Journal of Humanistic Mathematics

Volume 6 | Issue 1 January 2016

Wabi-Sabi Mathematics

Jean-Francois Maheux
Université du Québec à Montréal

Wabi-Sabi Mathematics

Cover Page Footnote

I would like to thank Draco Szathmary, Maude Bélanger and Franciele da Silva for their contribution to my thinking for this paper; and the reviewers whose comments helped me introduce these ideas in a more inviting way.

Wabi-Sabi Mathematics

Jean-François Maheux

Department of Mathematics, Université du Québec à Montréal, CANADA
jfmaheux@mail.com

Abstract

Mathematics and aesthetics have a long history in common. In this relation however, the aesthetic dimension of mathematics largely refers to concepts such as purity, absoluteness, symmetry, and so on. In stark contrast to such a nexus of ideas, the Japanese aesthetic of *wabi-sabi* values imperfections, temporality, incompleteness, earthly crudeness, and even contradiction. In this paper, I discuss the possibilities of "wabi-sabi mathematics" by showing (1) how wabi-sabi mathematics is conceivable; (2) how wabi-sabi mathematics is observable; and (3) why we should bother about wabi-sabi mathematics.

Keywords: *aesthetics; imperfection; education; epistemology; mathematical activity*

Wabi-sabi is a beauty of things imperfect,
impermanent, and incomplete.
It is a beauty of things modest and humble.
It is a beauty of things unconventional.
Koren (1994)

As a Way In

If beauty can take many names (e.g. [40]), the aesthetic dimension of mathematics often seems to be related to a very specific nexus of ideas. A proof is often called elegant when smooth and efficient, and while some mention qualities such as inevitability and non-triviality [17], most of those who reflect on the aesthetics of mathematics focus on elements such as symmetry and simplicity [44]. While the question of what we mean when we say that mathematics is, or can be beautiful is potentially controversial (take for example B. Russell's appeal to coldness and austerity as defining features of the beauty of mathematics [38]), most attempts to discuss mathematical aesthetic point in similar directions. There is on one side, the classical approach in favour of abstraction, universality, eternality, truth and purity; all things above the rawness of human experiences. Modernist thinking, on the other hand, values how mathematics can be seamless, smooth, and clean. Mathematics is appreciated because it is well made, useful, efficient, precise and so on [4]. Through this, we acknowledge the human labour at the origin of mathematical accomplishments; nevertheless, we get the feeling that this humanistic aspect is quickly obscured behind mathematics' sort of technical perfection. Pimm [32] argues that in this view mathematics has to be "purified" from all such mundane residues, just like the brushstrokes of a painting need to be made invisible.

I would like, in this article, to discuss a radically different approach to thinking about and experiencing aesthetics in mathematics: the *wabi-sabi* way. Wonderfully evoked in the quote opening this article, wabi-sabi is a Japanese perspective on aesthetics focusing on imperfections, incompleteness, temporality, fortuitousness, unconventionality, and so on. What about wabi-sabi mathematics? In the following sections, I dwell for a moment on the three following questions:

1. How is wabi-sabi mathematics conceivable?
2. How is wabi-sabi observable?
3. Why would anyone bother?

Warily revealing my own epistemological posture with regard to mathematics throughout, I will try however, to be as inclusive as possible. And it is clear to me though, that even if we recognize that (doing) mathematics has a wabi-sabi aspect, the idea of admiring this dimension of mathematics can still be provocative!

Conceiving (of) wabi-sabi mathematics

Wabi : raw, simple, rustic, imperfect
Sabi : withered, worn, impermanent

At the heart of wabi-sabi aesthetics is a resistance to generality that makes it unimaginable to define what wabi-sabi aesthetics exactly is. Wabi-sabi is instead, the opposite of exactitude. We might say that wabi-sabi is concerned with a range of experiences qualified with adjectives such as: earthy, irregular, imperfect, textured, intuitive, relative, ambiguous, contradictory, and so on. Wabi-sabi engages us to appreciate such attributes, finding beauty in the contemplation of objects in which we can recognize them. The classical example is that of roughly hand-made teacups we nowadays often find in Asian stores. Uneven and color patched, they can evoke impermanence, as opposed to the eternal perfection of well-made, seamless china (Figure 1).

Figure 1: What a wabi-sabi teacup might look like: a modern tea vessel made in the wabi-sabi style. (Source: http://en.wikipedia.org/wiki/Wabi-sabi)

In wabi-sabi, the incompleteness and imperfection of life are celebrated as highly aesthetic, drawing us not to some transcendent out-of-this-word purity, but to the ephemerality of life and human achievements. Wabi-sabi mindfulness grows in the appreciation of impermanence, defects and limitations. Even ambiguity and contradiction are, from a wabi-sabi perspective, aesthetically rich for they can contribute to opening and developing our sensitivities to the unpredictable and uncontrollable nature of nature, including human existence … and experiences.

At first glance, it might seem difficult to conceive (of) mathematics in such a way. One reason for this, I suspect, is that we are still generally inspired by what some authors [21] call the mythological "romance of mathematics" (page *xv*) in which mathematics is represented as abstract and disembodied, objective and inherently structured; logical, provable and therefore certain and universal. On the other hand, closer attention to the field reveals a quite different picture.

