and shapes. When discussing functions, for example, one distinguishes between identities and equations, and, for different subcontexts, such as matrix algebra, complex variables or vector analysis, particularized definitions are needed. The distinctions, and yet underlying unity, of these various equivalence relations are often only vaguely realized by learners and so could use more thoroughgoing discussion.

In the political realm, where equality is so central to our EuroAmerican views of democracy, justice, and fairness, *equality* is used mostly in regard to the rights or treatment of people vis-a-vis government, institutions, or businesses. While there is a long and ongoing history of philosophical and legal discussions of equality, when used in common catchphrases, it often means quite different things to different people.² Further, in common American -English usage, the connotations of *equality* and *equivalence* differ: *equality* "implies the absence of any difference," that is being *exactly* the same, while *equivalence* implies that, although there may be differences, "they amount to the same thing."³ These different realms of usage—the mathematical, the political, and the everyday—are, however, not strictly distinct. That they interact is too often ignored; the differing usages, no doubt, influence and support, or, at times, confuse each other.

To enlarge our thinking and stimulate discussion about what equality means, we add a quite different view. Among the Basque of Sainte-Engrâce, France, there is a concept *bardin-bardina* translated as "equal-

equal." Consideration of the Basque concept makes us realize that there is cultural variation in even as basic a concept as equality. In addition, elaboration of their concept, within the Basque context, can provide an opportunity to display mathematical ideas used in the promotion of cooperation. All too often, in an attempt to embed mathematical ideas in realistic-sounding contexts, we overlook that we are implicitly transmitting values as we present numerous examples of competition, winning, and financial gain. Here, instead, the focus is on how people organize their interactions to provide mutual assistance and receive mutual benefit.

The Basque concept of equality is underpinned by two operational principles that structure relationships so that everyone both gives and receives. The principles are referred to as *iingüirü* and *aldikatzia*. The former is translated into English as "rotation," in the sense of "moving around a centre," and the latter as "'serial replacement' as well as 'alternation.'" How these mathematical ideas apply in this context and how they relate to equality is best described in terms of their operation.⁴ Where we use some algebraic symbols in the description, the notation is ours and not that of the Basque. The symbols are introduced to succinctly capture and express, in terms familiar to us, the system involved and some of its logical implications. More important, the fact that this translation is possible highlights the mathematical nature of the ideas involved.

CONTEXT

The community of Sainte-Engrâce is in the Basque province of Soule, one of the nine Basque provinces in the Pyrénées-Atlantique which straddle the French-Spanish border. Although the exact origins of the Basque are unknown, it is generally agreed that they predate the French and Spanish-speaking peoples in the region around them by perhaps thousands of years. In the 1970's, at the time of a study of Sainte-Engrâce, there were about two million Basque, with

about three-quarters of them living in the Basque provinces in Spain, one-eighth in the Basque provinces in France, and the rest living in other areas of the world. Having their own language, a rich history, their own political and social organization, and long-held traditions, the recent history of the Basque has been marked by conflict with the nation states which encompass them. Nevertheless, the Basque way of life continues, particularly in a place like Sainte-Engrâce which, situated in the high mountains, is one of the most geographically and socially isolated communities in the region. Although the population declined from about 1000 people in the late 1900's to about 375 in the 1970's, the community remains self-reliant, centering on small farms and shepherding.

The mountains which surround the Sainte-Engrâce region range from about 1000 m to 2500 m. The Basque conceive of the region in which they live as enclosed by a circle of mountains with their households forming another circle within that. Whether or not this is actually the case, this spatial model forms the basis for their idea of circularity which pervades many of their interactions. In this circle everyone has neighbors to the left and neighbors to the right. No one is first and no one is last. Everyone's participation is involved in keeping the circle unbroken.

THE GIVING OF BREAD

Until the 1960's, a fundamental circular exchange was the giving of blessed bread. Each household regards its neighbor to the right as its first neighbor. (The directions right and left are as viewed from the center of the circle so that right is clockwise and left is counterclockwise.) The giving of bread took place weekly and was thought of as being given from first neighbor to first neighbor. That is, each Sunday a woman from one particular household, call it H_i , bought two loaves of bread to the church where it was blessed and partially used in a church ritual. Then, before sunset, a portion of the bread was given by H_i to her first neighbor, namely to H_{i+1} . The following week H_{i+1} was the bread-giver and H_{i+2} the bread-receiver. Thus, the giving (and receiving) of bread moved around the circle serially, taking about two years to complete one cycle of about 100 households. While each household was both a giver and receiver of bread, this mode differs from simple reciprocity; only if there were a total of two households would H_i and H_{i+1} directly reciprocate as each other's first neighbor.

FIRST NEIGHBOR OBLIGATIONS

In a more extensive, ongoing, cooperative arrangement, the exchange among neighbors is again predicated on the circular model, but this exchange involves several first neighbors. The *first* first neighbor of H_i is, as in the breadgiving, H_{i+1} , the neighbor to the right; the *second* first neighbor of H_i is the neighbor on the left (H_{i-1}); and the *third* first neighbor is the next on the left (H_{i-2}). Thus, for example, when there is a death in household H_i , the household calls upon its first neighbors for assistance. As a group H_{i-2} , H_{i-1} , and H_{i+1} help to keep the household going, but H_{i+1} provides particular assistance in specific preparations for the funeral. And, on the occasion of a home birth for H_i , it is a woman of household H_{i-1} who serves as the midwife.

Planting, harvesting, threshing, sheep shearing, and pig slaughtering all require the work of more than one person and so, there too, the first neighbors are called upon. These assistances are directly reciprocated by providing food and drink and by the giving of small gifts, but, primarily, the reciprocation is serial, that is, by assisting, when called upon, as the first neighbors of others.

A particularly interesting result of this mode of interaction in the farming yearly round is that households must schedule their work with the obligations of others and to others in mind. Also, for the same chore, each household gets to work with different groups of households and to play different roles within those groups. H_i , for example, works in groups (H_{i-2} , H_{i-1} , H_i , H_{i+1}), (H_{i-3} , H_{i-2} , H_{i-1} , H_i), (H_{i-1} , H_i , H_{i+1} , H_{i+2}), and (H_i , H_{i+1} , H_{i+2} , H_{i+3}), taking the roles of primary household, and first, second, and third first neighbors respectively. And, to avoid causing conflicting obligations for himself or any of his neighbors, H_i cannot schedule his household's work on the same day as the work of H_{i-3} , H_{i-2} , H_{i-1} , H_{i+1} , H_{i+2} , or H_{i+3} because, for example, H_i 's third first neighbor (H_{i-2}) is H_{i-3} 's first first neighbor and his first first neighbor (H_{i-1}) is H_{i+3} 's second first neighbor.

We note that the subscript arithmetic is mod n , where n is the number of households and $n \geq 4$. (For $n = 4$, this cooperative mode reduces to a group of 4 households which always work together but with rotating roles.) It is particularly important to recognize that the equivalence relations in modular arithmetic, usu-

ally referred to as congruence rather than equality, is

$$H_{i+nk} \pmod{n} = H_i \text{ for } k = 0, \pm 1, \pm 2 \dots$$

That is, to capture the circular nature of the Basque concept, we must involve the algebra and form of equivalence in modern mathematics that applies to cycles.⁵

We further observe that if a particular job takes a group of four households one day, it would take a minimum of

$$\left\lceil \frac{\left\lceil \frac{n}{4} \right\rceil}{4} \right\rceil \text{ days}$$

to complete the job for all n households.⁶ This minimum completion time has a minimum of 4, taken on when n is a multiple of 4, a maximum of 7 taken on when $n = 7$, and is equal to 5 for $n > 12$.

SUMMER PASTURING

By far the most intricate cooperative arrangement involves the shepherding and cheese-making groups that work and live together during the summer months. These groups of households share in the ownership of pasturage sites in the mountains. The origin and practices of these groups are part of a long tradition which was described in writing as early as the 1600's. Prior to the 1900's, the ideal ownership group consisted of 10 households, each contributing 50 to 60 ewes and 2 rams to the summer flock and one man to the working unity. The flock of about 550 sheep had to be driven up into the mountains in late May, watched over until they were driven down to the valley for shearing in July, then driven back up to be watched over until returning to their valley homes at the end of September. Additional important aspects of the May to July work were the twice daily milking of the sheep, and the making of cheese from the milk. Different roles were defined that encompassed the various jobs that needed doing, and a formal system of rotation was used to insure that everyone was equal in terms of work contributed, in terms of cheeses produced, and in terms of status.

The households, first of all, had a specific order in the ownership group that remained unchanged from year to year. For the May-July period, for the working

group of 10 men, there were 6 explicit roles which required 6 of the men to be together at the mountain site. Thus, calling the households' representatives H_1, H_2, \dots, H_{10} , and the work roles ranked in status order R_1, R_2, \dots, R_6 , once the sheep were safely at the mountain site, assuming the household count started with H_1 , the assignments were: $H_1 \rightarrow R_1, H_2 \rightarrow R_2, \dots, H_6 \rightarrow R_6$, and H_7, H_8, H_9, H_{10} returned home. After 24 hours, the rotation would begin: H_7 would ascend the mountain, keeping to the right, and then H_1 would descend, keeping to the left. Their ascent and descent is conceived of as taking place in a circle. Upon his arrival on the mountain, H_7 would take on role R_6 and each of the others would move up one role: $H_2 \rightarrow R_1, H_3 \rightarrow R_2, \dots, H_7 \rightarrow R_6$. Similarly, every 24 hours, at the end of day i , there would take place the rotation up and down of H_{i+6} and H_i , respectively, and the moving up by one role of the others: $H_{i+1} \rightarrow R_1, H_{i+2} \rightarrow R_2, H_{i+6} \rightarrow R_6$. With 10 men cycling through this rotation, the subscript arithmetic is, of course, mod 10. Thus on, say the 18th day, those present at the mountain site would be $H_8, H_9, H_{10}, H_1, H_2$, and H_3 in roles R_1 through R_6 respectively. Out of every 10-day period, each man spent 6 consecutive days at the mountain site and 4 days at home. Generally, from May to mid-July, each of the 10 men carried out each of the 6 roles about 6 times with, for reasons of equity to be explained later, an extra turn at R_1 for H_1 and H_2 .

From the time of shearing in July until the end of September, because milking and cheese-making were complete, the number of men needed at the mountain site was reduced to two with just two roles, R_1' and R_2' . For this, two men remained on the mountain for 6 consecutive days, alternating daily between roles R_1' and R_2' . After the 6-day period, the pair descended the mountain and the next pair in the cyclic order ascended. Thus, if the period began with H_1 and H_2 in roles R_1' and R_2' , then the next day $H_2 \rightarrow R_1'$ and $H_1 \rightarrow R_2'$, and so on until, on the 7th day, $H_3 \rightarrow R_1'$, while $H_4 \rightarrow R_2'$, or, in general, on the i th day of this second phase,

$$H(2 \left\lfloor \frac{i-1}{6} \right\rfloor + k) \pmod{10} = R_k' \text{ where } k=1 \text{ if } i \text{ odd,} \\ k=2 \text{ if } i \text{ even.}$$

Hence, during a 30-day period, each man spent 6 consecutive days at the mountain site, 3 of them as R_1' and 3 as R_2' , and 24 days at home. Usually, each man had two of these 6-day turns on the mountain.

