Journal of Humanistic Mathematics

Volume 6 | Issue 1 January 2016

Surprise and the Aesthetic Experience of University Students: A Design Experiment

Ofer Marmur
Technion-Israel Institute of Technology

Boris Koichu
Technion-Israel Institute of Technology

Recommended Citation

Marmur, O. and Koichu, B. "Surprise and the Aesthetic Experience of University Students: A Design Experiment," *Journal of Humanistic Mathematics*, Volume 6 Issue 1 (January 2016), pages 127-151. DOI: 10.5642/jhummath.201601.09 . Available at: https://scholarship.claremont.edu/jhm/vol6/iss1/9

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

Surprise and the Aesthetic Experience of University Students: A Design Experiment

Ofer Marmur

Faculty of Education in Science and Technology,
Technion - Israel Institute of Technology, ISRAEL
omarmur@tx.technion.ac.il

Boris Koichu

Faculty of Education in Science and Technology,
Technion - Israel Institute of Technology, ISRAEL
bkoichu@tx.technion.ac.il

Abstract

Little is known about instructional means by which the aesthetic experience of mathematics can be enhanced for undergraduate learners. This paper presents and discusses an iterative lesson design process towards creating an opportunity for students to appreciate the beauty of an unexpected solution to a challenging calculus problem. The lesson design draws on insights from both mathematics education research on aesthetics and research on aesthetic appreciation in music. The data were collected over the course of five lessons with different groups of calculus students in which the intended problem was presented in two different ways. In addition, stimulated-recall interviews were conducted with nine students who took part in the later lessons and exhibited strong emotions regarding the problem. The data suggest that the students' aesthetic response to the problem was essentially conditioned by the extent of their surprise as a result of revealing a clever solution to the problem after being exposed to repeated failed attempts. Implications for practice are drawn.

Keywords: *iterative lesson design; mathematical beauty; problem solving; stimulated recall interviews; undergraduate calculus*

1. Introduction

Providing students with opportunities to experience the beauty of mathematics is considered by many to be a highly important educational mission, yet it has proven to be a rather challenging and non-straightforward task (e.g., [4, 11, 34]). In particular, little is known about instructional means by which aesthetic experiences of mathematics can be enhanced for undergraduate learners.

This paper presents and discusses pedagogical decisions involved in an iterative lesson-planning process towards creating an opportunity for students to appreciate the beauty of an unexpected solution to a challenging problem. Our investigations occurred in what is frequently referred to as a "traditional" instructional context, a large-group tutorial in a university-level calculus course.

Our main goal is to elucidate how the creation of a strong and repetitive emotional surprise for students can serve as a mechanism for enhancing their mathematical aesthetic experience. By pursuing this goal we aim to contribute to the growing body of research on the interplay of affective, cognitive, and situational factors involved in students' aesthetic appreciation of elegant solutions to mathematical problems.

2. Theoretical Framework

2.1. Conceptualization of mathematical beauty

From a theoretical standpoint, the case of interest is situated at the crossroad of two general perspectives for conceptualizing mathematical beauty, objectivism and subjectivism, the latter also referred to as projectivism [6, 36]. From the perspective of objectivism, mathematical beauty is an intrinsic property of a problem, theorem, or proof, and is consequently independent of the observer and the socio-historical context surrounding it [6]. Dreyfus and Eisenberg [11] list the characteristics that contribute to the aesthetics of a solution or proof as: clarity, simplicity, brevity, conciseness, structure, power, cleverness, and surprise. Furthermore, in the literature one can find many additional characterizations of mathematical beauty, such as visualization, elegance, order, form, relations [39], transparency [8], symmetry, complexity, and reference to realistic applications [3]. The perspective of objectivism finds great support among many mathematicians, and is often accompanied by elitist views, which question the ability of non-mathematicians to possess or develop an aesthetic sense for mathematics (e.g., [29]).

The perspective of projectivism is on the opposing side. Clarifying this approach is McAllister [24], who regards beauty as "a value that is projected into or attributed to objects by observers, not a property that intrinsically resides in objects" (page 15). Under this approach, what is considered to be mathematically beautiful is influenced by personal taste, previous personal knowledge and experience, age of the observer, and socio-historical cultural context (compiled from [3, 9, 31, 34, 36, 38]). From an educational point of view, empirical research has shown evidence that students do in fact hold aesthetic sensibilities for mathematics, though their aesthetic views and judgments often differ from those of mathematicians, possibly as a result of different personal and psychological needs [3, 4, 19, 35].

Wishing to benefit from the best of both worlds, we accept in our research and practice McAllister's [24] claim that "mathematical beauty can play both a subjective role in the experience of mathematicians and an objective role in the appraisal of mathematical proofs and other results" (page 15). This means that on the one hand, we can use characteristics such as surprise, simplicity, cleverness, and so forth [11], in order to hypothesize whether a mathematical problem might evoke an aesthetic response. On the other hand, pedagogical considerations guiding the instructional-design process intended to increase students' aesthetic experiences stem from the affordance of the perspective of projectivism. Recognizing the educational contribution of bringing the aesthetic dimension into the classroom, we situate our research as one that originates from a student-centered approach. Accordingly, we accept Koichu and Kontorovich's [20] proposed definition, and consider a problem or solution to be mathematically beautiful if "it is evaluated as such by the poser of the problem, its readers or solvers, and if one's argumentation underlying the evaluation involves some of the aforementioned general characterizations" (page 73).

2.2. Mathematical beauty and surprise

In the aforementioned lists of characteristics, including clarity, simplicity, conciseness, and so forth, as pertaining to the aesthetics of a mathematical solution or proof, surprise was mentioned as just another characteristic.

