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Cover Page Footnote
This paper issues from a talk given at the Workshop on Beauty and Explanation in Mathematics, March 10-12, 2014, Umeå University, Sweden. I am most grateful to the organiser, Manya Sundström, and to the other participants, for their comments in discussion and for their talks, which modified my views. I am also very grateful to Josephine Salverda for insightful comments and sharp-eyed reading of an earlier version, which helped me improve the substance and eliminate errors.

Mathematical Proofs: The Beautiful and The Explanatory

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Abstract

Mathematicians sometimes judge a mathematical proof to be beautiful and in doing so seem to be making a judgement of the same kind as aesthetic judgements of works of visual art, music or literature. Mathematical proofs are also appraised for explanatoriness: some proofs merely establish their conclusions as true, while others also show why their conclusions are true. This paper will focus on the *prima facie* plausible assumption that, for mathematical proofs, beauty and explanatoriness tend to go together.

To make headway we need to have some grip on what it is for a proof to be beautiful, and for that we need some account of judgements of beauty in general. That is the concern of the first section. The second section faces the problem that it is far from obvious how abstract entities, such as mathematical proofs, can be beautiful, strictly and literally speaking. Reasons are given for the view that they can be. The third section introduces the distinction between proofs which explain their conclusions and proofs which do not. Finally, the question whether, for mathematical proofs, the beautiful and the explanatory tend to coincide is addressed. It is argued that we have reason to doubt that explanatory proofs tend to be beautiful, and insufficient reason to believe or disbelieve that beautiful proofs tend to be explanatory.

Keywords: *proof; explanation; beauty.*

Mathematicians sometimes judge a mathematical proof to be beautiful and in doing so seem to be making a judgement of the same kind as aesthetic judgements of works of visual art or literature. Mathematical proofs are also appraised for explanatoriness: some proofs merely establish their conclusions as true, while others also explain their conclusions. In this paper I first defend the view that mathematical proofs are open to aesthetic evaluation and some of them may be correctly judged beautiful. Then I assess the claim, brought to my attention by Manya Raman-Sundström[1], that, with regard to proofs, beauty and explanatoriness tend to go together.

The paper is organised as follows. As a necessary framework for defending the aptness of judgements of beauty in mathematics, Section 1 concerns judgements of beauty in general. Section 2 concerns judgements of beauty about mathematical proofs. Section 3 is about explanatory proofs. Section 4 assesses the claim that explanatory proofs are almost always beautiful and vice versa. Then there is a short concluding section.

1. Judgements of Beauty

A judgement of beauty is often a response to pleasure we experience in perceiving something, for instance a painting or a performance of some music. But these judgements are not statements about ourselves, about our hedonic states or our preferences. We allow that we can be corrected, either by being alerted to aspects of the work we may have missed (or misappreciated) or by instructive comparison with other works of the same genre. We allow that other people may be better judges than we are, having a more developed sensibility of the relevant kind, and that with further attentive experience we may become better judges than we are now. But we do not allow that others are better judges than we are about our own likes and dislikes. So a strictly subjectivist view of ascriptions of beauty is wrong.

Also wrong is a certain kind of objectivist view. For to judge something beautiful is not to attribute to it a property which is mind-independent.[2] The judgement cannot be correct unless the thing has a propensity to give significant pleasure to people with a developed sensibility of the relevant kind from mentally apprehending it (e.g. in visual experience if it is a painting,

[1]Manya Raman-Sundström, "On Beauty and Explanation in Mathematics", talk delivered at the conference *Mathematical Cultures 2*, De Morgan House, London, September 17–19, 2013.

[2]There are other senses of "objective", see Chapter 2 of [3]. In some of these other senses a judgement of beauty might be objective.

in auditory experience if it is a performance of some music). I take this connection of beauty to pleasure in mental apprehension to be essential to beauty.