By these rotations, the men's contributions were the same in terms of time spent at home, time spent at the mountain site, time spent in each of the six roles R_1, \dots, R_6 , and time spent in the roles R_1' and R_2' . The procedure also insured receiving an equal number of cheeses made from the milk of the sheep. These cheeses were an important part of a household's annual food supply. One responsibility that went with the highest status role (R_1) was making two cheeses and watching over the cheeses that others had previously made. With the exception of the first cheese made on the first day and the first cheese made on the second day, the cheeses made by a person were for his household's use during the year. (The first cheese was sold outside of the community with all the members of the group sharing equally in the profit, and the other was given to the priest or guard of the forest. The extra turns noted before, of H_1 and H_2 being R_1 and, hence, of making more cheese, were to compensate for these cheeses.) In general, a cheese weighed about 8 or 9 kilos. With six turns at being R_1 and making two cheeses on each of these days, each person took home about 100 kilos of cheese.

In cases where a household had fewer sheep than the ideal of 50 to 60, they could own a half share in the cooperative. In that case, two households together owned a full share and together contributed the standard number of sheep as well as two workers, one from each household. The two workers had to alternate their six-day mountain stay so that each did three

of the six stays in the May-June segment and one of the two stays in the July-September segment. In this way, they each did half as much work and got half as much cheese as did the others, but they did not modify the rotations up and down the mountain or through the various roles.

A larger cycle in which the annual cycles are embedded is the multi-year cycle. We noted that the ten households are in a fixed order H_1, H_2, \dots, H_{10} . The order remains fixed throughout time, but which household representative starts a year as R_1 rotates by one position each year. That is, in a hypothetical Year 1, H_1, H_2, \dots, H_6 are the first subgroup at the mountain site but then, in Year 2, the first subgroup would be H_2, H_3, \dots, H_7 , and so on, from year to year. (To reflect this in our previous statements involving H_i , i should be modified to $i+Y-1$ where Y is the year number of the cooperative's operation.)

Finally, we introduce the crucial issue of equality of status which becomes particularly significant for groups smaller than the ideal of ten. The six roles, from highest to lowest status are: R_1 = woman of the house; R_2 = master shepherd; R_3 = servant shepherd; R_4 = guardian of non-lactating ewes; R_5 = guardian of lambs; and R_6 = female servant. R_1 is the cheesemaker and is also in charge of cooking and of cleaning the hut in which the six men live. R_6 serves as his servant in the household chores. R_2 , the master shepherd, organizes and directs the work of R_3, R_4 , and R_5 . Because

| | Day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... | i | ... | i+5 |
|------|-------|-------|-------|-------|-------|----------|----------|----------|-----|-----------|-----|------------------|
| Role | R_1 | H_1 | H_2 | H_3 | H_4 | H_5 | H_6 | H_7 | | H_{i+1} | | H_{i+5} |
| | R_2 | H_2 | H_2 | H_4 | H_5 | H_6 | H_7 | H_8 | | H_{i+2} | | H_{i+6} |
| | R_3 | H_2 | H_4 | H_5 | H_6 | H_7 | H_8 | H_9 | | H_{i+3} | | H_{i+7} |
| | R_4 | H_4 | H_5 | H_6 | H_7 | H_8 | H_9 | H_{10} | | H_{i+4} | | H_{i+8} |
| | R_5 | H_5 | H_6 | H_7 | H_8 | H_9 | H_{10} | H_1 | | H_{i+5} | | H_{i+9} |
| | R_6 | H_6 | H_7 | H_8 | H_9 | H_{10} | H_1 | H_2 | | H_{i+6} | | $H_{i+10} = H_1$ |

Figure 1: The rotation through six roles with ten households. (Subscript arithmetic is mod 10.)

there is a decided hierarchy in the roles, the rotation is of special significance in preserving equality. Having ten men rotate through the six roles insures that no status hierarchy is consistently imposed. In particular, whoever serves as R_6 (house servant) when some H_i is R_1 (woman of the house) will serve as R_1 (woman of the house) when that H_i is R_6 (house servant). And, the Basque further note that this H_i will never be above those whom his R_6 will be above when he serves as R_1 . (This is seen in Figure 1 where, for example, on day 1, H_1 and H_6 are in roles R_1 and R_6 respectively, but on day 6 their roles are reversed. And, since H_6 is above some or all of H_7, H_8, H_9, H_{10} on days 2-6, H_1 is never above any of them.) In general, using mod 10 subscript arithmetic, on day i , $H_i = R_1$ and $H_{i+5} = R_6$, but on day $i+5$, their roles are reversed: $H_{i+5} = R_1$, $H_i = R_6$. Also, since H_{i+5} is above $H_{i+6}, H_{i+7}, H_{i+8}$, and H_{i+9} , H_i is never above them. Similarly, H_{i+5} is never above $H_{i+1}, H_{i+2}, H_{i+3}$, and H_{i+4} .

After 1900 the number of households in the cooperatives decreased as a result of the overall decrease in the number of community households. With fewer households in each, cycling through the various roles would still insure equality of time and work contributions, but the criteria for the equality of status would not be met without adjusting the number of roles. To view this generally, let n = number of households in the cooperative and, hence, use subscript arithmetic mod n , and let r number of roles. To insure the role reversal of woman of the house/house servant, that is, to insure that

$$\begin{aligned} H_i &= R_1 \text{ and } H_{i+r-1} = R_r \text{ on day } i \text{ and} \\ H_{i+r-1} &= R_1 \text{ and } H_{i+2r-2} = R_r \text{ on day } i+r-1, \end{aligned}$$

the following relationship between roles and households would have to hold:

$$\begin{aligned} i+2r-2 &= i(\text{mod } n) = i+n, \text{ or} \\ 2r-2 &= n. \end{aligned}$$

This relationship would also insure that there is no overlap between those whose roles are below those of H_i and those whose roles are below those of H_{i+r-1} since this criterion is satisfied whenever $n > 2r-2$.

Clearly, the relationship $2r-2 = n$ is satisfied for $r = 6$ and $n = 10$. And, while we do not know how the Basque arrived at the requirements, the Basques knew

that there could be at most 5 roles when there were 8 households, 4 roles when there were 6 households, and 3 roles when there were 4 households. To accomodate odd numbers of households and the situations where there were more than the necessary minimum of households, meeting the requirement $n > 2r-2$ became sufficient. In these cases, the stipulation of H_i and H_{i+r-1} being over non-overlapping groups is maintained, but the requirement of the complete role reversal of H_i and H_{i+r-1} is loosened. The number of roles were reduced, in about 1900, from six roles to five roles by deleting R_4 , and then, in about 1940, they were further reduced to four roles by deleting R_5 . In the 1960's and 1970, they were still further reduced by either reassorting the functions into three newly titled, but still hierarchically ranked roles, or by creating only two roles by combining into one the master ranks R_1 and R_2 , and into another the servant ranks R_3 and R_6 .

CONCLUSION

The Basque concept of equal-equal is evidenced by a variety of different circles and cycles. There is, first of all, the giving and receiving of bread in which H_i simply gives to H_{i+1} and the giving moves around the circle made up of all households in the community. There are also the ongoing first neighbor obligations for which the circle of all is divided into fixed, adjacent, overlapping sets of size 4. For the summer pasturing, the community separates into subunits of households, and each subunit is a circle which rotates within itself. There is the annual rotation in which shareholder H_{i+1} replaces shareholder H_i in the starting position of the season's cycle. Beginning with the designated starting household, the season's cycle is made up of two consecutive subcycles: in the first subcycle, one man (H_{i+6}) goes up and one (H_i) comes down the mountain daily, and, in the next subcycle, two (H_{i+2}, H_{i+3}) go up and two (H_7, H_{i+1}) come down every six days. Further, within these subcycles, there can be alternation within a single H_i of a pair of joint owners of the share. And, while on the mountain, the earlier occupants cycle once through $r \leq 6$ roles, and the later ones cycle three times through two roles.

The variety and interrelatedness of the cycles, as well as the cycles themselves, testify to the deep embedding of these ideas in the culture. We not only see algorithms of interaction involving cycles, sequences, and alternation, but a spatial concept of circle under-

pinning them, as well as an overarching concept of equality uniting them.

The overarching concept, “equal-equal”, is not a static relationship as is our conventional mathematical or everyday equality. It is a dynamic process of interaction in which an essential feature is that the participants know what is expected of them and they know what to expect from others. That is, the actors in the process move in synchronization, doing different things, at different times, but together making up a whole. If one were to stop the process at an arbitrary point in time, there would be inequities in what has been contributed, what has been received, and who is superior to whom. But, just as a circle is enclosed by a never-ending line, the process of creating an equal-equal relationship continues throughout the season and throughout the years.

In a previous discussion of the spatial ideas of several cultures,⁷ we noted that for many outside of our Euro-American stream, time and space are intimately connected and, what is more, the circle is as fundamental for them as lines and angles are for us. While it is surprising to think that these differences may pervade the concept of equality as well, it may, in fact, be that where equality is conceived of as a static point of balance separating more and less or better and worse, it is often too precarious to be stable or easily attained.⁸

NOTES

1. More specifically, an equivalence relation R on set S is one which satisfies the following for all elements a, b, c of set S :

Reflexive: aRa

Symmetric: If aRb , then bRa ;

Transitive: If aRb and bRc , then aRc .

2. For socio-political discussions of equality that were influential in Euro-American culture, see, for example, *Nicomachean Ethics*,

Book V, Aristotle, 4th century B.C.E.; Jean Jacques Rousseau's "A Discourse on the Origin of Inequality" (1754) and "The Social Contract" (1762); and John Stuart Mill's "On Liberty" (1859).

3. The phrases quoted appear under the synonyms for same on p. 1289 of *Webster's New World Dictionary of the American Language*, College Edition, World Publishing Co., N.Y., 1966.

4. My discussion of the Basque and their ideas is derived from *A Circle of Mountains: A Basque Shepherding Community*, Sandra Ott, Clarendon Press, Oxford 1981. Of particular relevance are pp. vii-viii, 1-41, 63-81, 103-106, 129-170, and 213-217. The few phrases directly quoted are from p. vii.

5. For a circle of, for example, 5 households, $n = 5$ and the households are $(H_1, H_2, H_3, H_4, H_5)$. When counting around the circle, the household identified as, say, H_{22} is the same household as H_2, H_7 , or H_{12} . For more about modular arithmetic, see, for example, Chapter 7 on congruences in *Invitation to Number Theory*, Oystein Ore, New Mathematical Library, MAA, Washington, D.C., 1975.