A good starting point is probably the foundational crisis of mathematics that resulted, in the early 20$^{\text{th}}$ century, from the search for a proper, solid foundation for mathematics (e.g. [1, 29, 34]). In a nutshell, many great mathematicians attempted to formalize the basis upon which mathematical objects and axioms were developed (e.g. Hilbert) in order to overcome some of the paradoxes that had been recently discovered (e.g. Russell's). Despite their efforts, no satisfying approach could be found, until it was finally proven, by Gödel's famous incompleteness theorems, that such a program could simply not be achieved. Others, after Gödel, continued working in that direction (Tarski, Turing, Novikov, Cohen), showing how mathematics is in many ways imperfect in the sense that not every mathematical statement can be proved to be true or false, that we cannot know beforehand which statements are provable or not, and that an "entirely provable" mathematics would imply that statements inconsistent with one another would, at some point, be accepted to be true.

In that light, mathematics, as a network of ideas[1], seriously breaks with the "perfect" image often associated with it, and widely opens the doors to a wabi-sabi approach. Quite threatening at first, as the expression "foundational crisis" powerfully evokes, finding beauty in the incompleteness of mathematics is far from inconceivable! One could argue that the impossibility of knowing if a conjecture, such as Fermat's last theorem or Goldbach's conjecture, is provably true or false before it is actually proved, is part of what keeps mathematics an open, workable field (and adds some excitement to it!). Wabi-sabi aesthetics offers us a chance to contemplate this fundamental incompleteness in a positive way, to find beauty in it, and embrace it as part of the profoundly rich nature and texture of mathematics.

With that in mind, looking back at the history of mathematics reveals how these phenomena are not so exceptional. From the discovery of the irrationality of the square root of 2 to the invention of negative or imaginary numbers, statistics, non-Euclidian geometry, non-standard analysis or computer-based proofs, mathematics keeps overflowing its own boundaries. The French philosopher of mathematics Gilles Châtelet [11, 12] calls this

[1] I use this expression to avoid entering here into a debate concerning the nature of mathematics, e.g. theorems mathematicians prove versus what those theorems are about. Is mathematics the same as what we (can) know about mathematics? Does mathematics exist outside of actual "mathematical doings"? Fascinating questions!

the *virtuality* of mathematics; its capability to go beyond what is merely "possible" in order to create new mathematical possibilities. From such a perspective, mathematics is always in an unfinished, transitory state. It is transforming in a way that is arguably not merely cumulative. The generality of its statements is in continuous decay. For example, we used to say that parallels never meet, and that square roots are always positive real numbers. What might once have appeared as mathematics' clearly ordered structure seems to become more and more messy: numbers are also geometric entities which one might analytically manipulate; number theory conjectures meet with observations about elliptic curves' modular forms allowing us to prove the arithmetic observations from which algebraic number theory was born, and so on. Furthermore, might not the vertiginously increasing specialization of mathematics, making some of it understandable by only a handful of individuals, question the possible meaning of mathematics' "conventionality" and "universality"?

All this, however, can also be appreciated, following the wabi-sabi way, as a reflection of our human condition. What first appears to us as clear and clean, soon is revealed, through observation and use, to be complex and ambiguous. These "flaws" do not prevent us from working with mathematical objects or ideas. As Weiner [49] once put it:

> The average mathematician neither knows, nor, I grieve to say, cares, what a number is. You may say if you like that his analysis is blunted and his work rendered unrigorous by this deficiency, but the fact remains that not only can he attain to a very great degree of comprehension of his subject, but he can make advances in it, and discover mathematical laws previously unknown. (page 569)

A wabi-sabi perspective suggests going even further, and being appreciative of such deficiencies. In a way, is not the ambiguity of mathematical ideas what makes mathematical work so necessary (e.g. [28])? Nobody seems to be able to give a truly satisfying definition of what a number is (while what "truly satisfying" means seems quite subjective), or to define a point otherwise than by the negative (a geometrical object with no dimension), but mathematicians' attempts to do so often ended up interesting in other ways. For example, Russell and Whitehead venture to correct Frege's mistakes played an important role in the birth of modern logic (e.g. [47]).

Beside such impressive contributions, geometry and number theory nevertheless developed with points and numbers, making-do with a temporary, dependant and uncertain understanding of what they might be. There is a real beauty in how such an "unfounded" construction nonetheless keeps standing, in how little patches here and there (e.g. the interdiction to define certain paradoxical sets in Cantor's theory) keeps the whole thing working.

To complete this section, I want to refer to the words of a contemporary commentator of mathematics, mathematician/computer scientist Gregory Chaitin. Chaitin is known for his work on Gödel's incompleteness theorem and the development of algorithmic information theory [8]. His research led him to recognize the tremendous importance of randomness in mathematics. More precisely, Chaitin realized that the probability that a "randomly constructed program" will halt is a specific number, Ω, whose digits are "algorithmically random." In a nutshell, there cannot be a program shorter than the number itself to calculate its digits (for a concise explanation, see [10]). Following this, Chaitin insists that mathematics comes to (increasingly) resemble the natural world, and should also be "explored" in the way physicists run experiments and collect "evidence" to sustain their theories. That is, he argues in favour of a more "empirical", data driven approach to mathematical research, including work on famous problems such as the Riemann hypothesis (e.g. [6, 9]). Through that lens, the wabi-sabi potential of mathematics becomes quite apparent. If mathematics is fundamentally rough and bumpy like the physical world, it has the potential to be appreciated in such a way. In that spirit, Chaitin's comments incite us to find pleasure and beauty in the randomness of mathematics, i.e. in what Franklin playfully calls "an embarrassment" to regard "deductive logic as the only real logic" [15, page 1].