Though a thing cannot be beautiful unless it has a pleasure-giving propensity, not any kind of pleasure will do. For example, the relevant kind of pleasure does not result from satisfying or exciting an appetite, nor from anticipating or imagining the satisfaction of an appetite, nor from one's appreciation of the object's instrumental goodness in serving some goal, fulfilling some function or promoting some (non- aesthetic) value.[3] In short, for the correctness of a judgement of beauty only disinterested pleasure is relevant.[4]

Also excluded are one-shot pleasures. However funny on first hearing, a particular joke loses its power to please the more often one hears it; for this reason jokes are not candidates for beauty. This is not so for fine musical compositions; provided that the gaps between occasions when one listens to some particular music are not too small, pleasure returns and often increases. Something is beautiful, then, only if it has a propensity to give disinterested repeatable pleasure.

It is well known that aesthetic judgements differ, and so do the degrees of pleasure experienced by different people in mentally apprehending the same thing. This divergence can be largely explained by a combination of three factors. First, people are affected differently by different *genres* of things. If works of a particular genre, say classical Chinese vocal solos, leave you cold because of their generic character, then of course you will get no pleasure from listening to any of them and you will not be in a good position to compare one with another for aesthetic quality. Nonetheless some of them may be beautiful. For an item of a particular genre to be beautiful it is not required that it has a propensity to please anyone who can experience it. But it must have a propensity to please some of those who can and often have got pleasure from items of that genre. Let us call those people *fan*s of the genre.[5]

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[3] I have in mind political and moral values, for example. I leave open the question whether the exclusion holds for aesthetic values.

[4] "Disinterested pleasure" is a Kantian expression, but I doubt that my use of it exactly coincides with Kant's use of the German original.

[5] The claim entails that something beautiful may fail to please a non-fan, not that it must do so.

So one factor accounting for differences of aesthetic judgements is simply that people differ with respect to the genres of things that may affect them. But even among fans of a particular genre there are differences of aesthetic judgement about things of that genre. Two other factors may have role here: differences in cognitive sensitivity, such as in powers of auditory discrimination, and differences in past attentive experience and education of the relevant kinds. The last factor, past experience and education, is likely to be most the important factor explaining differences over time in one's own aesthetic judgements about instances of a genre. The cumulative effects of attentive experience to many instances and of having one's attention drawn to differences between instances alters the pleasure one gets from them by making one more discriminating. Let us say that, relative to a genre, the fans of that genre with high sensitivity and substantial attentive experience to and education about things of the genre have a *well developed sensibility*. Let us call these fans the *connoisseurs* of the genre.

Substantial convergence of aesthetic judgements within a genre may be limited to its connoisseurs, i.e. to the fans of the genre with a well developed sensibility of the relevant kind. Where there is no convergence of aesthetic judgements — perhaps the ludicrously named field of *conceptual art* is an example — the belief that there are correct and incorrect aesthetic judgements within that field falls under suspicion. But for many genres there is substantial convergence of aesthetic judgement among connoisseurs of the given genre, and that makes possible an inter-subjectively validated kind of correctness.

For these reasons, I propose the following:

> A judgement of beauty about something of a particular genre is correct only if that thing has a propensity to give significant disinterested repeatable pleasure to connoisseurs of the genre from mentally apprehending it.[6]

That is not a sufficient condition. One reason why not is that there may be ways of mentally apprehending something which are not the primary intended mode of mentally apprehending it. For example, it is possible for

[6]There is absolutely no implication here that only connoisseurs of a genre are capable of appreciating the beauty of an item of that genre.

suitably trained people to get pleasure from reading the score of some music, or from reading the text of a play. But if they would not get pleasure from experiencing *performances* of the music or play (or from imagining such experiences in aural imagination), any pleasure they get from reading score or text would not be sufficient grounds for their judging the work to be beautiful. So to get a condition that is sufficient as well as necessary one would have to add at least that the pleasure in question is to result from apprehending the work in the primary intended way.

This account, with its reference to genres, works and connoisseurs, does not deal with aesthetic judgements of natural scenes, hence lacks full generality. Even confined to judgements of human works, the account is liable to need amending. But at this point adding further qualifications to rule out potential counterexamples has diminishing returns. I claim only that the account is roughly right for the restricted domain of human works, and good enough to give us bearings for consideration of aesthetic judgements about mathematical proofs.