6. The symbol $[x]$ denotes the greatest integer less than or equal to x . For example, $[4.0] = 4$, $[4.1] = 4$, and $[4.99] = 4$. The symbol $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . For example, $\lceil 4.11 \rceil = 5$, $\lceil 4.99 \rceil = 5$, and $\lceil 5.0 \rceil = 5$.

7. See Chapter 5, "The organization and modeling of space" in *Ethnomathematics: A Multicultural View of Mathematical Ideas*, Marcia Ascher, Brooks/Cole, Belmont, CA, 1991 (paper edition, Chapman & Hall/CRC, New York, 1994.)

8. Other mathematical ideas of the Basque are being studied extensively by Roslyn M. Frank. See, for example, "The Geometry of Pastoral Stone Octagons: The Basque *Sarobe*," R. M. Frank and J. D. Patrick, pp. 77-91 in *Archeoastronomy in the 1990's*, Clive L. N. Ruggles, ed., Loughborough Group D Publications, London, 1993, or "An essay on European ethnomathematics: the coordinates of the septuagesimal cognitive framework in the Atlantic facade," R. M. Frank, 78 pp., ms., 1995. Also, a special counting technique among the Basque living in California is described in "Counting sheep in Basque," Frank P. Arawjo, *Anthropological Linguistics*, 17 (1975) 139-145.

On Solving Equations, Negative Numbers, and Other Absurdities: Part II

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7. NEGATIVE NUMBERS

Yet the uneasiness won't go away. The history of mathematics is full of "impossible" objects that later became common, so much so that we wonder at the blindness of our ancestors. Irrational ratios horrified the Pythagoreans, but were quite understandable to the school of Plato. Imaginary numbers were simply not there at all, that is, there were no "numbers" x such that $x^2 + 1 = 0$, until such numbers somehow turned up "temporarily" in some of the cases of Cardano's sixteenth century solution of the cubic equation. Proper combination of such imaginary numbers turned out to deliver genuine, "real" answers, that "checked" in every detail.

Negative numbers, too, have had such a history, and that not so long ago. Even today, while we teach children the number line, positives to the right and negatives to the left (or positives up and negatives down, as the y -axis is marked in the Cartesian plane), and while we feel quite superior to those of our ancestors who said you couldn't subtract 9 from 7 (We know the answer to be -2; don't we?), let us consider our algorithm for the more difficult subtractions that we teach in the third or fourth grade:

We subtract 19 from 57; how? We can't take 9 from 7 so we regroup: Instead of subtracting $10+9$ from $50+7$, we subtract $10+9$ from $40+17$. Now 9 from 17 is 8 and 10 from 40 is 30, and our answer is $8+30$ or 38. In my day this was called "borrowing:" we borrowed the "1" — really 10 — from the 5 (really 50), and so on, with a certain way of placing the borrowed digit on the page. In effect, we replace the array

$$\begin{array}{r} 57 \\ -19 \\ \hline \end{array} \text{ by the new arrangement}$$

$$\begin{array}{r} 4(17) \\ -19 \\ \hline \end{array}$$

before performing the operation that produces 38 as the answer.

But this whole scheme is predicated on the notion that "you can't take 9 from 7," surely nothing other than the quaint prejudice we have a minute earlier been priding ourselves on having overcome! We can take 9 from 7 if we have the courage of our convictions. Damn the torpedoes; let us take 9 from 7 and get -2, and then take 10 from 50 and get 40, and then combine -2 with 40 to get 38, by golly, the correct answer! Here is the layout:

$$\begin{array}{r} 57 \\ -19 \\ \hline \end{array}$$

4(-2), i.e. 40-2, or 38.

Is there anything wrong with this?

Yet with no sense of inconsistency, teachers who tell children about negative numbers on the number line in Grade 2 say that "you can't take 9 from 7" in Grade 3, to introduce the apparent necessity for "borrowing". One can give a good reason for all this, in that the "regrouping" or "borrowing" scheme can be chained in a convenient manner for a longish problem, while combining the positive and negative differences over a multi-digit subtraction might prove more tedious, but this is probably not why we have the algorithm we do. Our 'regrouping' scheme as written above was invented four or five hundred years ago, in the early years of the European adoption of the decimal system, and in that era subtracting large from small numbers was suspect. Try it on an abacus, for example, which historically preceded the written algorithm but uses the same idea. In fact, there is not an arithmetic book in the Western world that

shows how to subtract 866 from 541 by placing the figures in this form

$$\begin{array}{r} 591 \\ -866 \\ \hline \end{array}$$

and then going through the “borrowing” ritual to the bitter end. Where on earth is the last “loan” to come from? Our schoolbooks even today evade the question by merely announcing that the difference $b-a$ is the negative of the difference $a-b$, and telling us to solve the subtraction problem printed above by computing the opposite difference

$$\begin{array}{r} 866 \\ -591 \\ \hline 275 \end{array}$$

in the approved manner, finally changing the sign of the positive result, to -275 , to answer the original problem.

8.1. DE MORGAN'S RESERVATIONS ABOUT NEGATIVES

Now, so recent and illustrious a mathematician as Augustus DeMorgan, while willing to go so far as to make temporary use of so ridiculous a notion as “ -2 ” as we did in the earlier subtraction of 19 from 57, was still unwilling to grant a negative number a real final existence. In his 1831 book, *On the Study of Mathematics* (reprinted in 1898 by the Open Court publishing company in La Salle, Illinois), Chapter IX is named “On the Negative Sign, etc.” Here (p.103) De Morgan cautions the beginner in algebra to beware of negatives:

If we wish to say that 8 is greater than 5 by the number 3, we write this equation $8-5=3$. Also to say that a exceeds b by c , we use the equation $a-b=c$. As long as some numbers whose value we know are subtracted from others equally known, there is no fear of our attempting to subtract the greater from the less; of our writing $3-8$, for example, instead of $8-3$. But in prosecuting investigations in which letters occur, we are liable, sometimes from inattention, sometimes from ignorance as to which is the greater of two quantities, or from misconception of some of the conditions of a problem, to reverse the quantities in a subtraction,

for example to write $a-b$ when b is the greater of two quantities, instead of $b-a$. Had we done this with the sum of two quantities, it would have made no difference, because $a+b$ and $b+a$ are the same, but this is not the case with $a-b$ and $b-a$. For example, $8-3$ is easily understood; 3 can be taken from 8 and the remainder is 5; but $3-8$ is an impossibility; it requires you to take from 3 more than there is in 3, which is absurd. If such an expression as $3-8$ should be the answer to a problem, it would denote either that there was some absurdity inherent in the problem itself, or in the manner of putting it into an equation. Nevertheless, as such answers will occur, the student must be aware what sort of mistakes give rise to them, and in what manner they affect the process of investigation...

I caution the reader here that De Morgan is not naive, and that he is making a philosophical point from which he wishes to derive the usual rules of algebra as we know and use them, including “negatives,” and that his general idea, as we shall see, is that playing with absurdities like $3-8$ AS IF they made sense can be made to lead to correct final conclusions. It takes him a full chapter to explain this.

It would be wise for present-day teachers to have some appreciation of the philosophical problem involved here, and its clever modern solution by “negative numbers” defined as equivalent pairs of such “impossible” subtraction pairings. But this process, which mathematicians call “embedding a commutative semi-group in a group,” while logically satisfying and consistent, does not really attack the problem of what the new numbers mean in applications to the world of apples and gardens. However, this is not something for the 9th grade to elucidate. The mere representation of negative numbers as they appear in practical life, debts as against credits, past as against future, and so on, will usually do the job without needing such sophistication.

De Morgan observes this himself later in the same chapter. He has set up a problem in which the answer has turned out to be $-c$, and the surprise is that we suddenly discover that c is positive. What are we to make of the absurd answer, $-c$? On page 55 he gives a simple example:

"A father is 56 and his son 29 years old. When will the father be twice the age of the son?"

Putting x a time when this *will* happen, i.e. in the future, he arrives at the equation $2(29+x) = 56+x$, i.e. twice the age of the son x years from now will equal the father's age x years from now. The solution is $x = -2$. It checks in the equation, but what does it mean? Unlike the problem of the rectangular garden above, this negative number is the *only* answer. Can it mean that the problem has no solution?

Today we would immediately construe this solution to mean that it was two years *ago* that the son *was* half the age of the father, and we would be done with it. To De Morgan this needed more explanation. It was a mistake, he explains, to have begun the algebraic formulation of the problem by putting the date in the future. The negative sign, an absurdity, tells us we have made such a mistake and have asked an impossible problem. We should instead let x be the number of years into the *past* that the doubling of age occurred. then $2(29-x) = 56-x$, i.e. twice the age of the son x years *ago* equals the father's age x years *ago*. The solution is $x = 2$, and De Morgan is philosophically satisfied.

Just the same, this kind of thing happens so often that there must be a simpler way to interpret what has happened. De Morgan announces his principle, his justification for the use of absurd numbers, on page 121:

...When such principles as these have been established, we have no occasion to correct an erroneous solution by recommencing the whole process, but we may, by means of the form of the answer [by 'form' he means negative or positive], set the matter right at the end. The principle is, that a negative solution indicates that the nature of the answer is the very reverse of that which it was supposed to be in the solution; for example, if the solution supposes a line measured in feet in one direction, a negative answer, such as $-c$, indicates that c feet must be measured in the opposite direction; if the answer was thought to be a number of days after a certain epoch, the solution shows that it is c days before that epoch; if we supposed that A was to receive a certain number of pounds, it denotes that he is to pay c pounds, and so on. In deducing this principle

we have not made any supposition as to what $-c$ is; we have not asserted that it indicates the subtraction of c from 0; we have derived the result from observations only, which taught us first to deduce rules for making that alteration in the result which arises from altering $+c$ into $-c$ at the commencement; and secondly, how to make the solution of one case of a problem serve to determine those of all the others...reserving all metaphysical discussion upon such quantities as $+c$ and $-c$ to a later stage, when [the pupil] will be better prepared to understand the difficulties of the subject.

8.2. DE MORGAN'S RESERVATIONS AS TO IMAGINARY NUMBERS

From this point onwards, De Morgan uses negative numbers without much shame, stating for example that a positive number has two square roots, one of them negative. On the other hand, he still does not use negatives entirely freely. In discussing the quadratic equation a few pages later he distinguishes six cases, viz.

$$ax^2+b = 0$$

$$ax^2-b = 0$$

$$ax^2+bx+c = 0$$

$$ax^2-bx+c = 0$$

$$ax^2+bx-c = 0$$

$$\text{and } ax^2-bx-c = 0.$$

This is to say that he is loath to permit a , b , or c to be negative, since, after all, there is no need. Whatever we today might call the signs of the coefficients is taken care of by letting the letters always represent positive numbers but having the equation take on the appropriate one of the six forms listed. This all leads to an analysis of the sign of the discriminant, b^2-4ac in some cases and of b^2+4ac in others, all very correct and difficult to remember. (In many American school algebra books of a hundred years ago students were asked to memorize the analysis of all six cases, and whether the roots in each case would be positive, negative, etc.) But worse is to come: When the discriminant is negative, a wholly new problem emerges: imaginary numbers.