However, some scholars argue that 'surprise' should in fact be regarded as a pivotal characteristic of mathematical beauty and a main contributor for a creation of an aesthetic experience. For example, with regard to the characteristic of 'simplicity': a simple solution to a problem is much more likely to be viewed as 'beautiful' when the initial expectation of the solver is that the problem possesses a complicated solution, and this expectation meets an unexpected simple path for solution. As stated by de Freitas and Sinclair [10]: "All the characteristics of the mathematical aesthetic [...] lack significant impact if a feeling of surprise is not also engendered" (page 187).

Surprise, additionally, is claimed to be a useful and powerful tool for the improvement of students' mathematics learning in a classroom. According to Movshovits-Hadar [26], teaching mathematics 'the surprise-way' (page 35) can serve as a useful route for gaining students' curiosity and interest, which in turn serves as motivation for learning and increases its successfulness. This view is also supported by Nunokawa [27] who analyzed lessons where surprising gaps between expectations and realizations in the 'mathematical world', the 'real world', and in-between the two, were utilized as opportunities to enhance students' interest in mathematical ideas. However, it should be noted that in both these cases, the type of surprise discussed is only addressed on an intellectual level, and with no reference to any affective or bodily aspects relating to the mere act of being surprised. Keeping in mind that affect plays a major role in mathematics education (e.g., [14, 25]), and that cognition, affect, and aesthetics are strongly intertwined during mathematical activities [33, 36], we recognize that further investigation is needed as to the role and impact of surprise at an emotional level.

In order to gain insight into the role of emotional or bodily surprise in the creation of an aesthetic experience, it seems natural to examine how this is achieved in the artistic domain. Analyzing mathematical aesthetics using an artistic lens is a known approach, as is demonstrated for example by Tymoczko's [37] aesthetic analysis of the Fundamental Theorem of Arithmetic. The analysis borrows musical terminology, such as the manipulation of tempo, density, rhythmical changes, tension build-up, and its release. This view of looking at how an aesthetic experience is built specifically in music seems especially relevant when investigating ways to achieve such an experience in the classroom. In music, an art-field that arranges sounds in time, aesthetic meaning is not derived from a single experience, but from a flow of experiences [18]. It is not merely a highlight-event that creates an aesthetic experience, but a buildup of tension and release, anticipation of what is about to come, playing with preparation, suspense, and resolution [18]. Similarly, throughout the duration of a mathematics lesson, students may flow through changing cognitive and affective experiences (see also [13], for affective pathways) which shape their accumulated and ultimate perception and experience of the lesson, as well as any potential aesthetic evaluation.

More specifically, Huron's [16] "contrastive valence" theory for explaining the aesthetic pleasure that arises from listening to a musical piece is based on research in the neuroscientific domain, and ascribes to surprise a prominent contributing role. According to Huron's theory, when we are surprised, two distinct types of responses occur in our brains: a reaction response and an appraisal response. While the reaction response to surprise is rapid, bodily, and from a biological perspective regards the surprise as a negative event (see also [21], especially on fear-conditioning), the appraisal response is a slow, reflective, and conscious evaluation of the situation. These two types of responses are not necessarily in accordance with each other. In fact, according to Huron's "contrastive valence" theory, a positive appraisal response following a negative reaction response is a mechanism for evoking musical pleasure. Moreover, the bigger the contrast is between these two types of responses, the bigger the pleasure is.

The relevance of this theory for mathematical aesthetic sensibilities can be justified by recalling that according to Rota [31], perceiving a proof as beautiful is dependent on all previous experiences, difficulties, and effort that went into the process of solving the problem. This implies that in order to create an aesthetic experience based on surprise during a lesson, careful planning is required, where a sufficient amount of time is given to the audience (or in our case: students) after the surprising event. This allows a slow positive appraisal process to "kick in", changing what is initially perceived as a negative surprising experience into a pleasurable one.

2.3. Surprise, beauty, and meta-affect

When reflecting on surprise as a characteristic of mathematical beauty, one may possibly link it to positive emotions, such as enthusiasm coming forth from a "wow" illumination moment. However, as suggested by Huron's [16] theory, surprise is oftentimes instinctively perceived as an emotionally negative event. Accordingly, when students encounter a surprising solution, which may be aesthetically appealing to their lecturer or teaching assistant, they may in fact respond to it negatively, perceiving it as tricky or unfair, wondering how they could have reached such a solution by themselves (this phenomenon is recognized in the literature [11, 19], as well as seen in class during many years of teaching by the authors of this paper). Subsequently, this may have a harmful effect on students' "openness" to seeing any beauty in the solution. Indeed, Brinkmann [3] suggests that a necessary condition for school students to perceive beauty in a mathematical problem is their "feelings of security and success" (page 377), i.e., their belief of being able to succeed in solving such a problem. A similar stand has been taken by Koichu *et al.* [19] with regard to students at the undergraduate level of education.

However, according to Goldin's [14] discussion on meta-affect, negative emotions during mathematics learning may actually have a positive impact on students, depending on the emotions students hold with regard to their negative emotions. For example, Goldin demonstrates that a feeling of frustration may be perceived by a student as negative if it encodes a fear of failure, yet it may be perceived as a positive experience if it encodes anticipatory pleasure of succeeding in solving a challenging problem. Indeed, negative emotions are important not merely as part of the creation of an aesthetic experience (as implied by the "contrastive valence" theory of Huron [16]), but they are an integral part of the learning process. The extensive research on math anxiety might have created the misleading impression that contemporary mathematics education should aim at evoking only positive emotions in students. However, learning is challenging, and as such not always solely a *pleasant* experience. Self confidence and belief are also built through dealing with difficulties, and feeling proud when succeeding to overcome them. Bandura [2] claimed that perceived self-efficacy is actually higher when experiencing setbacks but still noticing relative progress, in comparison to situations of constant success without feeling any improvement over previous performance. Indeed, as Goldin [14] clearly states, "up to a point, negative emotions en route foster greater eventual pride, pleasure, and satisfaction in having attained a concept or solved a problem" (page 409).