2. Ascriptions of beauty to mathematical proofs

There are pleasing visual representations of mathematical entities, but these are off-topic, as our concern here is with mathematical proofs. There are some diagrammatic proofs, such as those in Oliver Byrne's version of the first six books of Euclid's *Elements* [4] and the derivations of Nathaniel Miller's [9] formal system of part of Euclidean geometry as illustrated by his proof of Euclid's first proposition in Figure 1.[7] The latter is a genuinely visual proof, in the sense that the diagrams cannot be dispensed with: they are not merely illustrations and they cannot be replaced without changing the proof.

However, proofs which are described as beautiful by mathematicians are almost never genuinely visual proofs; some are presented without any diagram, and when diagrams do appear, they are usually illustrative aids to understanding the proof as given in the verbal-symbolic text. So the aesthetic

[7] This is only a proof of Euclid 1.1 given (a) the rules for obtaining one diagram from earlier diagrams in a derivation and (b) the intended semantic interpretation of the diagrammatic formalism.

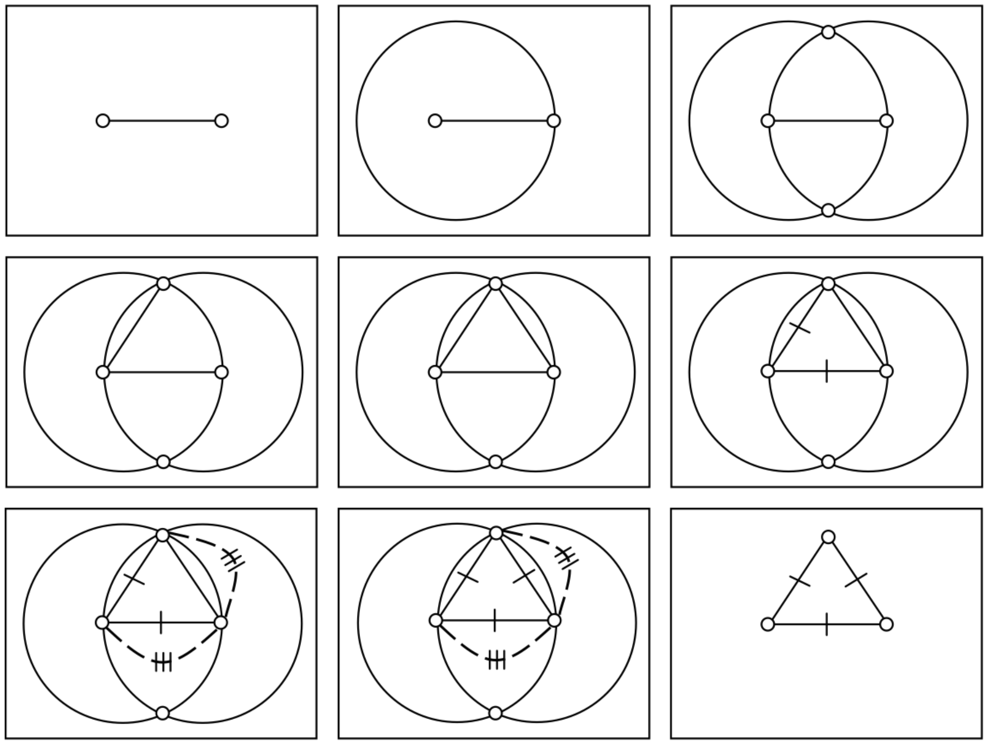

Figure 1: A diagrammatic proof of Euclid's first proposition.

quality of attendant diagrams is not going to account for mathematicians' positive aesthetic judgements about proofs.[8] But this leaves us with what I will call the *abstractness problem.* How can anything as abstract and austere as a mathematical proof be a genuine candidate for beauty? How can two proofs of a theorem be compared for aesthetic quality?