Another interesting example of the wabi-sabi beauty of mathematics can be found in ill-posed or inverse problems, for example [20]. Inverse problems are unstable problems characterized by having little tolerance to measurement and modelling imprecisions, or problems that have no solution in the desired class. Relatively common in science, engineering, economy, etc., they have led, since the 1950s, to the development of a branch of mathematics concerned with approximating the range of validity of specific answers while considering "similar" problems whose differences, however, suffice to make them impossible to solve in a direct way. Problems where the empirical observations of desired answers serve to define a range for which an equation (whose answers are similarly constrained) is offered as an "explanation" of the phenomenon (hence the term "inverse") are of that sort. Among many others, Kabanikhin also gives the example of summing a Fourier series, which actually consists in finding a function from its Fourier coefficients (the smallest variation of which radically affect the sum). From a wabi-sabi perspective, there can be beauty in the observations that straightforward or precise answers cannot be reached, and in the fact that some problems can be solved only through "rough" calculations that often remain practical approximations. How something as simple in appearance as "$x^5 - x + 1 = 0$" can only be imprecisely solved, reveals an irreducible complexity in mathematics, a sort of inner resistance to simplification. Inverse problems can take different forms, and have been addressed mathematically in various ways. There is a real richness in "that sort" of mathematics.

The challenge and power of the wabi-sabi aesthetic is to find beauty in quasi-solutions or empirical proofs (e.g. the four-color theorem) not only as "best possible answers" in relation to a sort of unreachable perfection, but to appreciate them in their own terms, for how they are an integral part of mathematics. There is beauty to be found in the strange and hardly readable entanglement of Wiles' proofs of Fermat's last theorem, just as much as, according to Netz [30] regarding Hellenistic mathematical writings, the Ancient Greeks seemed to appreciate laborious, abstruse calculations, or an unclear aim at the beginning of a demonstration (in order to create surprise when the reader comes to realize what has just been proven!). Beauty then could be found even in the fallacies Russel [39] finally admitted to be inevitable in mathematics:

> I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies [. . .] and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable (page 54)

So in the end, we might find it not even surprising at all that a wabi-sabi aesthetic would suit mathematics so well. After all, if the origin of mathematics is grounded in bodily sensory-motor experiences as suggested by [21], we may have reasons to think that somewhere in mathematics something would still be marked by these raw, temporal, subjective experiences. In what we call intuitionist tradition, people are familiar too with this idea of a contingent, physical origin of mathematics (e.g. [2]). However, the focus there is precisely toward moving away from such "limitations". Opponents of the "fallibilist" tradition often similarly argue that the presence of errors and oversights, historical displacements, contingence and limits to human knowledge of mathematics is inevitable. For them, mathematics is essentially about going beyond such imperfections (e.g. [14, 35, 42]). The opportunity created by wabi-sabi aesthetics is one of shifting our disposition toward those temporal, humanistic aspects of mathematics to see them as a source of joy and mindfulness, maybe in the way (why not?) athletes can find pleasure in the hardness of the training, and not only in the resulting performances. Something I suppose many mathematicians actually do, when they take pleasure in working through difficult problems. Could a wabi-sabi perspective on mathematics help those who do not have our appreciation of mathematics, find there, a kind of beauty?

In this section, I have attempted to sketch out some of the ways in which a particular perspective on the fundamental nature of mathematics might be conductive of/for a wabi-sabi aesthetic. How successful I was, probably depends on the reader's attitude toward the very deep questions at the heart of this proposition (not the least of which is "what is mathematics?"). It would be enriching if readers with different epistemological inclinations were to respond to my (necessarily opinionated) analysis, and offer other ways as to how wabi-sabi aesthetics can relate to mathematics. In the next section, however, I want to slightly move away from mathematics as a network of ideas to consider what wabi-sabi mathematics could look like in actual mathematical practice. To do so, I could turn to professional mathematicians' activities, or look into how other people use mathematics in the course of their work (e.g. physicians, biologists), or draw on the amazingly rich work of ethnomethodologists of mathematics (Livingston is one example, see for instance [22, 23, 24]). It is interesting to illustrate the potential fertility of the idea of wabi-sabi mathematics to instead take it to the other end of the spectrum and turn to students doing "ordinary" mathematics at the elementary and secondary level. Again, I do not pretend to be thorough in regard to what wabi-sabi aesthetics might contribute, but I intend to simply initiate some reflections on the possibilities of wabi-sabi thinking about mathematical activity at these levels.

Observing (of) wabi-sabi mathematics

Wabi-sabi is about private, intuitive, relative, personal experiences, as opposed to universal, general or prototypical ones.

Readers acquainted with my work in mathematics education are probably surprised, after reading the previous section, by how I am writing about mathematics as if it existed "in itself," i.e. outside of someone's actual, concrete "mathematical doings", as I like to say. I did so in the hope that even those committed to a "positivist" epistemology of mathematics (should it be realism, idealism, functionalism …) would be in a position to appreciate the idea of mathematics being wabi-sabi. For those familiar with the idea of mathematics simply taken as "doing mathematics" (e.g. [27]), a wabi-sabi aesthetic of mathematics should not be difficult to conceptualize. In this section, I want to discuss the observation of the wabi-sabi dimension of mathematics in actual mathematical doings.