2.4. Large-group calculus tutorials as a setting for enhancing students' aesthetic experience

At the university where this research took place (see Subsection 4.1), tutorials are classes attended by 50 to 100 students, where problems accompanying the theoretical material presented in the lectures are being solved. As such tutorials tend to be exam-oriented in nature, at least in the way that many students perceive them, it is legitimate to wonder whether tutorials are indeed the right instructional setting to insist on trying to lead students to see and appreciate mathematical beauty. From a psychological point of view, situational factors play a role in how human beings experience aesthetics. As Jacobsen [17] points out, people are most likely to respond differently to a piece of art if they see it in a supermarket as opposed to a museum. People in these two different places are as a rule in different mindsets, which impact the potential creation of an aesthetic processing. Stretching the above analogy, tutorials might be viewed as "supermarkets", where students come to "shop" for tools that can help them solve problems in their exams. One could conceivably argue therefore that the tutorial context could potentially not leave any room for an aesthetic experience. In light of this view, we wish to supply the following explanation in support of the importance and relevance of integrating the aesthetic dimension of mathematics into university tutorials.

Tutorials of high-level calculus courses, such as the course in which our study was conducted, already include problems that can be viewed as "beautiful" in terms of the aforementioned characteristics of mathematical beauty. Such problems often appear in homework assignments or final exams, and are therefore relevant to be taught in the lessons. Consequently, the question at hand is not whether to bring such problems to class, but whether to invest time and educational effort into creating opportunities for students to appreciate these problems as beautiful. Our answer to this question is "yes, at least sometimes." This is for the following reasons.

Firstly, going through an aesthetic experience can enhance students' enjoyment and interest in the material, which can in turn impact their academic engagement in the short and long term, as well as increase their motivation to learn [12]. Furthermore, neuroscientific research has shown that emotional events influence the encoding and storage of information in our brains, and consequently can enhance memory capabilities for the long-term [28]. Accordingly, it is reasonable to assume that students who get excited and enthusiastic about the beauty of a mathematical problem are more likely to remember the mathematical knowledge related to it, and will more likely be able to retrieve that knowledge in the future.

Secondly, putting time and effort into creating an enjoyable aesthetic experience for students can counteract strong negative emotions that are often triggered in students when presented with a surprising and tricky solution to such a problem, as discussed earlier in the paper (see also [11, 19]). In such cases students may not perceive the surprise in the solution as a pleasant one leading to aesthetic appreciation, but rather as a frustrating surprise that undermines their self-efficacy, i.e., their belief in their own ability to succeed in such tasks [2]. Applying Huron's [16] theory to such cases, it seems that students pass through the first stage of a negative reaction response, yet not through the second stage of an appraisal response, which reflectively re-evaluates the situation. Therefore, utilizing instructional means that have the potential to increase the likelihood of eliciting aesthetic appreciation from students in class can also help in monitoring student emotions as well as mending some of their negative feelings.

3. Research Questions and Research Approach

Our research relies on the premise that insights into the mechanisms of aesthetic appreciation in mathematics and art presented in Section 2 can gradually be transformed into design principles for lesson-planning that can enhance the experience of aesthetics in undergraduate students. Specifically, we pursue the following research questions:

- How can an unexpected/surprising solution to a challenging problem be presented in a large-group tutorial setting so that the solution would evoke an aesthetic response on the side of the students?
- What are the cognitive and affective characteristics of students' experiences during a solution process that evoked an aesthetic response, as reflected by stimulated recall interviews conducted one to two weeks after the tutorial took place?

The first question belongs to the realm of research questions that can be addressed by a design experimentation methodology. Accordingly (see [7] for an elaborated discussion on the main features of design experimentation), we utilize the interventionist methodology, present task design iterations, and attempt to precisely indicate how theoretical considerations can inform the design of the intended lesson.

The second research question is (mainly) addressed by means of a stimulated-recall interview methodology (see [5] for a discussion on its strengths and limitations). These stimulated-recall interviews with the students were utilized in order to gain a better understanding of the nature of the students' aesthetic responses expressed during the tutorials. In addition, they allowed us to inquire into why some of them felt that the problem was beautiful, as well as to gain insight into their emotions and thoughts leading to that "wow" moment.

4. Method

4.1. Context and Participants

The study was conducted in a setting of undergraduate first-semester calculus tutorials, which are given to students from the computer science and electrical engineering faculties of the Technion. The Technion is a highly selective university, and its computer science and electrical engineering faculties are among the most demanding faculties. Accordingly, the university considers the course participants as very capable students who are able to cope with challenging mathematical problems, whether in homework assignments or during exams.

The theoretical basis of the course (that is, definitions, theorems, and proofs) is presented to the students in the lectures. It is in the tutorials where the students are presented with problems on the taught topics for the first time.

The data were collected over the course of five semesters, with different groups of students attending the lessons of the same teaching assistant (the first author of this paper). These were large groups of students consisting of anywhere between 60 and a 100 students.

4.2. The problem of interest

The problem of interest has been used in the setting of calculus tutorials during five consecutive semesters. This problem asks students to show that the sequence defined by $a_1 = 1,\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ converges and to calculate its limit.

This problem appears in a suggested planned calculus tutorial on monotone sequences. Through dealing with similar looking questions on this topic students become aware that a standard way of solving such problems is to prove by mathematical induction that the sequence is monotone and bounded and so converges. In this problem, a calculation of several initial elements of the sequence gives reason to believe it is monotonically decreasing starting from $n = 2$, and bounded below, by 0 for example. Applying arithmetic limit laws shows that the limit is $\sqrt{2}$ if indeed we can prove that the sequence is monotone and bounded and therefore converges. At this point we invite readers to try to solve this problem themselves, since the solution process may influence the potential aesthetic evaluation of it, as argued earlier.