One possible answer is negative. One might argue that appraising a proof as "beautiful" is just loose talk. This would be the case if one said of one proof that it is "beautiful" (or more "beautiful" than another) for covert reasons that are not genuinely aesthetic. Many uses of the word "beautiful" are loose. The game theorist John Nash was described as having "a beautiful mind"; the actor Philip Seymour Hoffman was described as "a beautiful person in a

[8]Here I distinguish between a presentation of a proof and the proof itself.

hideous world." In neither case was the judgement genuinely aesthetic: Nash was praised for his mathematical intelligence and Hoffman for his caring nature. The word "beautiful" abounds as a general feel-good word, as in the song "It's a beautiful world", expressing a sentiment indistinguishable from that expressed by the earlier song "It's a wonderful world", or the film title "Life is beautiful" (*La vita è bella*). The problem of explaining how mathematical proofs could be beautiful disappears if we simply accept that applications of the word "beautiful" to proofs are, in every case, loose uses.

One kind of consideration against this negative view results from a closer look at the source of the problem, namely, the abstractness of proofs. In paradigm cases (music, sculpture ...) the beauty of an object arises from the combination of its sensory qualities; we appreciate its aesthetic quality from our apprehension of it in attentive sense experience. But when it comes to proofs, sense experience is not germane: the visual appearance of a presentation of a proof, however pretty, is usually irrelevant to qualities of the proof itself. The problem then is to say how it is possible for the proof to have an aesthetic quality. This problem seems acute because we focus on things whose aesthetic value is governed directly by the quality of our sense experience in perceiving them.

But if we move away from those paradigm cases, it becomes clear that sense experience does not always have this all-governing role. The aesthetic value of a poem partly depends on the sound and rhythm of an aural recitation of it, but only to the extent that it combines effectively with the content of the poem.[9] Even less does the quality of a prose work, such as a short story, an account of a historical event, or an essay on morals and manners, depend on the quality of our sense experience in hearing or reading it. In these cases, the mental apprehension from which we may get aesthetic pleasure consists in grasping the semantic content of the work, that is, in comprehending it. When reading a prose work, it is the quality of the non-sensory process of comprehending it that counts. That is why one can aesthetically appreciate a novel, for example, from a good translation of it.[10] Once we notice that in some cases aesthetic judgement of a work depends primarily on comprehen-

[9]This may need refining to deal with poems with non-standard content or without content, such as Lewis Caroll's *Jabberwocky*.

[10]There is much more to be said about these matters, but this is not the place for elaboration.

sion rather than perception or sensory imagination, the supposed problem of the abstractness of mathematical proofs fades away. A mathematical proof is, like an essay in morals and manners, an argument, though one without digressions. We can get pleasure in the process of grasping the argument, being intellectually sensitive to the way it unfolds, and this can be a proper basis for aesthetic evaluation. So we are not forced to accept the negative view that every application of the word "beautiful" to mathematical proofs is loose talk.

Another kind of consideration against the negative view may arise from one's own mathematical experience. We may find ourselves having a feeling about a proof that properly warrants calling the proof beautiful. To substantiate this claim, I will present a short proof that has certainly struck mathematicians that way. It is a proof of a theorem that answers a combinatorial problem in geometry. Put concretely, the problem is this. Let the n walls of a single-floored gallery make a polygon. What is the smallest number of stationary guards needed to ensure that every point of the gallery wall can be seen by a guard? If the polygon is convex (all interior angles less than 180°), one guard will suffice, as guards may rotate. But if the polygon is not convex, as in Figure 2, one guard may not be enough.

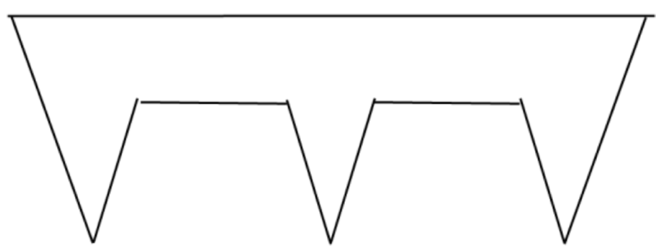

Figure 2: A non-convex polygon for the gallery problem.

The answer, first proved by Václav Chvátal [5], is this: for a gallery with n walls, $\lfloor n/3 \rfloor$ guards suffice, where $\lfloor n/3 \rfloor$ is the greatest integer less than or equal to $n/3$.[11]

Here is the admired proof of Chvátal's theorem, due to Steve Fisk [6].