De Morgan was writing in 1831, but in an insular England that was largely ignorant of recent developments in Continental mathematics. The Argand diagram for complex numbers had been known for 35

years, and Gauss and Cauchy had developed a science of complex numbers almost to the point of view taken today, but De Morgan makes no attempt in his book to develop a philosophy of their interpretation equivalent to what he has done for negatives. Perhaps he understood more than he was saying, but in this book, designed for teachers of children, he refrained from its elaboration. On page 151 he writes:

We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility. It depends upon the fact, which must be verified by experience, that the common rules of algebra may be applied to these expressions without leading to any false results...

Despite these pleasant features, he denies them any sense. He proposes two problems to distinguish his

meanings: The first is the problem of the ages of father and son described above, where a *negative* answer can be made to yield up some sense, either as a guide to a restatement of the problem, or by the device of interpreting such a number as the same as its positive opposite, taken in an opposite direction. The equivalence of the two devices is of algebraic and practical importance. But his second example, he thinks, yields no such practical interpretation. Here it is: "It is required to divide *a* into two parts, whose product is *b*. The resulting equation is $x^2 - ax + b = 0$..., the roots of which are imaginary when *b* is greater than $a^2/4$." Try as he may, he cannot get out of this one. If he replaces *x* by $-x$ in the problem the roots are still imaginary when *a* is too small. (For De Morgan, "imaginary" means what we call complex.) He concludes that there is an essential difference between mere negative numbers, which can be repaired by a reinterpretation of the problem, and imaginary numbers, which for all that they obey the usual algebraic rules, cannot be made to represent anything sensible.

Of course, he has a physical prejudice in the back of his mind here. The problem of dividing *a* into two parts whose product is *b* is an ancient one, Babylonian but put into geometric form in Euclid, where it is con-

strued as asking for a segment of length *a* to be partitioned into two segments which are sides of a rectangle of given area. (We would say "of given area," whereas Euclid remains purely geometric, and exhibits as the datum "*b*" a triangle to which he wants the resulting rectangle to be equivalent in his own sense of "equals." There are no numbers at all, hence no "areas" in our sense, in Euclid's formulation of such problems.)

Euclid's theorems provide a *construction* by which the point of partition may be found, but he notes a limitation: If the triangle *b* is larger than the square built on $a/2$ (i.e. half the segment *a*), then the necessary point of partition cannot be found. And that's the end of it: impossible. Euclid's "impossibility condition" is pre-

cisely our criterion concerning the discriminant, as it turns out. It says that the given length *a* is simply too short to accomplish the asked-for job, no matter where you divide it.



Complex numbers are absurd if construed as line segments — or are they?

Neither Euclid nor De Morgan construes this problem in any other way; it is plain that the number *a*, which is to be partitioned in De Morgan's problem, looks to him like a line segment, and that there is plainly no solution, not even one that can be reinterpreted as an "opposite" when it turns out negative, when *b* is larger than the square upon $a/2$. Yet today, we often take a different point of view.

To us, to "divide *a* into two parts" when *a* is a *number*, means nothing other than to find two *numbers* whose sum is *a*, and this can be done in such a way that the product is any given *number* (not area) *b* is easy, when complex numbers are allowed as answers. Complex numbers are absurd if construed as line segments — or are they? Remember, -10 was also absurd, when construed as a length.

9. THE NEGATIVE ROOT IS NOT ABSURD!

But this is not the only interpretation of the number -10 that turns up in our gardening problem.* Ah, how much wiser we are, or think we are, than our forefathers! Let us return to the problem of the garden, whose area is to be 600 square yards, and one side of which is 50 more than the other. We put *x* for the

*Introduced in Part I.

“length” of the garden, and found that x had to be 60 or -10, if anything. We rejected -10 as absurd, and solved the problem: 60 was the length, 10 the width.

Now where is this garden to be located? Here: One corner of it is under my feet, and the length is to be taken to the east, the width to the north. We can walk around the garden by walking 60 yards east, ten yards north, 60 back to the west and 10 south again and here we are. What about -10? Suppose we use that absurd solution as De Morgan, poor, simple De Morgan, suggested. We now surround what piece of land? Well, $x = -10$ and the “width” is 50 yards less, or -60, so: We walk -10 yards east, i.e. +10 yards *west*, then -60 yards north, i.e. 60 yards *south*, then back ten yards and back 60 yards and here we are at the origin (original corner). It is a totally different piece of land, to be sure, lying in the fourth (Cartesian) quadrant rather than the first. Its east-west dimension is of absolute value 10 rather than 60, so that “length” might be considered a strange description of that part of the boundary; but it, with the “width” of absolute value 60, satisfies all criteria of the problem. Its length is — as a *number* — indeed fifty more than its width (-10 is greater than -60 by 50, is it not?), and its area is 600, if “area” is the product of the numbers that describe the sides.

The answer the teacher expected was then 60 yards east by 10 yards north. But the stupid kid who insisted on “checking” the impossible answer $x = -10$, and got it to “check” at an area of 600 had just as good an answer, only his garden had a different orientation and position. I wonder what a Babylonian would have said to that.

One lesson that comes from all this is summarized by the title of a famous paper by the physicist Eugene P. Wigner, “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” (*Comm. in Pure & Appl. Math.* v.13 (1960), 1-14). The present example, interpreting the ‘absurd’ second solution of the quadratic equation, is trivial compared to the sort of thing Wigner mainly had in mind, but it is of the same nature: the equations arrived at by scientists to describe some part of the physical world often seem to contain more information than the inventors thought they had put into it, and that it is one of the wonders of the life of science to discover such a thing in practice. But one also has to know how to look.

10. THE ANALYTIC GEOMETRY OF THE GARDEN

How did anyone ever think of that second solution to the garden problem? It sounds like a stretching of the meaning of “-10” to suddenly start talking east and west, north and south, but in truth we do talk that way in the 20th century all the time. Here is a reformulation of the garden problem which will *automatically* make sense of the “absurd” solution as well as the usual one. The word “analysis” was used above to describe the process of algebra we were using; well, the reformulation has to do with analytic geometry. Any child can do it:

PROBLEM: Let a rectangle in the plane have one corner at (0,0) and the opposite corner at (x,y) , where $y = x - 50$. Find all the corners if the area is to be 600.

ANSWER: Notice the problem does not insist on (x,y) being in the first quadrant. The area is clearly the absolute value of xy , whatever quadrant (x,y) is in. Since $y = x - 50$, we set $|x(x-50)| = 600$ and hope such an x can be found, as above. Then either $x(x-50) = 600$ or $-x(x-50) = 600$, according to whether the number inside the absolute value signs turns out to be positive or negative. The first of the two equations gives $x = 60$ or $x = -10$, as earlier, and produces the corners (60,10) and (-10,-60) to define two rectangles (whose opposite corners are at the origin) that do the job. How easy! Of *course* $x = -10$ has a meaning, once we set the thing up on the coordinate plane.

But wait, what about the other equation, “ $-x(x-50) = 600$ ”? This one has solutions, too, and they are $x = 30$ and $x = 20$, producing opposite-corner points (30,-20) and (20,-30), either of which, with the origin, sure enough forms a rectangle of area 600. Goodness, the more we want to make sense of the problem, the more answers turn up! But if you look at these last two “solutions,” do they “check” when we try to prove they satisfy the conditions of the problem? They do: they give the correct area, and $y = x - 50$ as demanded. The trouble here is that we probably have stated the problem badly.

If all we wanted was that the *number* that is the y -coordinate of the opposite-corner point should be 50 less than the *number* that is the x -coordinate of that point, these last two solutions check out in every detail. But surely this is a poor statement of the original problem, where the architect doubtless intended the

length of one side of the garden to be 50 more than the length of the other side. The condition “ $y = x-50$ ” is not a statement of that condition, while $|y| = |x|-50$ is the point (either that, or $|x| = |y|-50$).

With this restatement we can go back over the whole problem and find that the third and fourth “solutions” do *not* check. On the other hand, the new conditions on length, expressed in terms of absolute value, give rise to some new possibilities, and it will perhaps surprise nobody that there are eight solutions, with the “opposite-corners” at (10,60), (60,10), (10,-60), (60,-10), (-10,60), (-60,10), (-10,-60), and (-60,-10), that is, all the possible ways you can place a sixty-by-ten rectangle with one corner at the origin and sides parallel to the axes.

Pandora’s Box is now open: what if the rectangles are not parallel to the axes? There are answers to that one, too, but they go beyond simple algebraic equations and their meaning. It were best now to cut our losses and go back to the beginning: “Sixty by ten” is doubtless the best answer. But intellectually we have found something out: negative numbers, just as De Morgan said, *can* be made to mean something valid. We have found something else out, too, just as De Morgan said, which is that we must understand that we are *making* them mean something, and that the process of associating these invented numbers with some scientific or architectural *use* is not as simple or obvious as it might seem when they are presented axiomatically. Logic is not only a matter of reasoning from axioms for a field, it is also a matter of reasoning from life.

11. EVEN IMAGINARY SOLUTIONS ARE NOT NECESSARILY ABSURD

Finally, let us return to the partition of a segment of (positive) length **a** into two pieces forming adjacent sides of a rectangle of area **b**. (This discussion will be rather condensed, compared to what has gone before.)

We suppose x is a length that does the job, i.e. x and $a-x$ are the two side-lengths. We blindly set up the quadratic equation $x(a-x) = b$ and find two solutions (both of which check in the equation if not the prob-

lem): They are

$$(a/2) + \sqrt{\left(\frac{a}{2}\right)^2 - b} \text{ and } (a/2) - \sqrt{\left(\frac{a}{2}\right)^2 - b}.$$

So, if there is a solution it has to be one of these two numbers. (Actually, since these solutions add up to a , this pair of numbers is the only possible solution, i.e. if x is the first, $a-x$ is the second, and if x is the second, $a-x$ is the first.)

When $(a/2)^2 > b$ all is well; we get two positive numbers which add to a and which solve the problem. We can draw a picture of the resulting rectangle, and we have no negative solution to have to interpret. But what happens when $(a/2)^2 < b$? Can we, with Wigner, discover the “unreasonable effectiveness of mathematics”

by finding that there really is a genuine visible rectangle that solves the problem even when **a** is too small to partition properly, i.e. to produce the sides of a rectangle with desired area **b**? Sure.