Continuing with the solution, it is rather simple and straightforward to prove that the sequence is bounded below by 0, yet in this specific case the routine method of mathematical induction as taught in class fails to prove monotonicity. Algebraic manipulations on $a_{n+1} \leq a_n$ show that monotonicity can be proven if we choose a better lower bound (in fact, the best lower bound) and first prove that $a_n \geq \sqrt{2}$ for $n \geq 2$. The latter statement, perhaps surprisingly, again cannot be proven with the same routine method of mathematical induction. Yet if we recall the inequality of arithmetic and geometric means (henceforth to be called *the AM-GM inequality*), then we can already see the light at the end of the solution-tunnel:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{a_n + \frac{2}{a_n}}{2} \geq \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2}.$$

In terms of characteristics of mathematical beauty, this solution can be viewed as beautiful since it possesses features of surprising simplicity and cleverness (see Subsection 2.1). While at the onset this problem seems to have a familiar and technical method of solving using mathematical induction (as is done in other similar problems), later on it is surprisingly discovered that this method fails to achieve a solution. However, if students recall the AM-GM inequality which was taught earlier during the semester, this yields an unexpected quick and easy solution to the problem. Indeed, the potential for an aesthetic response to this problem and its solution is what originally attracted us to it, in the hope that the students would share our excitement. Yet this was not the case in the several years the first author of this paper had taught this problem in class, prior to the beginning of a research on the mechanisms underlying students' mathematical aesthetic evaluation.

4.3. General guiding principles for the lesson design

Stimulated by the insights presented in Section 2, we formulated two main principles that we believed could enhance the likelihood of evoking an aesthetic response in class. Our first principle was to allow students the opportunity to experience a familiar method first, knowing in advance that it would lead to a dead-end. Not only is this important for students' development of heuristic problem-solving behavior (e.g., [32]) to see for themselves hands-on why a certain method fails, but following Rota's [31] line of thought, it can also enhance the aesthetic appreciation of the alternative solution route that will follow. Moreover, in terms of Huron's [16] theory, we assumed the dead-end to play the role of a negative surprise for the students. This, and the subsequent passing of time till the highlight-moment of a resolution for the problem in class, can allow for the creation of a reflective process, ultimately leading students to an aesthetic appreciation of the solution. Additionally, this instruction principle was also conceived as a response to a phenomenon described by Dreyfus and Eisenberg [11], where students did not acknowledge any beauty in a short and elegant solution, while resistantly claiming: "So what! My way works too" (page 7). In our case students would first become convinced that "their way" in fact did not succeed, and as such we expected them to be more open to exploring a new route, as well as see its beauty.

The second principle was to legitimize students' feelings and expressions of negative emotions in class, even encouraging them, especially surrounding the surprising event of reaching a dead-end (the event was surprising since a traditional calculus tutorial at the Technion usually presents only successful solutions). Referring again to Huron's [16] "contrastive valence" theory, the more negatively the initial surprise is perceived by the students, the bigger their ultimate excitement at the end of the process could potentially be. Additionally, encouraging students to vent their frustration with the difficulty of the solution process can allow them to deal with their negative emotions. By conveying in class that emotions such as frustration or fear are natural in a challenging problem-solving process, students' meta-affect with regard to these negative emotions can be influenced, leading them to experience them in a positive manner (cf. [14]). Ultimately, the goal is to create a setting where students can "release" some of their negative emotions regarding the difficulty of the solution prior to reaching the highlight-moment. When subsequently such a moment is presented, students can be more "emotionally available" to not only focus on the negative, but also see the potential beauty.

4.4. Data collection

We report on two groups of iterations, initial iterations and further iterations, each with corresponding data collection tools. The *initial iterations* took place over two consecutive semesters. During this stage we attempted to implement the above principles on the aforementioned problem of interest, as well as on other problems with surprising solutions which we believed the students could potentially see beauty in. During this stage, the data were collected by audio recordings of the lessons and the first author's research-diary, which was structured in accordance with Liljedahl, Chernoff, and Zazkis' [23] suggestions for documenting phases of iterations in a design experiment: predictive analysis, trial, reflective analysis and adjustment. Additionally, we asked every student who said in class that she or he found the problem "beautiful" to write us a short email explaining why she or he thought so.

The *further iterations* took place over three subsequent semesters and addressed only the problem presented in this paper. In the first semester of these three, the data were still collected in the same way as before. Yet a readjustment of the implementation of the guiding principles during that lesson led to what we regarded as successful results (as will be elaborated on below). Consequently, during the two following semesters, two similar versions of the lesson were video-recorded and subsequent stimulated-recall interviews were held with nine students who explicitly expressed their appreciation of the problem in the lesson.

During the stimulated-recall interviews, the students were presented with a 15-20 minute segment of the filmed lesson in which the problem of interest was taught. The segment started from the moment leading to the first incidence of reaching a dead-end in class, and ended about two minutes after the surprising use of the AM-GM inequality. At the beginning of the interview, we explained to the students that the video served as a tool to help them "relive" the lesson. They were instructed to stop it whenever they had a particular recollection of what they thought or felt at that moment. Additionally, they were told that if they would not stop the movie for an extended period of time, the interviewer (the first author of this paper) might stop the video occasionally and ask them whether they have any recollections on that specific moment (it should be noted that with eight out of the nine interviewees there was barely any need to implement this type of intervention). When students stopped the video to share their memory of that moment in class, the interviewer sometimes asked small clarifying questions, mainly in the form of: "Can you explain *why* you thought/felt this way at that moment?". At the end of the interview, students were encouraged to share any additional memories, thoughts, and/or feelings they had with regard to the problem during the lesson itself or the period of time between the lesson and the interview. The interviews were conducted 5 to 19 days after the lesson (6 interviews up to 9 days after the lesson, 2 interviews after 12 days, and one interview after 19 days). The duration of the interviews ranged from 30 to 60 minutes, depending on the level of detail that was shared by the student. The interviews were audio-recorded and transcribed.