One way to go about this is to play with the idea that doing mathematics in many ways *inherently* implies dealing with imperfections, disorder, or impermanence; and that this could be a good thing. To illustrate this, I offer a short excerpt, in which a primary school student is visibly engaged in mathematical work. Again, the point here is not that this should be surprising, but to think of how and why it could be appreciated (and maybe even cherished!) in a wabi-sabi aesthetic. In the following fragment, we find the young Liu conversing with a research assistant about the solids she made out of "Polydrons" (interlocking plastic shapes, see Figure 2), telling us how they are both similar and different:

RA: What is the name of that one? [*while Liu puts the prism aside and picks up the square-based pyramid*]. What's the difference between those two?

Liu: That's... that one [*the pyramid*] is flat … this one is flat like that [*puts down the pyramid, sitting on its rectangular face*] and this one's [*holding the prism between her palms*] hmm … is tall [*puts the prism behind the pyramid but still holds it between her hands*] … it stands [*briefly places the prism on the apex of the pyramid*] … this one is like really tall and

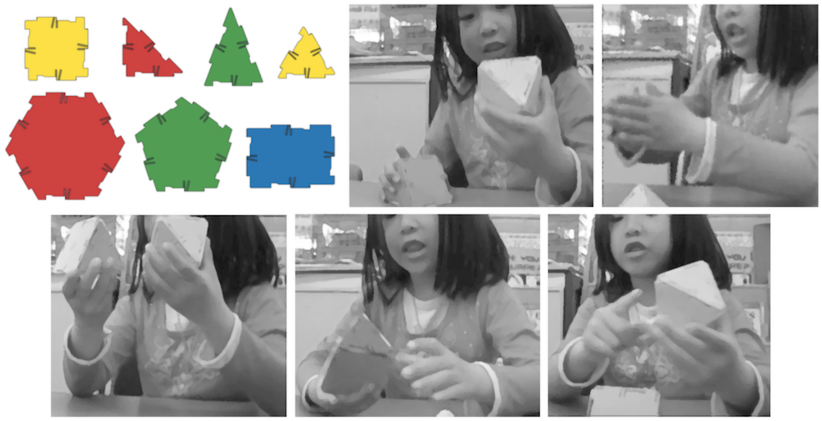

Figure 2: Polydrons and Liu comparing the solids she made with them.

this one [*lifting the pyramid and looking at its square face*] is kind of down [*touching the top of the pyramid with two fingers repeatedly*] there and [*touching the prism*] ... hmmm, these aren't really the same [*pressing firmly the sides of the prism and lifting it up looking at one of its rectangular sides*] but ... hmmm [*puts the prism on the desk*] but they do have both have squares [*placing one hand on each solid*]

RA: Oh! Both have ...

Liu: But ... hmm [*They speak simultaneously*]

RA: Can you show me where?

Liu: ... triangles triangles! ...[*touching the triangular sides of the prism*] Huh! And they have [*holding the pyramid, she turns it to look at its base*] ... this has a square [*showing the square face of the pyramid*] this has a square [*picking up the prism and showing its square face*]

Liu: ... But hummm ... these ... these has more ... this one has more squares [*taking the prism in her right hand*] than this one [*taking the pyramid in her left hand*], but they have the ... [*putting them back on the desk*] hmm ... they have ... this one [*looks at the pyramid*] has more triangles than this one [*points at the prism*] but this one [*points at the prism again*] has more squares [*showing the pyramid*] than this one [*putting the pyramid back on the desk*]

Comparing shapes is quite a mathematical challenge for most third grade students, and it is, in fact, listed as an "expected outcome" or "essential knowledge" in relation to classroom mathematical activity in many curricula. In this case, comparing involves both qualitative and quantitative observations of the properties of the objects (e.g. kinds of faces and number of faces), a language to situate those observations (here: in space), and some methodological elements to organize the (timely) observations and put them in contrastive or corresponding association. These are abilities that keep developing through mathematical activities, for example, when students are asked to compare expressions or functions and they certainly play an important role when professional mathematicians attempt to classify objects such as curves, or sets, or programs and so on.

Now, when I examine Liu's mathematical work, I immediately see how her various "statements" (in words and gestures) are private, intuitive, temporary and ambiguous. Her comparison starts with something about being flat in different, unspecified ways ("that one is flat... this one is flat like that"). Size is then evoked when the "tall" prism is contrasted with the pyramid being "kind of down". At the same time, her gestures are pointing out other aspects of the shapes for which no word seems to be forthcoming. Holding the prism between her palms (second frame in Figure 2) can remind us of the parallelism of the two triangle sides of the solids, a characteristic that is not present in the pyramid. Liu is also drawing attention to the apex of the pyramid with a gesture (when briefly placing the prism on top of the pyramid), and this can also be a defining feature in the comparison. Liu's talk then drifts toward the observation that the shapes "both have squares," while one "has more squares" than the other. Every time something is said, something is added; new points of comparison come forth "despite" the vagueness of her previous observations. Liu's talk is frenetic and approximate, as when one tries to grasp a fleeing, vanishing idea. The words and actions through which mathematics is made are best when evocative, and the mathematical ideas themselves are but evocations of bodily experiences, of touching and seeing, and going from one object to another.