Let a four dimensional Euclidean space have its axes labelled x, y, u, v , with the point $(x, y, 0, 0)$ representing the number $x+yi$ when this is the solution of a quadratic equation using the sign “+” in the quadratic formula, and the point $(0, 0, u, v)$ representing the number $u+vi$ where this is the solution of the same quadratic equation using the “-” sign in the quadratic formula. Observe that in our problem, where a and b are positive, the numbers x, y, u and v obtained from our quadratic will always be positive when the discriminant forces us into complex roots. Thus $x+yi$ can be pictured as a vector, or rather an arrow with tail at the origin and arrow-head in the first quadrant of the xy plane, and similarly for $u+iv$ in its plane, which is perpendicular to the xy plane. The vectors are (when you disregard the frill of the arrowhead) perpendicular segments in 4-space, and the area of the rectangle they subtend — a genuine, visible rectangle — is their inner product $xu+yv$. Work it out; it is **b**.

How come? In this problem, **a** was “too small” to admit such a partition, or to put it in other terms, **b**

“

The process of associating these invented numbers with some scientific or architectural use is not as simple or obvious as it might seem when they are presented axiomatically

was “too big” for a rod of length **a** to be broken for the purpose of making a rectangle that big. But what are the lengths of the vectors that made the sides of our rectangle in 4-space? They are $\sqrt{x^2 + y^2}$ and $\sqrt{u^2 + v^2}$, or \sqrt{b} in each case; hey! — we’ve even got a square, not just a rectangle! Those are pretty long segment lengths, big enough so that the square they build in 4-space is sure enough of area **b**. But we found earlier that long things like that can’t partition a segment of length **a**. Indeed, the sum of these two *lengths* is $2\sqrt{b}$, which is certainly not **a**.

Well, what was the problem? Did we ask for **a** to be partitioned into two pieces whose lengths add to **a**, or did we ask for **a** to be partitioned into two numbers whose sum was **a**? We solved the latter problem, by finding complex numbers whose (complex) sum was **a** but whose lengths were big enough to make a square of size **b**.

Do I hear someone cry fraud?

“Fraud!” cried the maddened thousands, and echo answered fraud; But one scornful look from Casey and the audience was awed.

Partitioning **a** into *complex* pieces that make, in a suitable geometric interpretation of complex numbers, a

suitable real rectangle is no more fraudulent than interpreting the garden problem as one of finding coordinates of a point in the Cartesian plane, rather than lengths of wall, or using negative numbers in the manner of DeMorgan to represent the past instead of a putative future.

We all know there is no date at which the son *will* be half the age of the father; it’s too late for that already. In De Morgan’s time it was still questionable whether using a negative answer amounted to a swindle. Unfortunately, “hardly a man is now alive,” (to quote from another narrative poet) who still appreciates the intellectual effort it took to overcome this natural disinclination to treat mathematical artifices as if they had real significance, and it is a rare teacher who recognizes there is even a problem.

A garden plot with negative sides is really every bit as silly, at first glance, as a square with complex sides. But you can get used to these things after a while. The important thing is to understand just what it is you are getting used to.

Editor’s Note: In the last issue of the Humanistic Mathematics Network Journal Mr. Raimi’s e-mail address was incorrect. It should be: rarm@db1.cc.rochester.edu, with a 1 instead of an l. We apologize for the error.

Mathematics Found in Poetry

continued from page 20

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Architecture and Mathematics: An Introduction for Elementary and Middle School Children

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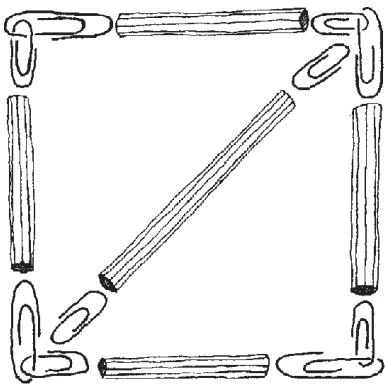
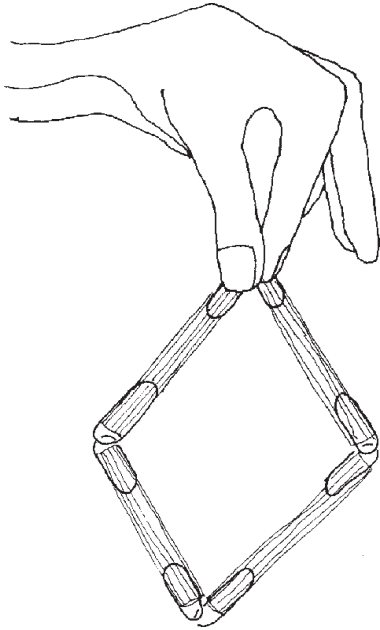
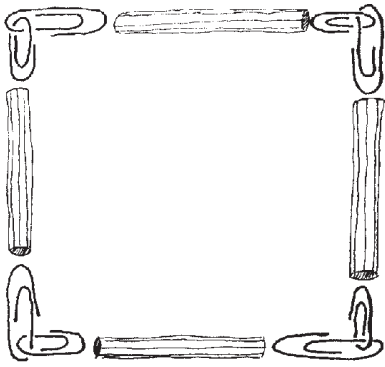
In June 1996 I directed an international conference on architecture and mathematics entitled "Nexus '96: Relationships Between Architecture and Mathematics."¹ The conference venue was my "home town," Fucecchio, a medium-sized (population 18,000) industrial town on the Arno river, about midway between Florence and Pisa (and ten minutes from Leonardo's home town, Vinci). Conferences, mathematics, architecture, professors all of these are almost unheard of in Fucecchio. In order that the conference not remain an "ivory tower" event in the town, I proposed a series of lessons to fourth and fifth graders in the elementary school and to sixth, seventh and eighth graders at the middle school. The idea was to show them some of the ways in which architects use mathematics. In return, I asked them to become architects, using geometrical shapes to "design" buildings. Their compositions would then be exhibited during the conference. In this article I want to share with you the brief program I put together for those lessons.

I introduced myself to the younger children by telling them how I became an architect. My family moved to Houston in 1958, and I grew up with the city. With the boom in oil prices, Houston in the 1960's grew like wildfire. I was inspired by all the new buildings in town to study architecture. I learned that architects design buildings not just using their imaginations, but also by following the tradition of architectural design that, in our western world, began as far back as ancient Egypt. The ancient Roman architect Vitruvius said that good architecture has to satisfy three conditions: firmness, commodity and delight.² Firmness means that a building has to stand up, supporting its own weight and protecting the people and things inside. Commodity means that the building has to serve its purpose in a convenient and appropriate way: a church or temple has to remind us that we are in the house of God; a house should feel like a home and make our daily lives comfortable and healthy. Delight

means that architecture should be beautiful. Mathematics is an important tool of the architect for achieving all three of these qualities. In these lessons, we shall see some of the ways in which mathematics is used.

One of the most fundamental building blocks of architecture is shape. Because buildings have shape, it is important that architects know geometry. Looking around the classroom, you can notice any number of differently-shaped rectangles: the door, the windows, the light fixtures, the blackboard, the desktops. What other shapes are used in architecture? We find circles, circular segments and spheres used in round windows, arches and domes. The cylinder is found in columns and in towers. Triangles are found in many roofs. Cones are found in the roofs of the towers of castles. Some shapes are important to the architect because they are stronger than others. The triangle is a very important shape for the architect, because it is "rigid." We can best understand the rigidity of a triangle through an experiment.

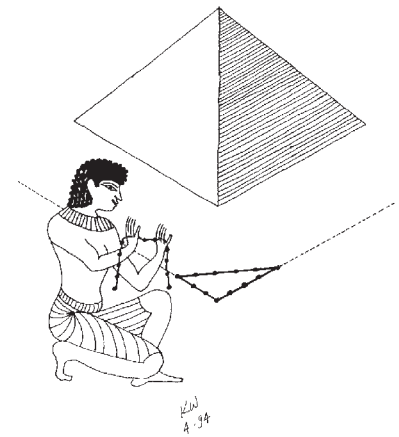
To experiment with the special properties of the triangle, we built squares out of pieces of drinking straws and linked paper clips.³ The squares were constructed by linking two paper clips to form each corner, then slipping the paper clips into the ends of the drinking straws (Fig. 1a). Once we had a square shape, it was easy to see that it didn't hold its shape: we could make it into a rhombus or flatten it altogether. In technical terms, when we put pressure on one of the angles, the square "deformed under stress." Would this be a good shape for holding up a house? The students all agreed it would not. Next we added a third paper clip to two opposite corners and added a fifth drinking straw (Fig. 1c). Now we could no longer deform the square: in structural terms, it had become "rigid." What had happened to the square? We had actually transformed it into two triangles, and triangles are



Figures 1a, 1b and 1c

The construction of a square with paper clips and drinking straws. You can make a square ... but it doesn't stay a square! When a fifth drinking straw is added as a diagonal, the square is transformed into two triangles and the shape becomes rigid.

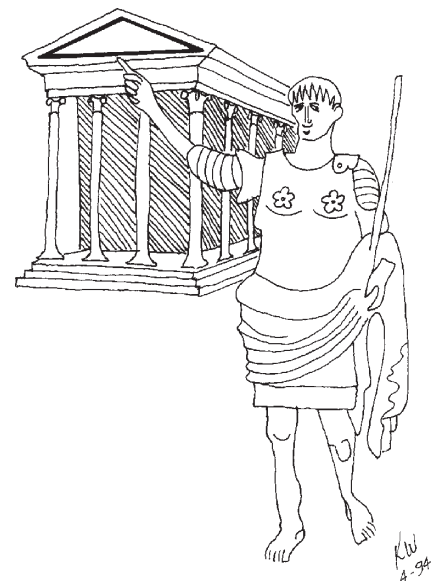
Figure 2
The triangle was used
in the Pyramids of the
Egyptians.