5. Findings

5.1. Initial iterations

As earlier presented, aiming at evoking aesthetic responses from students, the first guiding principle of the lesson design was to try the familiar solution method in class first, even when the teacher knew this would ultimately reach a dead-end. Our problem of interest contains two opportunities for doing so while using mathematical induction: one for monotonicity and the second for $\sqrt{2}$ as a lower bound. Due to the time constraints in tutorials, as well as the fact that these two inductions fail in a similar way, we decided during the initial-iterations stage to lead students in class only through the first failed attempt at induction, while deferring the second failed induction to be checked by the students at home.

During the initial stage of the lesson-planning iterations, it became clear it was *feasible* to have students evaluate mathematical problems as "beautiful", though usually only one to two students at a time. This stage revealed that the use of characteristics such as simplicity, cleverness, and surprise, to explain mathematical aesthetics, was not merely a way for us to evaluate which problems could be perceived as beautiful, but that this notion was shared by some of the students. Though the amount of students alluding to the beauty of the problem was little each time, these responses from students were in most cases spontaneous, and subsequent explanations showed great enthusiasm. These included words and phrases such as "amazing", "wow", "impressive", "I was so excited when I saw it [the solution]", and "I will never forget this proof".

For illustration, see the following excerpt from an email sent to us by a student as an explanation for her enthusiasm for the solution of the problem presented in this paper (translation into English by authors):

> *In high-school I had a teacher that told me that in math you follow the "laziness principle" — the shortest and smartest way is the way to go. Usually I don't figure out the short and smart way by myself and therefore I really appreciate it when someone shows it to me. The AM-GM inequality is something I knew and didn't think of using here. And I loved it how you just pulled it out of the 'bag of tricks' and it solved the problem within a second — the laziness principle at its best. It was just beautiful. I wasn't planning on saying it out loud [during the lesson], but I guess such a particularly silly smile was put on my face in light of the magical solution, that you had to stop the lesson in order to ask me why I'm smiling to myself. So I had to share :)*

Reflecting on the small amount of aesthetic responses during the initial-iterations stage, at that point in time we had to admit we were not able to claim in certainty a relation between our instruction manipulations and students' responses. Nor were we able to isolate the different impacts of our aforementioned suggested principles on the creation of an aesthetic experience. This raised the question whether such efforts and extensive use of valuable lesson-time were worth the modest results. However, the high level of excitement in the responses we did get, combined with the observation that we had received such responses on many more occasions when implementing our theory-based principles during that year (in comparison to almost no aesthetic responses in a previous year), led us to believe it was worth continuing our iterative process with the goal of increasing the amount of aesthetic responses.

5.2. Further iterations

5.2.1. Predictive analysis on the adjustment of the lesson-design

For the further-iterations stage we decided that even though it took more time in a lesson, it was worth going through *two failed attempts* with the students using mathematical induction *prior* to revealing the simple solution that utilizes the AM-GM inequality.

Looking through the lens of Huron's [16] theory, we expected the instruction manipulation of creating two consecutive surprises to intensify the negative responses to them. We suspected that after one surprising failed attempt during the solution process, students may be even more inclined to believe the second attempt at induction must succeed. When also this attempt fails to live up to its expectation, the second surprise would consequently be bigger. Since, according to Huron, initial responses to a surprising event are negative in nature, the increased negative surprise could serve as a more extreme contrast to the upcoming slower re-evaluative process, ultimately leading to an intensified aesthetic experience.

Looking through the lens of Goldin's [14] discussion on meta-affect, we assumed that going together in class through two failed-attempts at the solution may reveal to students the real ways of the working mathematician, often failing again and again when trying to reach a proof before reaching success [15, 30]. Consequently, we hoped students would be more inclined to positively address their negative emotions (such as fear of failure at solving the problem) as a natural part of the solution process. By doing so, students' overall experience and sense of well-being in class might be improved, leaving them more "open" to see beauty in the solution, even if they did not reach the solution on their own.

5.2.2. A general description of the findings

During the first semester of this particular stage of iterations, the only change we introduced was to add a second failed attempt at solving the problem, yet it created a big impact. Not only was there a general high-level of enthusiasm and excitement in class revealed through spontaneous responses of "how beautiful!" and many students smiling, but additionally there was a dramatic increase in the amount of students evaluating the problem as "beautiful". Where earlier there were usually one to two students at most raising their hands to the explicit question "who finds this problem beautiful?", in this specific lesson about a third of the class (approximately 20 out of 60 students) answered positively. Subsequent "same version" teaching of this problem in two additional consecutive semesters led to similar results.

An additionally noteworthy phenomenon was that a substantially bigger amount of students raised their hands to the question "who thought to herself or himself: how (the hell) was I supposed to solve this problem on my own?"

During the three semesters of this stage approximately a third to half of the students confirmed this feeling. Putting this into context, during the initial-iterations stage there never existed an intersection between groups of students responding to these two questions, i.e., finding the problem beautiful yet insecure about their ability to solve it on their own. However, during the second stage of the lesson-planning iterations not only did an intersection arise, but it was even substantial. This finding can be explained based on the aforementioned rationale for the lesson-design, which addresses students' meta-affect during challenging mathematics learning.