A wabi-sabi perspective impels me to find beauty in those "imperfections", and I do! With wabi-sabi, I realize that the vagueness of Liu's talk, for instance, is not an impediment to (her) mathematical activity. On the contrary, it is its very condition! Without the possibility to simply evoke and perhaps later on try and "perfect" (complete, clarify, refine, improve) mathematical ideas, would doing mathematics still be possible? A quick look at Wittgenstein's *Tractatus Logico-Philosophicus* or Whitehead and Russell's *Principia Mathematica* whose intentions were precisely that, shows how difficult (if possible at all) this would be. In a different way, there is also a lot to contemplate in the mismatched approximation in which Polydron shapes (Figure 2) can be called a triangle or a square despite their jagged edges. Doing mathematics is also about cutting corners (Liu is also not mentioning the weight or the color of the objects). Although, upon closer examination, Liu's talk is quite imprecise, we nevertheless have a sense of what she is saying; something becomes visible, and we feel like we "know what she means" (perhaps in part because of what we are looking for in her talk!)

Now, one might ask, is this beauty in the mathematics, in Liu's rendering of it, or perhaps in the process of learning mathematics? This touches to very deep and potentially controversial questions (which is not a bad thing!) as to the interest, the necessity or the possibility of distinguishing between mathematics, mathematical activity and the discovery/creation of mathematical ideas or texts. In an observation of secondary level students, Rowland [37] found that vagueness plays an essential role in the mathematics talk of both students and professional mathematicians. He argued that vagueness can be viewed as "a subtle and versatile device which speakers can and do deploy to make mathematical assertions with as much precision, accuracy or as much confidence as they judge is warranted by both the content and the circumstances of their utterances." (page xiii). Livingston's (e.g. [22, 23, 24]) analysis of how mathematical proofs are obtained, written, and read also suggest that the lines between inventing, the inventer and the invented are not as clear as it might seem, and shows how mathematics can be viewed as a highly suggestive and evocative discipline.

Liu's comparison has little to do with the idealistic image of achieved mathematical truth, but it is vivid (second grader's) mathematical work in the making. A true picture of doing mathematics as tinkering and bricolage (e.g. [3]) is remindful of what de Freitas and Sinclair [43] describe as the process through which a mathematical idea "becomes a highly animate concept made vibrant and creative through the indeterminacy buried in it" connected to a shift of attention "from an emphasis on logical necessity towards an opening for contingency, ambiguity and creativity" (page 466). We also get a strong sense of how what Roth [36] calls the "living/lived mathematical work" can be seen, in the spirit of wabi-sabi aesthetics, as a constellation of private, intuitive, relative experiences. This is something we also get from young students' "*creations mathématiques*" in Freinet Pedagogy, and what is sometimes termed "children's mathematics" (e.g. [51]). Wabi-sabi aesthetic invites us to appreciate the beauty of "good-enough" mathematics [52] in its own way, as a delightful way to do mathematics, or see mathematics come to life.

This is but a brief illustration of how and where the wabi-sabi beauty of mathematics can be observed. A high school teacher who is also conducting a master's degree research project in mathematics education and with whom I discussed the idea of wabi-sabi mathematics, emailed me a few days ago. He said:

> Here is a problem I did with my students. It was a lot of fun and it produced huge conversations within the class. It goes straight to the heart of creativity in mathematics and of course reasoning. I am fairly convinced that this problem is an example of wabi-sabi. Very exciting!

Curious to hear from him on what basis a problem could be described as wabi-sabi, I inquired further. He answered:

> The problem's wabi-sabi nature wasn't at first noticeable, but it occurred to me that both teachers and students were disturbed by the problem, because of its unorthodox nature. I then asked myself why? The problem is stated incompletely and the solvers must make assumptions, and based on solvers' assumptions different solutions arise. Teachers are so used to dealing with polished questions, a problem which is imperfect becomes a thorn as it requires an approach that is not textbook or ministry mandated.
>
> It was also wabi-sabi because it [. . .] led to multiple creative solutions, among which one solution nevertheless stood out above the others because it generalized the problem, and approached it with elegance of thinking.
>
> What I am seeing is that the idea of wabi-sabi is moment of aesthetics in mathematics, particularly its pedagogy, where teachers can inform and work with their students thus creating mathematical community and cultivate awareness of perspectives and context in mathematics.

I believe we have here yet another entry point to wabi-sabi mathematics. Emphasis here is, in part, on the condition in which mathematics is made or experienced. There is also something rich in the experience of a multiplicity of ways to work on a problem, transform it, and contemplate an array of possible solutions. Wabi-sabi incites us to find beauty in such contingences, in the "humbleness" of local, particular answers. Even an awkward, convoluted, unconventional resolution has a charm, and can be seen as an opportunity to dwell upon the complexity of (doing) mathematics.

In pursuing the wabi-sabi aesthetic, one might even be tempted to deliberately offer students ill formulated problems, as some researchers have done occasionally (e.g. [33, 50]). Others have similarity suggested devoting more time in mathematics to modelling complex situations (e.g. how to optimize the boarding of a plane?) with an emphasis on "making something out of it" even though the developed answers remain very imprecise, local, etc. (e.g. [31]). We can find beauty in simply making something work where a "true answer" seems to be impossible. Furthermore, there is beauty in how it is precisely the imperfection of the model that allows it to work! I see another possibility to engage students with concepts and definitions in the way Borasi [5] experimented with the definition of a circle, or Zaslavsky [53] with the concept of "slope". Again, we do so with an emphasis on the pleasure and the mindfulness that is born from these irregularities.