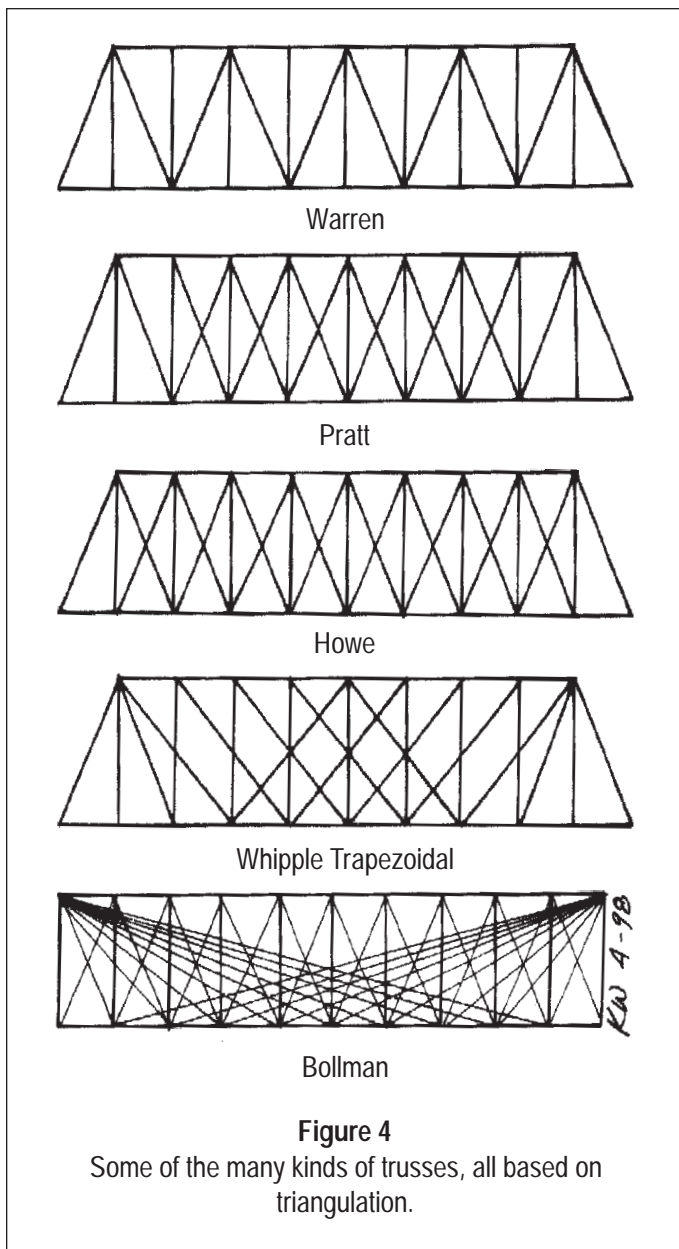


naturally rigid shapes—they do not easily deform under stress. This is the property that makes them important for architects. We find triangles in the Pyramids, for example, where four triangles lean against each other at the apex (Fig. 2). We also find them in the pediments of Greek and Roman temples, where the triangle shape was useful in supporting the roof (Fig. 3). An important use of triangles is in the construction of trusses, which are lightweight structural systems that derive their strength from triangulation. Although there are many kinds of trusses, all are based on triangles (Fig. 4). Trusses are used in bridge building and also in the construction of large buildings with open spaces where columns might get in the way of the action, such as in a basketball arena.⁴

In ancient architecture, different shapes were important because of what they symbolized. The square, for instance, represented the earth, because it has four

Figure 3
Greek and Roman
architects used
the triangle to
make strong roof
structures.





sides like the earth has the four directions, north, south, east and west. The cube also represented the earth, because it could be precisely measured. On the other hand, the circle and sphere represented heaven, because their diameters and circumferences can only be expressed using the irrational value, and not by human, rational measurements. It was believed that the irrational values belonged only to God. This is why we find the dome used over the altars in churches and temples, because the spherical shape of the dome represented heaven.

Having looked at the uses of shapes in architecture, we tried our first experiment in becoming architects.

We divided the class into work groups of 3 to 4 children, and each received a package of shapes (squares, rectangles, triangles, circles, half-circles) that had been cut out from construction paper at random. They were asked to use the shapes to design their first "buildings." This was the end of the first lesson for the elementary school children.

The first lesson for the middle school children was similar to that for the elementary students in scope but geared for the older student. In order to introduce the idea of mathematics in architecture, I began with architecture in art, using hands-on experiments with the Moebius strip as an introduction to the art of M.C. Escher.⁵ After experimenting directly with the Moebius strip, we could understand better the complexity of Escher's etchings. We had learned from the Moebius strip that while a strip of paper has two surfaces, a simple twist can make it a loop that has one surface only. Surfaces in architecture are important,

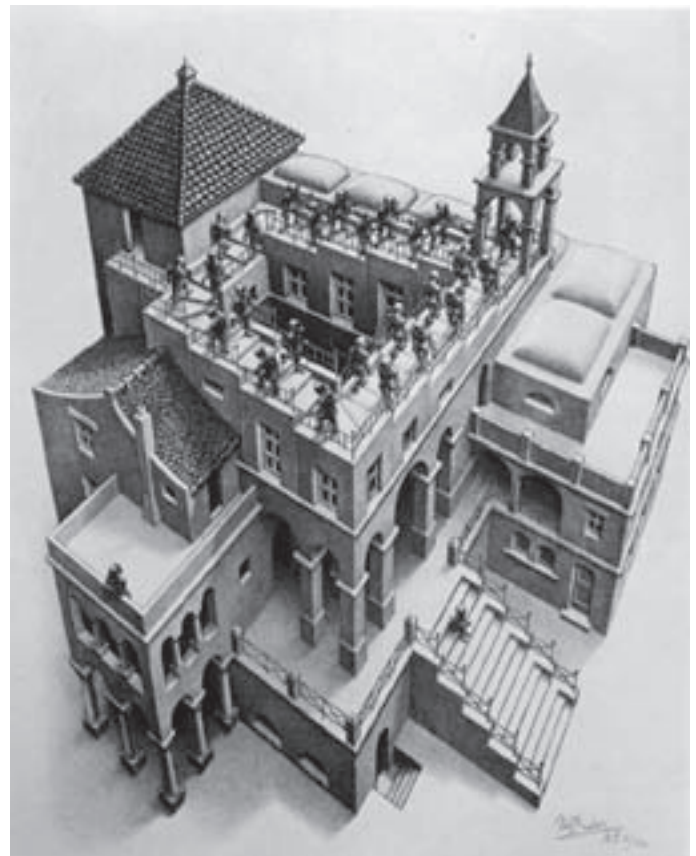
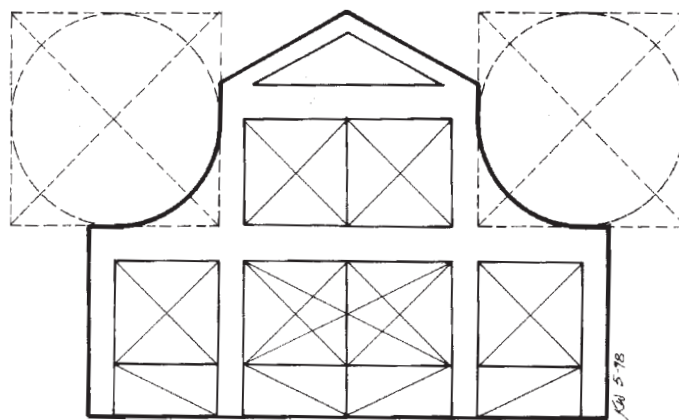


Figure 5
M.C. Escher's impossible stairs in the etching *Ascending and Descending*. All M.C. Escher's works copyright Cordon Art B.V. - Baarn - the Netherlands. Reproduced by permission.

too, as they enclose the architectural space. Could the students imagine what might happen if we distorted the surfaces in architecture? Escher did just that in some of his other etchings. At the beginning of his career Escher studied architecture in Holland, and he remained a very careful observer of architecture, often depicting it in his work. We looked at some of his etchings of “impossible” architecture, in which, for instance, he depicts stairs that lead both up and down at once (Fig. 5). He creates this illusion by depicting surfaces and shapes that appear at first to be normal. Upon closer examination, however, it can be seen that it is by distorting the shapes that he creates his illusions. This then led to the discussion of the importance of shape in architecture, much as I had discussed it with the younger students.

As a first exercise for the middle school children, we examined some of the buildings in our town to see if we could identify shapes in them (Fig. 6a and Fig. 6b).⁶ Afterwards, the students were given the random collection of shapes and asked to design their first building.

The subject of the second lesson was proportion.



Figures 6a and 6b
The facade of the church of S. Maria delle Vedute in Fucecchio and the analysis of its shapes.

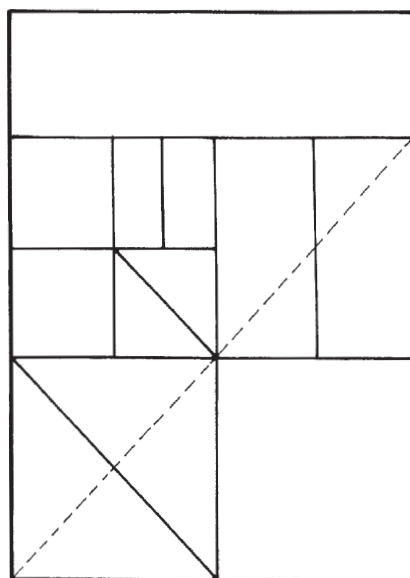
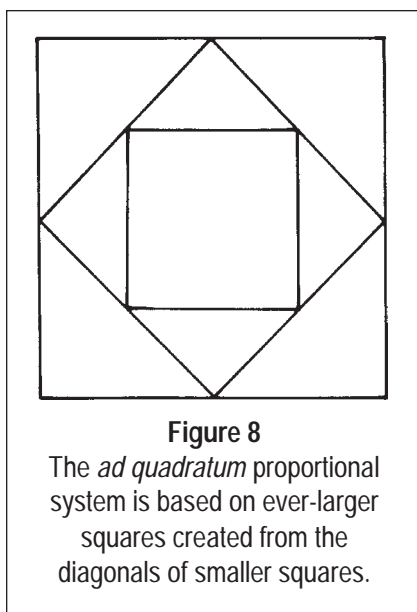


Figure 7
The proportional system of shapes created from paper folding.

Proportion is a comparative relation between sizes of elements. We used their first compositions as a starting point to discuss what proportion is- how big is a window in relation to the door, and how big is the door in relation to the whole wall? Proportion determines the relationships between parts of a building. Sometimes proportions are important in making sure that a building has, as Vitruvius says, firmness. If a building is many stories tall, its columns must be heavier to support its greater weight. If it is only one story high, the columns can be proportionately lighter. Proper proportions can help satisfy Vitruvius’ requirement that architecture have “commodity:” the wall must be bigger than a door, but a door can be either bigger or smaller than a window.

Proportion is also important because some shapes are believed to be naturally more beautiful than others. How were these shapes discovered? Ancient architects used to compare the proportions of architecture to those of the human body. Just as we can tell in our own drawings of a person when we have made the head too big or the arms too long, the architect could tell when the columns were too tall or the



doorways too narrow. Over time, architects analyzed what they considered to be the most beautiful buildings in order to be able to record in numbers what the perfect proportions were. These special proportions were used to satisfy Vitruvius' essential requirement that architecture be beautiful.

Finally, if the architect designs his whole building using a proportional system, he can also make sure that all the parts fit together. Having the parts fit means that the building will work well structurally and functionally and also be beautiful.