5.2.3. Findings from stimulated-recall interviews

During the stimulated-recall interviews, the students used common characteristics associated with mathematical aesthetics also used by professional mathematicians (as discussed in the Theoretical Framework section). This was in line with the findings from the initial iterations. Students who perceived the solution as beautiful referred to the fact that during the lesson there had been a significant effort to solve the problem using mathematical inductions, and ultimately it was solved in seconds using a surprisingly easy and clever way. When asked why they thought the solution was beautiful, words that repeated in the students' reports were: non-trivial, genius, short, simple, clear, unbelievable, amazing, surprising, and unexpected. One student in particular described this in terms of a triumph over the problem:

> *You feel at some moment that finally we have managed to overcome this "monster" of unsuccessful inductions, where we tried for two hours to solve [the problem] using mathematical induction, and at the end it was solved in a minute and a half in a much easier way.*

Unsurprisingly, a motive that appeared during the interviews was the *surprise* students experienced during the lesson.[1] Students gave clarifications as to why they were surprised, as well as descriptions regarding the nature

[1] It should be emphasized in this context that no intentional question on surprise was planned by the researchers, but only clarifying questions were asked on what students had already shared (elaborated in Subsection 4.4).

of their surprise in emotional terms. As for the reason underlying their surprise, many students reported that when reaching a dead-end while solving the problem in the tutorial, they expected the teaching assistant to suggest a small fix in the previous line of the solution, which would solve the issue and allow the solution-process to immediately continue forward. A repetitive description heard in the interviews was that they did not expect entire boards to be crossed-out (with what was believed so far to be a solution for the problem), let alone having this occur in the lesson twice in a row. As exemplified by one of the students:

> *I thought you'd simply make some kind of change and we would move on. That there will be some small nuance you will use and we'll be able to continue. But here it was extremely dramatic. You just erased it all!*

Regarding emotions triggered by the surprising event(s), expressions of students referring to this included: "I was in real shock", "It was horrible!", "We never experienced anything like this [in class]", "I felt frustrated", "I got annoyed", and "It was infuriating". Students referred to the surprise in emotional terms, in most cases of a negative nature, admitting it made them extremely annoyed or even angry, especially since this occurred *twice* over the course of the lesson. One student decided to demonstrate to us how infuriated she was with the situation by showing us that the cross-outs she made in her notebook over the second failed attempt were so strong that they left carving marks on the page. It should be reminded however that the same students who felt displeased, shocked, or angry during the process of the solution in class, used words like "beautiful" and "amazing" to describe the solution at the end of the lesson. Theoretically this can be explained by Huron's [16] theory, where the creation of a more contrasting negative base serves to amplify the upcoming pleasure as a result of the clever and easy solution. This explanation is also supported in the data, as several students reported feeling more driven to listen after the second surprise had taken place, and being more curious and intrigued to discover how the problem would actually be solved.

With regard to being confronted with unsuccessful attempts at solving the problem in class, many students testified it made them think about their struggle of repeated attempts at trying to solve homework assignments outside of the classroom environment. Seeing failed efforts occur in class as well gave them "a look into how things really work" as one student put it, and as another student stated: "I learned not to give up". For one student this was what made this lesson plan unique and memorable:

> *It was "the lesson with the cross-outs" [...] The non-standard lesson. The lesson where it wasn't just someone solving problems in front of me on the board, but the lesson where I was explained what to do when something doesn't succeed. Like, that it* ***is***[2] *possible to go on, maybe at the end it will work!*

For another student, being able to see her teacher as a struggling problem-solver helped her in directing her meta-affect to a positive outcome:

> *I felt a rise in self-confidence that this can happen, maybe it can also happen to you* [the teaching assistant], *or any other student* [...] *who I think is smart. It can happen to anyone . . . that we try something and it doesn't work.*

As discussed earlier, implementing Huron's [16] theory for the design of lessons aimed at evoking aesthetic responses, calls for allocating time in the lesson plan for the slow appraisal process to take place. This implies that the solving of a beautiful problem in class should take more time than usual, or at least until reaching the ultimate highlight-moment of the resolution. Therefore, we also looked in the data for statements that referred to affective or cognitive events students experienced as a result of the extra time given during that lesson. A phenomenon that repeated in descriptions of strong students was that they used the extra time after a surprising failed attempt to try to find an alternative solution on their own while the teaching assistant was answering questions of other students. Some students stated that time that was used for discussion after each failed attempt gave them the opportunity to relax and reflect on the situation. The following two excerpts are taken from an interview with the same student addressing discussions in class after each surprising event:

[2]Emphasized in the student's intonation.

> *This* [time for discussion] *made it a bit easier for taking this turn. It gave some time to* **digest**[3] *it. And to understand that this is actually not the way to go, and that we're going to do something else now.* [After the first failed attempt]
>
> *I think that the conversation you held again after [the failed attempt] softened a bit the surprise.* [After the second failed attempt]

Lastly, during the interviews several students specifically referred to the lesson as having a dramatic nature, illustrated by the following excerpts:

> *When the [mathematical] material leads to enjoyment, it attracts you more to it and triggers more fun. [...] In this lesson there was passion. There was passion and there was desire. When you tell yourself: wow, this can't be!*
>
> *For me, personally, it made it more interesting to see what's going to happen. Because there are two shows here. One show is the problem and show no. 2 is that you're fooling us every time.*

6. Discussion

In this paper we presented an iterative process of lesson design aimed to create an aesthetic experience for students in a "traditional-instruction" calculus tutorial. The general lesson-design principles came from varied theories on cognition, affect, aesthetics, and the combination thereof. While the initial iterations showed it was feasible to elicit aesthetic responses from a small amount of students through the use of these principles, in further iterations the particular incorporation of *two* surprising events prior to the presentation of the aesthetic highlight of the lesson repeatedly managed to serve as a teaching method leading to an aesthetic experience of many students. This reoccurring surprise of reaching a dead-end in class may be especially strong in context of the undergraduate classroom, where results are usually presented as a "perfect theory" with no room for a trial-and-error process of problem solving [1].

[3]Emphasized in the student's intonation.