[11] If this does not sound to you sufficiently like a mathematical theorem, it can be restated as follows: Let S be a subset of the Euclidean plane. For a subset B of S let us say that B *supervises* S if and only if for each $x \in S$ there is a $y \in B$ such that the

A short argument by mathematical induction shows that every polygon can be triangulated, i.e. non- crossing edges between non-adjacent vertices ("diagonals") can be added so that the polygon is entirely composed of non-overlapping triangles. So take any n-sided polygon with a fixed triangulation. Think of it as a graph, a set of vertices and connected edges, as in Figure 3.

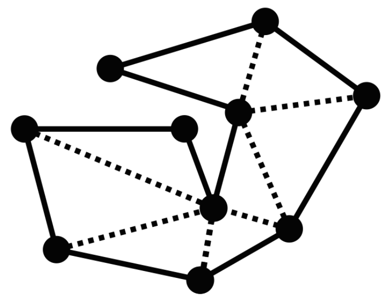

Figure 3: A triangulated non-convex polygon thought of as a graph.

The first part of the proof will show that the graph is 3-colourable, i.e. every vertex can be coloured with one of just three colours (red, white and blue, say) so that no edge connects vertices of the same colour.

The argument proceeds by induction on $n \geq 3$, the number of vertices.

For $n = 3$ the statement is trivial. Assume it holds for all k, where $3 \leq k < n$. Let triangulated polygon G have n vertices. Let u and v be any two vertices connected by diagonal edge uv. The diagonal uv splits G into two smaller graphs, both containing uv. Give u and v different colours, say red and white, as in Figure 4.

By the inductive assumption, we may colour each of the smaller graphs with the three colours so that no edge joins vertices of the same colour, keeping fixed the colours of u and v. Pasting together the two smaller graphs as coloured gives us a 3-colouring of the whole graph.

The second and final part of the proof shows that $\lfloor n/3 \rfloor$ or fewer guards can be placed on vertices so that every triangle is in the view of a guard. Let b, r and w be the number of vertices coloured blue, red and white respectively. At least one of these numbers will be no greater than the other two. Suppose

segment xy lies within S. Then the smallest number $f(n)$ such that every set bounded by a simple n-gon is supervised by a set of $f(n)$ points is at most $\lfloor n/3 \rfloor$.

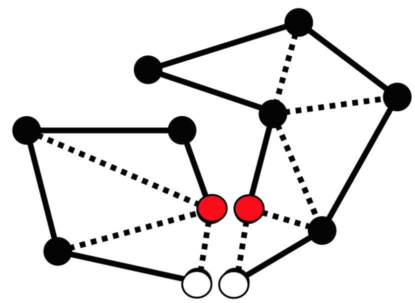

Figure 4: Coloring the vertices of the diagonal uv.

for concreteness that b is: $b \leq r$ and $b \leq w$. Then $2b \leq r + w$ and so $3b \leq b + r + w = n$. In other words we have $b \leq n/3$ and so $b \leq \lfloor n/3 \rfloor$. We can place a guard on each blue vertex and we are done.

This proof is described as "truly beautiful" by mathematicians Martin Aigner and Günter Ziegler in [1, page 232]. I agree that it is beautiful. But even those who do not, were they to compare it with Chvátal's original proof, would almost certainly regard Fisk's proof as more beautiful than Chvátal's (though Chvátal's is not bad at all). This weighs against the negative view that any use of the word "beautiful" in connection with mathematical proofs is just loose talk.

But if it is not loose talk, we should be able to specify some properties of the proof which we think contribute to its aesthetic merit.[12] If we cannot, the suspicion will lurk that the claim of beauty is just an expression of pleasure, hence loose talk; but if we can specify properties that we regard as contributing aesthetically, the claim is less likely to be a mere expression of pleasure, because by specifying properties we open the claim of beauty to debate and thereby reveal it to be a judgement that may be corrected or confirmed by relevant others. I think we can specify contributing properties, and I will try to do this for Fisk's proof.