One way to create a proportional system is by coordinating all the shapes we use in the composition. This provides us with a "vocabulary of shapes." Just as we use a vocabulary of words to make up a sentence, we can use a vocabulary of shapes to design a building. As an exercise, we created a proportional system through paper folding. Starting with a regular piece of construction paper, we folded one corner down to create a perfect square, then cut or carefully tore the remaining rectangle away. (With standard European paper, it isn't necessary to discard the remaining rectangle, because A4-format paper has the shape of a root-2 rectangle). We folded this "reference square" into four smaller squares; these were then subdivided into either two triangles, two rectangles, or four smaller squares. Finally, we said that circles could be cut out from any square as needed (Fig.7). These coordinated shapes became our "vocabulary of forms." By examining our shapes, we discovered that some of them could be added together to make a shape we already had (such as using two rectangles to make a square) but also that some of them could be added to make new shapes (as when we added a square to a rectangle to make a longer, narrower rectangle). With the older students, I reintroduced the idea of the irra-

tional quantities, looking at the diagonals of the rectangles as a way of creating new forms. We discovered that by dividing one of the squares along its diagonals into 4 triangles and rearranging them, we could create larger squares that were not related to the rational lengths of the original reference square, a system often used by Roman architects known as *ad quadratum* (Fig. 8). We could also create rectangles that had one rational side and one irrational side.

At the end of the second lesson, the students were invited to make new compositions with the new "vocabulary of shapes." These compositions were more refined than the first, due to the more sophisticated set of forms as well as to the confidence the children had gained from their first compositions.

The fifty compositions that resulted from the time spent with all the children were exhibited in the restored medieval palace where the conference was held (Figs. 9 and 10). Some days before the conference opened we had an "art opening" for the students and their parents and teachers. The experiment was deemed a success from all points of view. The students were very interested in architecture and the fact



Figure 9
An example of the students' work.

that mathematics isn't just something that has to be studied in the classroom, but is of great value as a creative tool as well. We also found the lessons provided a new way for the children to understand the built environment, using mathematical tools that are part of their normal curriculum. This is especially important because there is little or no architectural education at the levels of elementary and middle schools. The teachers were very pleased with a dem-

very much appreciated by all conference participants. Who could help but admire such "mathematical" architecture?

NOTES

1. For a review of the collection of essays that resulted from this conference, cf. Joseph Malkevitch "Book Review: Nexus: Architecture and Mathematics," *Humanistic Mathematics Network Journal*, 16 (November 1997), 52-53.

2. Cf. Vitruvius, *The Ten Books on Architecture*, Morris Hicky Morgan, trans. (New York: Dover Publications, Inc., 1960) book I, chap. III, sect. 2, 17. In this translation, "firmness, commodity and delight" are rendered "durability, convenience and beauty."

3. This experiment was described in Kim Williams, "How Buildings Take Shape," *Highlights for Children*. For the technique of building with drinking straws and paper clips, I am indebted to Howard Jacobs, *Mathematics: A Human Endeavor*, 2nd ed. (New York: W.H. Freeman, 1982) 267-269.

4. There are other ways of demonstrating structural rigidity through form through experiments with simple pieces of paper. Cf. Mario Salvadori, *The Art of Construction*, 3rd ed. (Chicago: Chicago Review Press, 1990), 109-118.

5. Some hands-on experiments with the Moebius strips are described in Jacobs, *Mathematics: A Human Endeavor*, 605-606.

6. This experiment can be done without leaving the classroom, by showing slides or xeroxed photographs of buildings and asking the students to analyze the facades in terms of shape.



Figure 10
Another example of the students' work.

onstration of geometric principles applied to a "real" activity. I was more than gratified by the enthusiasm of the students, not only for this activity, but for all facets of my work as an architect. The drawings remained on display during the conference, and were

Changing Ways of Thinking About Mathematics by Teaching Game Theory

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"It saddens me that educated people don't even know that my subject exists."

Paul R. Halmos

Halmos' words (1968), which served as the motto of the article, "Mathematics Today" (Steen, 1978), are equally relevant to this present project. It originated with the recognition of the narrow and limited conception of mathematics prevalent among the general public. Very few have any idea of what mathematics really is.

One of the main purposes of teaching mathematics in schools is to contribute to the enrichment of the mathematical world-view of the students. As in any scientific discipline, a mathematical world-view is formed by means of personal experience. The broader and richer the individual's experience, the more enriched and profound will be the corresponding world-view. Accordingly, the more diversified the encounter with mathematics, the richer the mathematical world-view.

Schools act as the crucible in which the student's world-view of scientific disciplines is formed: they are responsible for the impoverished mathematical world-view common among students and graduates. When the subject is art or "the arts" (music, painting, sculpture, literature), teachers make every effort to ensure that students experience art in the broadest sense. They try to introduce them to as many kinds of art as possible; in the teaching of mathematics, however, no real concern is shown for what mathematics really is.

We agree with Aumann (1985) that "the case for thinking of mathematics itself as an art form is quite clear"

and "if one thinks of mathematics as art, then one can think of pure mathematics as abstract art, like a Bach fugue or a Pollock canvas . . . ; whereas game theory and mathematical economics would be expressive art, like a cubist painting or Tolstoy's *War and Peace*."

In order to help students to sense the spirit of mathematics, effort must be made to introduce them to as many kinds of mathematics as possible. This may be accomplished by means of new curricula and new approaches to instruction. In Israel, a mathematics curriculum for high school upper grades composed of a combination of compulsory courses and 90 hours of

elective studies was approved in 1975. The change in curriculum structure gave rise to the idea of creating an elective in game theory. Game theory both satisfies the criteria of the elective mathematics curriculum and exemplifies a branch of the discipline which may contribute to a

change in attitudes and approaches to mathematics.

A course in game theory was created such that it is constructed of four topics dissimilar in character and bearing little mathematical relation to each other. The four topics were elected on the basis of their being of special interest beyond their mathematical content, not demanding specific prerequisite knowledge in mathematics, and providing general knowledge about game theory and its concerns (see Gura, 1995).

The first chapter of the course is called "Mathematical Matchmaking" and deals with the stable marriage problem that was raised by Gale and Shapley (1962). They concluded their paper with the following comment:

***“When the subject is art or “the arts”...teachers make every effort to ensure that students experience art in the broadest sense...in the teaching of mathematics, however, no real concern is shown for what mathematics really is.*”**

Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which non-mathematicians believe mathematics to be concerned with. Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with 'a head for figures' or that they 'know a lot of formulae.' At such time it will be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical What, then, to raise the old question once more, is mathematics? The answer, it appears, is that any argument which is carried out with sufficient precision is mathematics.

Following Gale and Shapley, we strove to create a course which would employ a predominantly verbal mode of presentation and which, at the same time, would be sufficiently varied in content and level of explication.

RESEARCH

Game theory as a high school course is new to the existing curricula; therefore, the research requires the methodology of a case study. We carefully examined whether game theory can be taught at the high school or equivalent level and, if so, at what level of explication? At what depth? In order to answer these ques-

tions, we focused on topic selection and the teaching of these topics. We planned an intensive study which would investigate the teaching of game theory in its natural environment, the classroom. There were no special control conditions, only the tools of the classroom framework — exams, written work and questionnaires. In order to obtain convincing results and conclusions, the research was conducted on three different types of classes — one high-level mathematics high school class, two pre-academic preparatory classes studying equivalent low-level high school mathematics and one teachers' college class majoring in arithmetic instruction for elementary school classes studying at high school level mathematics. The structure of the research corresponded to Yin's (1984) definition of a case study. It was an empirical study investigating the teaching of game theory within the framework of the real context, the classroom, in which the boundaries between content, quality of teaching, approaches to teaching, the class situation and teacher-student interaction are quite vague.

Data were gathered via exams, questionnaires, and detailed journals of what went on during the lessons. Our primary interest was qualitative information, although we also compiled quantitative results. There were only four small classes and therefore the statistical analysis is limited; emphasis was placed on the qualitative analysis. From teaching in the classroom, we learned that several topics in game theory can be taught at both levels of mathematics. Although there were difficulties, students at all levels were successful in dealing with the course material. In the two

| Class | N | Initial Grade \bar{x} | Final Grade \bar{x} |
|-------|----|----------------------------|--------------------------|
| C1 | 22 | 50.0 | 66.1 |
| C2* | 19 | 52.9 | 78.3 |
| C3 | 22 | 52.9 | 71.8 |
| C4 | 20 | 56.1 | 74.8 |

Table 1 : Average grade (in percentages) of the c level (the lowest) classes in the pre-academic courses, 1897
*studied game theory

| Class | N | Initial Grade \bar{x} | Final Grade \bar{x} |
|-------|----|----------------------------|--------------------------|
| C1 | 20 | 70.2 | 80.9 |
| C2 | 19 | 76.7 | 74.4 |
| C3 | 18 | 72.4 | 81.9 |
| C4* | 16 | 75.3 | 88.7 |

Table 2 : Average grade (in percentages) of the c level classes in the pre-academic courses, 1988
*studied game theory

| | Class a | | Class b. | | Class c | | Class d | | Total |
|-------------------|---------|------|----------|------|---------|------|---------|-------|-------|
| | pos. | neg. | pos. | neg. | pos. | neg. | pos. | neg. | |
| Before the Course | 3 | 11 | 13 | 7 | 12 | 7 | 2 | 6 | 60 |
| | 21.4 | 78.6 | 65.0 | 35.0 | 61.1 | 68.9 | 25.0 | 75.0 | |
| After the Course | 1 | 9 | 6 | 14 | 5 | 11 | 0 | 8 | 54 |
| | 10.0 | 90.0 | 30.0 | 70.0 | 31.2 | 68.8 | 0.0 | 100.0 | |
| Total | 4 | 20 | 19 | 21 | 16 | 18 | 2 | 14 | 114 |

Table 3 : The distribution of positive and negative answers to question 1 (in absolute numbers and percentages) before and after the course

| | Class a | | Class b. | | Class c | | Class d | | Total |
|-------------------|---------|-------|----------|------|---------|------|---------|------|-------|
| | pos. | neg. | pos. | neg. | pos. | neg. | pos. | neg. | |
| Before the Course | 0 | 14 | 13 | 7 | 16 | 6 | 5 | 3 | 60 |
| | 0.0 | 100.0 | 65.0 | 35.0 | 66.7 | 62.5 | 62.5 | 37.5 | |
| After the Course | 0 | 10 | 7 | 13 | 6 | 10 | 2 | 6 | 54 |
| | 0.0 | 100.0 | 35.0 | 65.0 | 37.5 | 62.5 | 25.0 | 75.0 | |
| Total | 0 | 24 | 20 | 20 | 18 | 16 | 7 | 9 | 114 |

Table 4 : The distrubution of positive and negative answers to question 2 (in absolute numbers and percentages) before and after the course

preparatory classes we also were able to compare the average grade in mathematics of four parallel classes when three of the classes did not study game theory.

Note that the class which studied game theory did not have a higher grade average from the start but had the highest average final grade. Variance analysis shows:

Table 1: $F = 10.660$ with significance level of 0.001

Table 2: $F = 16.706$ with significance level of 0.001

The game theory courses indirectly contributed to raising the level of grades in mathematics in general. The object of my discussion here is to offer some of the results which point out the effect of this course in game theory on changing ways of thinking about mathematics. The results are gathered from a questionnaire designed to ascertain the subject's worldview of mathematics and from an attitude survey.