Bothering (with) wabi-sabi mathematics

> *[wabi-sabi] is fundamentally open, present-oriented, and directed toward the expansion of awareness through exposure to ambiguity and contradiction, instead of focusing on goals, achievements, progress, and so on.*

My final question is to the interest, if any, of being concerned with the wabi-sabi side of mathematics, or the wabi-sabi aesthetical appreciation thereof. I will try to keep my argument as short and simple as possible.

For *me*, as a mathematics educator, a wabi-sabi aesthetical appreciation of mathematics is remindful of the importance of placing the doer (the learner, the teacher) before the mathematics. A controversial claim aligned with my own ethical and epistemological inclination to believe (and keep in mind) that without people doing mathematics, there wouldn't be any mathematics at all [26]. As Pimm [32], referring to Brookes [7], playfully puts it: "the most neglected existence theorem in mathematics is the existence of people" (page 173). One of the advantages of wabi-sabi is the opportunity it gives us to bring back (or keep) the people in mathematics from an aesthetical perspective, and this is especially important from an educational standpoint. In the first part of this article, I have brought forth elements related to the wabi-sabi aspect of mathematics "as a network of ideas", where "imperfection" might be found in various ways. However, wabi-sabi "as a way of living" [19] speaks profoundly to my sensitivity to, and interest in, the present, actual, day-to-day and moment-to-moment nature of doing mathematics. It speaks to the significance of offering students (and prospective or in-service teachers for example), living, breathing mathematical experiences where they can make mathematics even if it is not in the so called perfect, finished, stable or universal form we (and they) often have in mind. I find it fundamental to epistemologically ground my actions as a mathematics educator and present my students with such a view. Wabi-sabi challenges me to grow in awareness of how those students' struggles are perhaps reminders of the ambiguity of mathematical ideas, and appreciate how this is precisely where mathematical work takes place. If we can let go of the official need to see progress, to achieve goals, to instruct and qualify, we can simply be there, present and open, to do mathematics with them. Could we develop a culture of awareness to this, e.g. in mathematics educators, curriculum developers, teacher trainers, and so on?

I also suggest that "bothering" with wabi-sabi mathematics is important because presenting mathematics as elegant, smooth and efficient is simply not what mathematics actually looks like nor how it is done. Wabi-sabi can allow us to break from what Walkerdine [48] describes as mathematics' production of certainty, order and rationality "out of terror" (page 200). If mathematics is recognized and even appreciated as uncertain, disordered, or irrational; if doing mathematics is celebrated (at least in part) as the ambiguous work of evoking approximate ideas, it might not be as threatening to those who fear it. It might not be as despicable for those who hate it. Wabi-sabi is also crucial in my own appreciation of why I like mathematics. There is of course, something extremely enjoyable in how mathematical ideas can follow one another and take us to places that seemed at first unattainable. I also realize the joy I experience in doing mathematics in the way Tolstoy, in *Anna*

Karenina, cleverly phrases it: "Some mathematician, I believe, has said that true pleasure lies not in the discovery of truth, but in the search for it." [46, page 192].

There is also an important case to make, in relation to the rich body of work conducted in what we call "critical mathematics education" (e.g. [45]). Mathematics in society is generally perceived to be complete and to have only one truth [13]. Mathematics as such, embodies the supreme idea of objectivity and thus not only grinds itself, but also legitimate science and economics into "facts:" Numbers don't lie (e.g. [16])! A wabi-sabi appreciation of mathematics could be a rich way to challenge this impression, and develop the notion that math can be subjective, and even twisted to satisfy various agendas [41]. Including the wabi-sabi side of mathematics as part of a "culture of mathematics," might open the door for teachers at least to show bias and imperfections, and promote awareness of the dangers of these to the learner... yet at the same time release the creativity that can take place while doing mathematics. One example could be a discussion of the challenges of producing and interpreting statistical diagrams, and the ways in which some diagrams may be misleading [18].

From a research perspective, the idea of wabi-sabi mathematics is timely and appropriate for me. It highlights where my epistemologically oriented research over the last nine years or so, increasingly reveals itself as a deconstructive practice. Rethinking teachers' and students' mathematical activity in terms of fruitful, beautiful, inescapable imperfection is a new starting point. What might it mean to prepare for and conduct mathematics lessons on such a basis? How can it alter the way in which we conceive of teaching and learning? These are questions, which I intend to dwell upon and investigate, over the next few years. A study which, of course, also entails questioning further the very idea of wabi-sabi mathematics, poking and rattling it, so to say, pushing it further. I envision, over the next few months, experimenting a series of activities in which students will be presented with situations designed to bring forth the "imperfection" of (doing) mathematics.

As a Way Out

Wabi-sabi mathematics might at first be heard as an oxymoron. Even once one recognizes that (doing) mathematics has a wabi-sabi aspect, being in certain ways imperfect, transient, incomplete and unconventional, the idea of admiring this dimension of mathematics can still be provocative.