In the continuation of this section we would like to formulate some practical conclusions stemming from the presented study. The first conclusion regards the planning of time in class for the solution-process of potentially beautiful problems. As Andy Warhol once said, “The idea of waiting for something makes it more exciting”. Accordingly, for students to appreciate beauty of a solution, they first need to be in a state of waiting for it. Referring to the musical domain, the catharsis of resolution does not come without time designated for preparatory anticipation [18]. This means that mathematical problems that can be regarded as beautiful should be given more time when planning a lesson that wishes to include their aesthetic potential. This in fact is one outcome with regard to our reported problem that naturally occurred when we added an extra failed attempt in class. As found in the data, students can use the extended time given for the solution-process to figure out why certain ideas do not work, try out alternative routes, as well as be able to process their emotions. Additionally, Sinclair [34] identified one particular mechanism underlying the aesthetic response of professional mathematicians as establishing a personal and intimate relationship with what they are working on. In a traditional instructional style of university tutorials, where the problem is solved by a teaching assistant on the board, we suggest that the allocation of additional time for the variety of cognitively- and affectively loaded episodes students may go through, can allow for the students to be more connected and involved with the process, and serve as a base for them to create such a relationship with the problem.

The second conclusion regards the use of creating drama in a lesson as a supporting tool for evoking aesthetic appreciations. Looking at the data, we suggest that the difference between initial and further iterations lay in the extent of drama in the lesson and the intensity of emotions students experienced. With the intent of increasing the overall aesthetic experience of students, we focused in later iterations on readjusting the implementation of the first guiding principle, which ultimately caused the “reaching a dead-end” procedure to be perceived by students in a more extreme and provocative way. We would like to suggest that this could potentially also be done for the second guiding principle. For example, instead of a teacher calmly telling students in class that “it is ok to feel frustrated right now”, the teacher could provocatively ask: “Who feels extremely angry right now as a result of this annoying and unsolvable problem?”

The third conclusion regards the creation of a classroom environment where students are freed from a dichotomous problem-solving way of thinking in terms of "I succeeded to solve it" or "I failed to solve it". In a paper focused on students' points of view on mathematical beauty, Brinkmann [3] argued that "it is desirable that students' aesthetic feelings are not only restricted to those problems they feel they can solve by themselves" (page 378). Subsequently Brinkmann concluded that "we should create phases in classrooms, which have an atmosphere that is not predominated by the demand of success" (page 378). As it has hopefully been evident, the presented lessons enabled students to see the teacher in a more human light, as a person who might also struggle with solving problems he or she encounters. It is in place to recall that having a university tutorial that involves deliberate and planned failed attempts of solving a problem is a rather rare event. However, such an event reveals a working method that students can *relate to and identify with* from their experience with solving homework assignments by themselves, repeatedly trying different routes, and getting stuck more than once. Consequently, showing students that also an expert would try familiar methods first, and look for other options *only* when such methods fail, could alleviate students' possible sense of failure and frustration when dealing with homework problems on their own (cf. [14], for discussion on meta-affect).

7. Concluding words

The considerations presented in this paper imply that successful episodes in which students see beauty in a mathematical problem or solution require more instructional time in a lesson. Especially in university courses where time is in short supply, it is clear that aesthetic responses from students cannot be expected on a regular basis. However, in line with Liljedahl's [22] assertion on the vast impact of a single "Aha!" experience on undergraduate students, we similarly claim that the positive emotions that emerge as a result of a single "Wow, how beautiful!" moment are far greater and more influential than the emotions that emerge during routine problem-solving actions. Consequently, planning lessons aimed at creating an aesthetic experience even once or twice per semester can have a significant influence on students' beliefs, attitudes, motivation, and enjoyment, as well as on the memorability of the material and problem-solving methods taught in those lessons.

However, these long-term benefits on students' learning as a result of experiencing mathematical aesthetics should be subject to potential follow-up research. We believe that the presented case of one lesson is a step in the direction of reaching a better understanding of how to help students develop an aesthetic appreciation of mathematics in a "traditional" instructional setting. As undergraduate courses are for many students the final encounter with mathematics studies, and as such a "last chance" to influence their thoughts and emotions on the subject, such steps are additionally important for the enhancement of students' motivation, as well as the improvement of their general attitude towards the (inevitably) exam-oriented undergraduate-level mathematics.

References

[1] Alsina, C., "Why the Professor Must be a Stimulating Teacher: Towards a New Paradigm of Teaching Mathematics at University Level", in *The Teaching and Learning of Mathematics at University Level*, edited by Holton, D., Artigue, M., Kirchgräber, U., Hillel, J., Niss, M., and Schoenfeld, A. H., Springer, New York, 2002, pages 3–12.

[2] Bandura, A., "Self-efficacy: Toward a Unifying Theory of Behavioral Change", *Psychological Review*, Volume **84** Issue 2 (1977), pages 191–215.

[3] Brinkmann, A., "Mathematical Beauty and its Characteristics - A Study on the Students' Points of View", *The Montana Mathematics Enthusiast*, Volume **6** Issue 3 (2009), pages 365–380.

[4] Brown, S. I., "Mathematics and Humanistic Themes: Sum Considerations", *Educational Theory*, Volume **23** Issue 3 (1973), pages 191–214.

[5] Calderhead, J., "Stimulated Recall: A Method for Research on Teaching", *British Journal of Educational Psychology*, Volume **51** (1981), pages 211–217.

[6] Cellucci, C., "Mathematical Beauty, Understanding, and Discovery", *Foundations of Science*, Volume **20** Issue 4 (November 2015), pages 339–355.

[7] Cobb, P., Confrey, J., diSessa, A., Lehrer, R., and Schauble, L., "Design Experiments in Educational Research", *Educational Researcher*, Volume **32** Issue 1 (2003), pages 9–13.

[8] Davis, P. J., *Mathematical Encounters of the Second Kind*, Birkhäuser, Boston, 1997.

[9] Davis, P. J. and Hersh, R., *The Mathematical Experience*, Birkhäuser, Boston, 1981.