One such property is clarity: the central idea(s) of the proof and its structure are clear. (Argue by induction on the number of vertices that any triangulated polygon can be three-coloured; argue arithmetically that a

[12] But we cannot be sure exactly what (weighted) combination of properties is causally responsible for the aesthetic pleasure we get in this or any other case.

colour that occurs no more often than the others occurs at most $\lfloor n/3 \rfloor$ times.) Clarity is diminished by the use of unintuitive tricks or technical handles, to use an expression of Manya Raman-Sundström. It is also diminished by case-branching, that is, dividing the situation into a number of separate cases and providing distinct arguments for each case. Case-branching is a feature of Chvátal's proof but not of Fisk's, which may partly explain the relative appeal of the latter. The importance of clarity can be appreciated by contrast with obscurity. The mathematician Hermann Weyl makes the point as follows [10]:

> We are not very pleased when we are forced to accept a mathematical truth by virtue of a complicated chain of formal conclusions and computations, which we traverse blindly, link by link, feeling our way by touch. We want first an overview of the aim and the road; we want to understand the idea of the proof, the deeper context.

Another aesthetically positive property, for which I have no good descriptor, is a combination of brevity and simplicity of methods. This needs explaining. A proof of a substantial theorem which uses only elementary methods is liable to be undigestibly long and complicated.[13] For a shorter proof, we use more advanced mathematics, but that may make the proof inaccessible without a long run up of intensive study. An example of the latter might be Zagier's one-sentence proof of Fermat's Two Squares theorem [11].[14] Of course, if you have done the intensive study and internalised the advanced mathematics, there will be no problem of inaccessibility for you. But even so, you may feel that the use of advanced techniques is over-heavy for the theorem proved, and that is an aesthetic matter. So the property I have in

[13] A proof using only elementary methods may also take longer to find. The proofs of the prime number theorem using only elementary methods by Selberg and Erdös were published in 1949, more than fifty years after the proofs of Hadamard and de la Vallée Poussin using complex analysis.

[14] Here is what one mathematician said of this proof. "As a practicing number theorist who has devoted an inordinately large amount of time to polishing various proofs of the Two Squares Theorem ... I must say that I have always found the Heath-Brown/Zagier proof to be both contrived and confusing." Contribution to `Mathoverflow` discussion of Pete L. Clark, July 8, 2010: `http://mathoverflow.net/questions/31113/zagiers-one-sentence-proof-of-fermats-theorem`, accessed on January 24, 2016.

mind is a combination of brevity and simplicity of methods, or an optimal trade-off between them. But these virtues are relative to the depth of the theorem. We may feel that a proof of a given theorem is too long-winded for that theorem or that it uses a sledge-hammer to crack a nut. Perhaps it is better to say: of two proofs of a given theorem, the one with the better balance of length and accessibility is preferred to the other, *ceteris paribus*.

Fisk's proof of Chvátal's theorem has both of these qualities, clarity and the short-and-simple combination, and on both scores it has the edge over Chvátal's original proof. Fisk's proof has another property that contributes to its aesthetic appeal, namely, that it is imaginative, and not an obvious way to go, given the content of the theorem. This is no less important than the properties mentioned before, and it is a feature of many proofs described as beautiful. Here Fisk's proof does significantly better than Chvátal's. (The way I have presented the comparison, property by property, may be misleading: it would be a mistake to think that the aesthetic value of a proof is the sum of values contributed by each property separately; it is rather a function of the properties in combination.)

Let me summarise. Some mathematical proofs satisfy the necessary condition to be correctly judged beautiful, namely, they have a propensity to give significant disinterested repeatable pleasure to the connoisseurs of the genre from mentally apprehending them. We have grounds for this, because mathematicians do sometimes express such pleasure, by describing a proof as beautiful. But one may worry that this is just loose talk, because mathematical proofs are essentially abstract and non-sensory. Against this, prose works of literature can be strictly and correctly judged to be beautiful, that is, to have high aesthetic value, even when the pleasure we get from them does not arise from our sense experience in apprehending them. So the abstractness of proofs is no bar to aesthetic merit. Secondly, one's own experience may provide evidence of a distinctly aesthetic kind of pleasure in grasping a proof. Fisk's proof of Chvátal's theorem was presented in the hope of eliciting this kind of feeling. Finally, I specified properties of Fisk's proof which in combination may reasonably be taken to contribute to its aesthetic value, thus showing that the claim that it is beautiful is a judgement that is open to debate, and to correction or confirmation, rather than a mere expression of pleasure or approval.