Why celebrate our flaws? On the other hand, thinking in terms of the mathematical experiences most (if not all!) people doing mathematics encounter, being positive about the often non-functional, contradictory, uncontrollability, organic face of mathematics might help us reconcile with those moments. This aspect also aligns mathematics more closely with the other sciences, to reference a point made earlier in the paper. In addition, it seems to me particularly important since as a society, we seem to insist upon exposing all of our children to mathematical activity from an early age, and for many, many years.... Perhaps we can address this situation by developing a dialectical approach to consider both the "perfection" and the "imperfection" of (doing) mathematics. Luitel [25] also seems to be looking for this when he talks about conceptualizing mathematics as an "im/pure" knowledge system in order to move away from "exclusive emphasis on an ideology of singularity, epistemology of objectivism, language of universality and logic of certainty" (page 65). Thinking about the aesthetics of the im/perfection of (doing) mathematics could then similarly pave the way to a more dialogical aesthetic with/in/through mathematics; an aesthetic sensitive to its historical and cultural groundings.

Acknowledgement

I would like to thank Draco Szathmary, Maude Bélanger and Franciele da Silva for their contribution to my thinking for this paper; and the reviewers whose comments helped me introduce these ideas in a more inviting way.

References

[1] Krzysztof R. Apt "*Logicomix An Epic Search for Truth*, by Apostolos Doxiadis, Christos H Papadimitriou, Alecos Papadatos and Annie Di Donna," *Mathematical Intelligencer*, Volume **32** Issue 3 (September 2010), pages 51–52.

[2] Mark Van Atten, *On Brouwer*, Wadsworth, London, 2005.

[3] Heinrich Bauersfeld, "Theoretical perspectives on interaction in the mathematics classroom," in *Didactics of Mathematics as a Scientific Discipline*, edited by R. Biehler, R. W. Scholz, R. Strässer and B. Winkelmann, Kluwer Academic Publishers, Dordrecht, 1994, pages 133–146.

[4] Roseanne Benn, "Mathematics: Certainty in an Uncertain World?," in *Perspectives on Adults Learning Mathematics*, edited by Diana Coben, John O'Donoghue, and Gail E. Fitzsimons, Springer, Netherlands, 2002, pages 109–118.

[5] Raffaella Borasi, "Appreciating the mathematical element of mathematical content: the case of definitions," in *Essays in Humanistic Mathematics*, edited by Alvin White, Mathematical Association of America, Washington DC, 1993, pages 123–139.

[6] Jonathan M Borwein and David H Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A K Peters, Natick, MA, 2008.

[7] W. Brookes, *Mathematical Reflections*, Cambridge University Press, Cambridge, 1970.

[8] Gregory J. Chaitin, *Conversations with a Mathematician: Math, Art, Science and the Limits of Reason*, Springer, London, 2002.

[9] Gregory J. Chaitin, "Thoughts on the Riemann hypothesis," *The Mathematical Intelligencer*, Volume **26** Issue 1 (Winter 2004), pages 4-7.

[10] Gregory J. Chaitin, "Randomness in arithmetic," in *Information, Randomness & Incompleteness: Papers on Algorithmic Information Theory* (Collected Papers of Gregory J. Chaitin), Second Edition, World Scientific, Singapore, 1990, pages 14–19.

[11] Gilles Châtelet, *Figuring Space: Philosophy, Mathematics, and Physics*, Kluwer Academic Pub, Dordrecht, 2000.

[12] Gilles Châtelet, "Interlacing the singularity, the diagram and the metaphor," in *Virtual mathematics: The Logic of Difference*, edited by Simon B.. Duffy, Clinamen Press, Manchester, 2006, pages 31–45.

[13] Bruno De Finetti, "La prévision: ses lois logiques, ses sources subjectives," *Annales de lInstitut Henri Poincaré*, Volume **7** Issue 1 (1937), pages 1–68.

[14] Paul Ernest, "Certainty In Mathematics: Is There A Problem?," *Philosophy of Mathematics Education Journal*, Volume **28** (2014). Available at http://socialsciences.exeter.ac.uk/education/research/centres/stem/publications/pmej/pome28/index.html.

[15] James Franklin, "Non-deductive logic in mathematics," *British Journal for the Philosophy of Science*, Volume **38** (1987), pages 1-18.

[16] Steven Goldberg, "Numbers Don't Lie: Men Do Better Than Women," *New York Times*, Opinion, July 5, 1989.

[17] Godfrey Harold Hardy, *A Mathematician's Apology*, Cambridge University Press, Cambridge, 1992.

[18] Douglas M Hill, "Difficulties with Diagrams," *Journal of Science and Mathematics Education in Southeast Asia*, Volume **11** Issue 2 (1988), pages 32–40.

[19] Andrew Juniper, *Wabi Sabi: the Japanese Art of Impermanence*, Tuttle Publishing, Boston, 2011.

[20] Sergei Igorevich Kabanikhin, "Definitions and examples of inverse and ill-posed problems," *Journal of Inverse and Ill-Posed Problems*, Volume **16** Issue 4 (2008), pages 317–357.

[21] George Lakoff and Rafael E Núñez, *Where Mathematics Comes from: How the Embodied Mind Brings Mathematics into Being*, Basic Books, New York, 2000.

[22] Eric Livingston, *The Ethnomethodological Foundations of Mathematics*, Routledge, 1986.