[10] de Freitas, E. and Sinclair, N., *Mathematics and the Body: Material Entanglements in the Classroom*,Cambridge University Press, New York, 2014.

[11] Dreyfus, T. and Eisenberg, T., "On the Aesthetics of Mathematical Thought", *For the learning of mathematics*, Volume **6** Issue 1 (1986), pages 2–10.

[12] Frenzel, A. C., Goetz, T., Lüdtke, O., Pekrun, R., and Sutton, R. E., "Emotional transmission in the classroom: Exploring the relationship between teacher and student enjoyment", *Journal of Educational Psychology*, Volume **101** Issue 3 (2009), pages 705–716.

[13] Goldin, G. A., Affective Pathways and Representation in Mathematical Problem Solving, *Mathematical Thinking and Learning*, Volume **2** Issue 3 (2000), pages 209–219.

[14] Goldin, G. A., "Perspectives on Emotion in Mathematical Engagement, Learning, and Problem Solving", in *International Handbook of Emotions in Education*, edited by Pekrun, R. and Linnenbrink-Garcia, L., Routledge, New York, 2014, pages 391–414.

[15] Hadamard, J., *The psychology of invention in the mathematical field*, Princeton University Press, Princeton, 1945.

[16] Huron, D., *Sweet Anticipation: Music and the Psychology of Expectation*, The MIT Press, Cambridge, Massachusetts, 2006.

[17] Jacobsen, T., "Bridging the Arts and Sciences: A Framework for the Psychology of Aesthetics", *Leonardo*, Volume **39** Issue 2 (2006), pages 155–162.

[18] Johnson, M., *The Meaning of the Body: Aesthetics of Human Understanding*, The University of Chicago Press, Chicago, 2007.

[19] Koichu, B., Katz, E., and Berman, A., "What is a beautiful problem? An undergraduate students' perspective", in *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, edited by Woo, J. H., Lew, H. C., Park, K. S., and Seo, D. Y., PME, Seoul, pages 113–120.

[20] Koichu, B. and Kontorovich, I., "Dissecting success stories on mathematical problem posing: a case of the Billiard Task", *Educational Studies in Mathematics*, Volume **83** Issue 1 (2013), pages 71–86.

[21] LeDoux, J., *The Emotional Brain: The Mysterious Underpinnings of Emotional Life*, Simon & Schuster, NY, 1996.

[22] Liljedahl, P. G., "Mathematical discovery and affect: the effect of AHA! experiences on undergraduate mathematics students", *International Journal of Mathematical Education in Science and Technology*, Volume **36** Issues 2-3 (2005), pages 219–234.

[23] Liljedahl, P. G., Chernoff, E., and Zazkis, R., "Interweaving mathematics and pedagogy in task design: a tale of one task", *Journal of Mathematics Teacher Education*, Volume **10** Issues 4-6 (2007), pages 239–249.

[24] McAllister, J. W., "Mathematical Beauty and the Evolution of the Standards of Mathematical Proof", in *The Visual Mind II*, edited by Emmer, M., The MIT Press, Cambridge, 2005, pages 15–34.

[25] McLeod, D. B., "Beliefs, Attitudes, and Emotions: New Views of Affect in Mathematics Education", in *Affect and Mathematical Problem Solving: A New Perspective*, edited by McLeod, D. B. and Adams, V. M., Springer-Verlag, New York, 1989, pages 245–258.

[26] Movshovits-Hadar, N., "School Mathematics Theorems: An Endless Source of Surprise", *For the learning of mathematics*, Volume **8** Issue 3 (1988), pages 34–40.

[27] Nunokawa, K., "Surprises in Mathematics Lessons", *For the learning of mathematics*, Volume **21** Issue 3 (2001), pages 43–50.

[28] Phelps, E. A., "Human emotion and memory: interactions of the amygdala and hippocampal complex", *Current Opinion in Neurobiology*, Volume **14** (2004), pages 198–202.

[29] Poincaré, H., "Mathematical creation", in *The World of Mathematics*, edited by Newman, J. R., Simon and Schuster, New York, 1956, pages 2041–2050.

[30] Polya, G., *Mathematical discovery: On understanding, learning, and teaching problem solving*, Wiley, New York, combined edition, 1981.

[31] Rota, G.-C., "The phenomenology of mathematical beauty", *Synthese*, Volume **111** Issue 2 (1997), pages 171–182.

[32] Schoenfeld, A. H., *Mathematical Problem Solving*, Academic Press, Orlando, 1985.

[33] Silver, E. A. and Metzger, W., "Aesthetic influences on expert mathematical problem solving", in *Affect and Mathematical Problem Solving: A New Perspective*, edited by McLeod, D. B. and Adams, V. M., Springer-Verlag, New York, 1989, pages 59–74.

[34] Sinclair, N., "The Roles of the Aesthetic in Mathematical Inquiry", *Mathematical Thinking and Learning*, Volume **6** Issue 3 (2004), pages 261–284.

[35] Sinclair, N., *Mathematics and Beauty: Aesthetics approaches to teaching children*,Teachers College Press, New York, 2006.

[36] Sinclair, N., "Aesthetics as a liberating force in mathematics education?" *ZDM Mathematics Education*, Volume **41** (2009), pages 45–60.

[37] Tymoczko, T., "Value judgments in mathematics: Can we treat mathematics as an art?", in *Essays in humanistic mathematics*, edited by White, A. M., Mathematical Association of America, Washington, DC, 1993, pages 67–77.

[38] Wells, D., "Are these the most beautiful?" *The Mathematical Intelligencer*, Volume **12** Issue 3 (1990), pages 37–41.

[39] Whitcombe, A., Mathematics Creativity, Imagination, Beauty, *Mathematics in School*, Volume **17** Issue 2 (1988), pages 13–15.