3. Proofs which explain their conclusions

A proof explains its conclusion if and only if anyone who can properly understand the proof could come to know why its conclusion is true by following the proof. One may wonder whether in the realm of pure mathematics there is such a state as knowing why a proposition is true, as distinct from merely knowing that it is true. I will attempt to convince you by presenting two proofs of Pascal's Rule. Before that, to show that the idea of coming to see why a proposition is true is not just a philosopher's invention, here is Michael Atiyah, one of the foremost twentieth century mathematicians [2]:

> I remember one theorem that I proved and yet I really could not see why it was true. It worried me for years ... I kept worrying about it, and five or six years later I understood why it had to be true. Then I got an entirely different proof ... Using quite different techniques, it was quite clear why it had to be true.

Before presenting the proofs of Pascal's Rule (far simpler, no doubt, than Atiyah's theorem), it will help to make explicit a preliminary argument. Let n and k be integers, $0 < n$ and $0 \leq k \leq n$. To get a sequence of k things drawn from n things there will be n possible choices for the first component, $n-1$ for the second component, $n-2$ possible for the third component, ... , and $n-(k-1)$ for the kth component. So the total number of possible sequences of k things drawn from n will be

$$n \times (n-1) \times (n-2) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}.$$

As one can arrange a set of k things into exactly $k!$ sequences, it follows that the number of sets of k things drawn from n things will be the number of sequences of k things drawn from n things divided by $k!$. Using the bracket / combination notation for the number of sets of k things drawn from n things, we arrive at this:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The theorem to be proved is Pascal's Rule:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof 1.

We use the formula for the number of k-membered subsets of an n-membered set proved above and then apply simple rules such as $r(r-1)! = r!$:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!((n-1)-k)!} \\
&= \frac{(n-1)!k + (n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{(n-1)!(k+(n-k))}{k!(n-k)!} = \frac{(n-1)!n}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\end{aligned}$$

Proof 2.

Let S be any n-membered set and let α be one of the members. Then each k-membered subset of S either (i) contains α or (ii) does not contain α.

(i) Each k-membered subset of S containing α contains exactly $k-1$ members of S other than α, and there exactly $n-1$ of those. So the number of k-membered subsets of S containing α is equal to the number of $(k-1)$-membered subsets of $S\backslash\{\alpha\}$, which equals $\binom{n-1}{k-1}$.

(ii) The number of k-membered subsets of S not containing α is equal to the number of k-membered subsets of $S\backslash\{\alpha\}$, which equals $\binom{n-1}{k}$.

So $\binom{n}{k}$, the number of k-membered subsets of S equals the sum $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Both proofs are clear and easy to understand. But only the second proof explains its conclusion. The first proof, by using only rules of symbol-manipulation, reduces the risk of error from combinatorial thinking — this is the advantage of formal, "syntactic" methods over "semantic" thinking. But it is only by thinking semantically, in this case thinking in terms of cardinalities of sets, rather than proceeding by rules of symbol-manipulation, that one could come to grasp why the conclusion is true.

While it is easy to sense the difference between explanatory and non-explanatory proofs, it is difficult to say what features of a proof will enable those who understand it to grasp why its conclusion is true. Quite a few accounts have been offered, but as yet there is little agreement, as one can see from sections 4, 5 and 6 of Paolo Mancosu's entry [8] on explanation in mathematics in *The Stanford Encyclopedia of Philosophy*.[15] Fortunately for the concerns of this paper, we do not need a substantive account of what makes a proof explanatory; we just need to be sensitive to the difference between proofs which explain their conclusions and those which do not.

4. Do beauty and explanatoriness tend to co-occur?

The expression "tend to" is used here to accommodate the fact that no one has suggested that the beautiful proofs are exactly the explanatory proofs. But for the proposition in question to merit investigation and, if true, to be worth seeking to explain, the hypothesised co-occurrence must be something stronger than a mere tendency; the thought must be that there is no large class of proofs which have just one of the two attributes, or that proofs which have just one of them are very rare, or something similar. This is how I will read the expression "tend to" in this context.