[23] Eric Livingston, "Cultures of proving," *Social Studies of Science*, Volume **29** Issue 6 (1999), pages 867–888.

[24] Eric Livingston, "The Disciplinarity of Mathematical Practice," *Journal of Humanistic Mathematics*, Volume **5** Issue 1 (2015), pages 198–222.

[25] Bal Chandra Luitel, "Mathematics as an im/pure knowledge system: symbiosis,(w) holism and synergy in mathematics education," *International Journal of Science and Mathematics Education*, Volume **11** Issue 1 (2013), pages 65–87.

[26] Jean-François Maheux, *How Do We Know?: An Epistemological Journey in the Day-to-day, Moment-to-moment of Researching, Teaching and Learning in Mathematics Education*, Ph.D. Thesis, University of Victoria., 2010.

[27] Jean-François Maheux and Jérôme Proulx, "Doing mathematics: analysing data with/in an enactivist-inspired approach," *ZDM*, Volume **47** Issue 2 (2014), pages 211–221.

[28] Ami Mamolo, "Polysemy of symbols: Signs of ambiguity," *The Mathematics Enthusiast*, Volume **7** Issue 2 (2010), pages 247–262.

[29] Paolo Mancosu, *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*, Oxford University Press, Oxford, UK, 1998.

[30] Reviel Netz, *Ludic Proof*, Cambridge University Press, Cambridge, 2009.

[31] Nilima Nigam, "Mathematics in Industry, Mathematics in the Classroom: Analogy and Metaphor," in *Proceedings of the 2014 Annual Meeting of the Canadian Mathematics Education Study Group*, edited by Susan Oesterle and Darien Allan, published by CMESG/GCEDM, 2015, pages 25–35.

[32] David Pimm, "Drawing on the image in mathematics and art," in *Mathematics and the Aesthetics*, edited by N. Sinclair, D. Pimm and W. Higginson, Canadian Mathematical Society / Springer-Verlag, New York, 2007, pages 160-190.

[33] Ewa Puchalska and Zbigniew Semadeni, "Children's reactions to verbal arithmetical problems with missing, surplus or contradictory data," *For the Learning of Mathematics*, Volume **7** Issue 3 (1987), pages 9–16.

[34] Hilary Putnam, "Mathematics without foundations," *The Journal of Philosophy*, Volume **64** Issue 1 (1967), pages 5–22.

[35] Gian-Carlo Rota, "The phenomenology of mathematical beauty," *Synthese*, Volume **111** Issue 2 (1997), pages 171–182.

[36] Wolff-Michael Roth, "Researching living/lived mathematical work," *Forum: Qualitative Social Research*, Volume **12** Issue 1 (2010), available at http://www.qualitative-research.net/index.php/fqs/article/view/1604/3115, accessed on January 24, 2016.

[37] Tim Rowland, *The Pragmatics of Mathematics Education: Vagueness in Mathematical Discourse*, Falmer Press, New York, 2000.

[38] Bertrand Russell, *Mysticism and Logic and other Essays*, Longmans, Green and Co., London, 1918.

[39] Bertrand Russell, *Portraits From Memory And Other Essays*, Simon & Schuster, New York, 1956.

[40] Crispin Sartwell, *Six names of beauty*, Routledge, London, 2013.

[41] Charles Seife, *Proofiness: The Dark Arts of Mathematical Deception*, Viking, New York, 2010.

[42] David Sherry, "On mathematical error," *Studies in History and Philosophy of Science*, Volume **28** Issue 3 (1997), pages 393–416.

[43] Nathalie Sinclair, Elizabeth De Freitas and Francesca Ferrara, "Virtual encounters: The murky and furtive world of mathematical inventiveness," *ZDM*, Volume **45** Issue 2 (2013), pages 239–252.

[44] Nathalie Sinclair, David Pimm and William Higginson, *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity*, Canadian Mathematical Society / Springer-Verlag, New York, 2007.

[45] Ole Skovsmose and Lene Nielsen, *Critical Mathematics Education*, Springer, Netherlands, 1996.

[46] Leo Tolstoy, *Anna Karenina*, Barnes & Nobles, New York, 2003.

[47] Jean Van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press, Cambridge, 1967.

[48] Valerie Walkerdine, *The Mastery of Reason: Cognitive Development and the Production of Rationality*, Taylor & Francis / Routledge, Florence, KY, US, 1988.

[49] Norbert Wiener, "Is mathematical certainty absolute?," *The Journal of Philosophy, Psychology and Scientific Methods*, Volume **12** Issue 21 (1915), pages 568–574.

[50] Uri Wilensky, "Paradox, programming, and learning probability: A case study in a connected mathematics framework," *The Journal of Mathematical Behavior*, Volume **14** Issue 2 (1995), pages 253–280.

[51] Maulfry Worthington and Elizabeth Carruthers, *Children's Mathematics: Making Marks, Making Meaning*, SAGE, London, 2003.

[52] Vicki Zack and David A Reid, "Good-enough understanding: Theorising about the learning of complex ideas (part 1)," *For the Learning of Mathematics*, Volume **23** Number 3 (November 2003), pages 43–50.

[53] Orit Zaslavsky, Hagit Sela and Uri Leron, "Being sloppy about slope: the effect of changing the scale," *Educational Studies in Mathematics*, Volume **49** (2002), pages 119–140.