First, two questions from the questionnaire which was filled out by students before and after the course will be discussed:

1. What is your opinion about the statement that is often heard that the activities of the mathematician are based on computations?
2. What is your opinion about the statement that mathematics is nothing but a game of symbols and formulas that were invented by the human mind?

The table indicates that there is a connection between

the type of answer (positive or negative) and the time it was answered (before or after the course). Calculating χ^2 for the whole population shows $\chi^2 = 7.274$ with significance at level of 0.01. We may say, therefore, that there is a significant relation between the type of answer and the time it was given. It would appear that the course in game theory contributes to reducing the number of students who relate to mathematics as mainly technical.

Question 2 explores whether the approach to mathematics is a formalistic one. A positive answer would attest to a formalistic, at least to some degree, approach to mathematics. The table shows that after the course in game theory the number of answers representing a formalistic approach to mathematics decreased. A χ^2 test calculated for the whole population shows $\chi^2 = 4.954$ with the significance at a level of 0.05. Clearly, there is a significant relationship between the rejection of the formalistic approach to mathematics and the studying of the course in game theory.

A real change in students' attitudes and approaches to learning mathematics was also observed and is best illustrated in the following questions from the attitude survey:

1. Was the course in game theory very interesting, interesting or not interesting?
2. Did the introduction of a new mathematical subject change anything in your overall approach to mathematics? If so specify what sort of change.
3. Do you think that the course in game theory can

help you in areas which require abstract thinking and are not connected to mathematics? Explain.

Answers to question 1 indicate the student's degree of interest in the course and those aspects of the course which stimulated this interest. In their answers the students relate to the technical aspect of mathematics, its relevance to life, to the different way of thinking — the very components that are significant in the overall mathematical world-view.

Examples of answers to question 1:

1. "Very interesting — a different approach from the usual one at school; the emphasis is on mathematical principles in addition to techniques."
2. "The course was very interesting because it was a new subject and because it deals with aspects of life which are concrete and more attractive to me."
3. "The course was interesting, more than any other subject I have studied in mathematics so far."
4. "Very interesting. This course enabled me to become acquainted with a 'different' mathematics and think in a different way."
5. "Interesting, different from other mathematics courses, theoretical and requires logical thinking."

The answers to question 2 revealed that more than 60 percent of the students changed their approach to mathematics following the course in game theory.

Examples of answers to question 2:


1. "The introduction to a new mathematical subject changed my attitude towards mathematics because I saw that mathematics is not just numbers and arithmetic operations, but there is something deeper and it is much more developed than I ever thought."
2. "Yes, by understanding the idea of proofs as a basis for mathematics."
3. "Yes, it changed my attitude because I understood that mathematics may be found in almost every area; mathematics is not just plain drill."
4. "To some extent yes, because the idea of mathematics being a useful instrument to the social sciences is new to me. I feel the same way about the use of verbal explanations alongside numbers."
5. "Yes, I don't like mathematics because it says nothing to me, but game theory interests me because in game theory there are real life subjects and therefore it is easier for me and it is more interest-

ing...suddenly..."

6. "Yes. This subject was more interesting than any other subject I've studied and has given me a strong desire to study more subjects of this kind in mathematics."
7. "I did not change. I've always loved mathematics."
8. "I did not change but my interest in the subject was strengthened."
9. "The introduction of a new subject based on mathematics did not change my attitude but familiarized me with mathematics as a whole which was much broader than I'd realized."
10. "No change. In fact it is hard for me to accept game theory as mathematics."

Answers 7-9 were given primarily by students studying high-level mathematics, who brought with them an interest in mathematics from the outset. The change in attitude was more pronounced in those students studying low-level mathematics.

Examples of answers to question 3:

1. "Yes. The course helped me see that every problem has several relevant aspects.  I also understood that there are several approaches to a solution and that sometimes one has to choose a specific one. I think that the course developed abstract thinking."
2. "Of course. Game theory requires abstract and analytic thinking in order to discover proofs and corresponding processes to pre-stated principles."
3. "Yes. Game theory enables me to look at a subject from a wide perspective, that is from above, from different angles and various possibilities."
4. "The course helped me realize that mathematics can be brought closer to the social sciences with the help of mathematical thinking which gives precise results."
5. "The course is based on logic and analysis of real situations and can therefore help in areas of abstract thinking."
6. "Of course. This material, with its deep proofs, develops abstract thinking."

One could qualify our conclusions by saying that the students' answers do not guarantee an improvement in abstract thinking following the course in game theory and therefore this is not factual evidence. However, we must take into account the feelings expressed

by the students.

The range of answers included those ideas which were anticipated as a direct result of studying game theory. Among the answers: a feeling that mathematics is everywhere, an understanding of the usefulness of mathematics, its relevance to life, its depth, the proof as a basis for mathematics — in general, a change in the modification of the mathematical world-view. Answer number 10 to question number 2 is atypical, but exists nonetheless; the perception of mathematics as technical may be so strongly rooted in some people that even this course could not change it. The answers to the attitude survey seem to validate our choice of game theory to enrich and enlarge the student's conception of mathematics.

In conclusion, our hopes for the course were realized. As a result of the course, the number of students with an open-minded attitude to mathematics increased; the students were able to see mathematics as not only technical and computational, but also as an expanding and developing world of its own. Students discovered that the world of mathematics is much richer than they had previously thought. Indeed, it appears that the very encounter with a new sphere of mathematics in and of itself creates a new receptivity in the students to the assimilation of new concepts and values.

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Nine and One Third Circles of Rejoicing

Michael Capobianco
St. John's University
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Alleluia!
Noble word
We greet you with joyful
Jubilations
You and your three thousand three hundred sixty
Permutations.
Let each one inscribed be
Round a circle on each degree
Then with clarity will be heard
The sound of nine circles and a third.

Manifesto on Mathematical Education

Saunders Mac Lane
University of Chicago
Chicago, IL

Distributed at the math meetings in Baltimore, MD; January, 1998.

All this continued noise and agitation for “Reform”! We are fed up with it. STOP!

Yes, calculus was, is and will remain alive and applicable. Teaching it is always hard—both a problem and a pleasure. But it does not require any fashionable “rethinking.” The insights and the achievements are always there.

Slogans without substance fly about freely. In actual fact, the calculus is BOTH a pump and a filter—because it is both profound and hard.

“Mathematics for all” is about politics, not about ideas. It may help down under K-12, but even there it is badly overworked. But all college teachers know that in fact not all college students will “get it.” Some are not at all prepared, others are unwilling or unable to learn. What really matters is a goodly measure of mathematics for all those who are ready to learn and willing to study for this end. Not all students succeed. There are still grades of “F.”

From K to 12 the tradition presents the hopeful intent that the teacher can bring any and all students to learn. But beyond 12, a wise tradition states that it is up to

the college student to decide for herself whether and what to learn. It is this basic location of the initiative which is behind the much touted economic value of college education.

Yes, there are many new texts, and the NSF grants provide funds and other help to produce many more of them. But getting an NSF grant for such a purpose is not a certificate of accomplishment or a title to preach to others—it is just a license to try and to realize that no text has it all. In the old days, Granville’s calculus was later powered up by Smith and Longley down at Yale, which then gave way to Thomas and later to Hille and company and many other bids for royalties. Consorting with an Ivy League cachet can now present nothing really new.

The real struggles continue to take place in all those classrooms where teachers engage and tempt each generation of new students with the wondrous uses of limits. The calculus is still there in all its glory, from definitions to the fundamental theorem connecting differentiation with integration. The orbits of the planets and those water pressures on that dam all serve to illustrate what really gives.

This is where proof, precision and understanding lie, ready for action. Get off the pulpit and get on to tempting the students with the ideas!

from *My Dance Is Mathematics:* *Poems From a Mathematics Teacher*

JoAnne Growney
Bloomsburg University

GEOMETRY CLASS

Yesterday, some visitors
interrupted geometry class --
angry voices raged around the room,
unwilling to stay caged within my head,
while I spoke pleasantly
of axioms of incidence,
placements of parallels,
numbers of degrees
in the angles of rectangles.

*Wake up. This is not difficult --
no hungry mouths to feed, no
bleeding wounds to heal. Adopt
a polygonal attitude. Examine
an assumption. Abandon the postulate
that says, don't ever question.
You were not born knowing.
Your mind won't get dirty
on a tangent of hyperbolic thought.*

*Open up.
Let one eye watch
the parallels
that meet.
Shift to a point
of perspectivity.
Draw those lines
that cross
at your heart.*

My students
ignored these voices,
so I dismissed them
and went on --
politely coaxing
obtuse angles
to square up
and respond.

FINDING TIME

Points chase points
around the circle,
anti-clockwise,
fighting time.
You know time's a circle,
rather than a line.

Make a line a circle!
Pick a center.
Wrap and wrap and wrap
the line around the rim.
How do the ends
get tucked in?

Cut a circle open,
stretch into a line.
Does the cut destroy
a point or fit
between a pair?
If the cut's midway

from now to Tuesday,
how do I get there?
Do I move on
by going back,
or may I
skip a space?

A square is neither line
nor circle; it's timeless.
Points don't chase around
a square. Firm, steady,
it sits there and knows
its place. A circle
won't be squared.

ISAMA 99

First Interdisciplinary Conference of
The International Society for
The Arts, Mathematics and Architecture
7-11 June 1999
San Sebastian, Spain

INVITATION AND CALL FOR PAPERS

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Fields of Interest

The main purpose of ISAMA 99 is to bring together persons interested in relating mathematics with the arts and architecture. This includes teachers, architects, artists, mathematicians, scientists and engineers. As in previous conferences, the objective is to share information and discuss common interests. Hopefully new ideas and partnerships will emerge which can enrich interdisciplinary education. In particular, we believe it is important to begin interdisciplinary education at an early age, so one component of ISAMA 99 will be teacher workshops for K-12 in addition to college level courses.

ISAMA will focus on the following fields related to mathematics: Architecture, Computer Design and Fabrication in The Arts and Architecture, Geometric Art and Origami, Music, Sculpture and Tessellations and Tilings. These fields include graphics interaction, CAD systems, algorithms, fractals and graphics within mathematical software (Maple, Derive, Mathematica, etc.) There will also be associated teacher workshops.

Call for Papers

Papers are invited on the topics outlined and other topics which fall within the general scope of the Conference. Abstracts should be submitted to the Conference secretariat by December 15, 1998. Abstracts should not be longer than 300 words, contain a list of keywords, and clearly state the methodology, purpose, results and conclusion of the final paper. All lectures will be in English. Participants may wish to give their presentation in the form of slides and/or video. The final paper should contain explanatory text and a selection of images.

In Future Issues...

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A New Family of Irrational Numbers With Curious Properties

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