It will be helpful to factor the question into two and address them in turn:

Do explanatory proofs tend to be beautiful?

Do beautiful proofs tend to be explanatory?

A reason why one might expect a positive answer to the first question is this. A proof is explanatory only if it can enable a person to grasp why the fact proved is true by following the proof. That is, the proof must provide people able to understand the proof both knowledge and understanding of the fact proved. Getting this understanding is pleasing because it satisfies a major intellectual desire. (Recall the remarks of Atiyah, cited earlier). So an explanatory proof can produce pleasure in people with a suitably developed mathematical mind by their direct intellectual grasp of the proof. Moreover, it is disinterested pleasure, independent of appetites or instrumental goals.

[15] For a recent view see Marc Lange's article "Aspects of Mathematical Explanation" [7] and his contribution to this special issue of the *Journal of Humanistic Mathematics*.

So the necessary condition for a proof to be beautiful seems to be met, if the proof explains its conclusion.

Precisely because an explanatory proof can provide disinterested pleasure to people who grasp the proof, it is reasonable to think that its being explanatory contributes to its aesthetic value. But something is beautiful, strictly and literally, only if it has a high or very high degree of aesthetic value. This is one reason for doubts about the generalisation that explanatory proofs tend to be beautiful. For in many cases, it would be an exaggeration to describe an explanatory proof as beautiful. Consider for example the second proof of Pascal's rule. While one might prefer it to the first proof, no one to my knowledge judges it a beautiful proof. In fact there are many explanatory proofs which are too facile to qualify for this accolade. Here is one of many explanatory proofs involving diagrams that illustrate the point.

Theorem. For real numbers a and b $(a+b)^2 = a^2 + 2ab + b^2$.

Proof. *We use the fact that a rectangle with adjacent sides of lengths y and z has area yz. Then a square with side length $a+b$ can be decomposed exactly into a square with side length a plus two rectangles with side lengths a and b plus a square with side length b, as shown in Figure 5.*

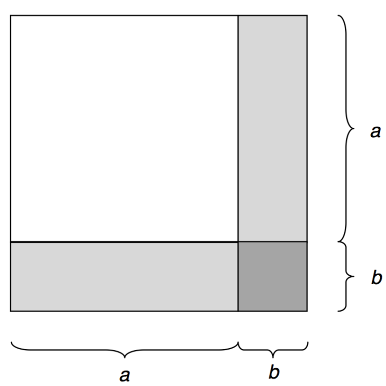

Figure 5: An argument in terms of areas.

This proof, like the explanatory proof of Pascal's rule, can be contrasted with a proof by symbol manipulation of the same theorem, and might be aesthetically preferable. But it does not satisfy the condition for being beautiful given earlier, that it should have a propensity to give significant disinterested repeatable pleasure to connoisseurs (i.e. to mathematicians, in this context).

There is a myriad of explanatory proofs, no less elementary, which for the same reason do not qualify as beautiful. Many of them rely on some geometrical or quasi-geometrical fact with an obvious spatial aspect. An example is this familiar proof of the commutativity of addition for positive integers: mn is the number of dots in an array of m rows of n dots. Rotate the array through a right angle to get n rows of m dots, making nm dots in total. As no dots are lost or gained in rotation, $mn = nm$.

The prevalence of such proofs, explanatory but facile, gives good reason to doubt the claim that explanatory proofs tend to be beautiful. Can we salvage something here by making the modified claim that degree of beauty of explanatory proofs is positively correlated with degree of explanatoriness? This is not going to work, because plain facile proofs of the kind just considered are no less explanatory than the explanatory proofs which mathematicians do judge beautiful, such as Fisk's proof of Chvátal's theorem.

Now we consider the proposition that beautiful proofs tend to explain their conclusions. While this is *prima facie* plausible, confidence wanes with exposure to exceptions. I will present an example, modified from [1], the last of six proofs of the infinity of primes. Let p_1, p_2, p_3, $\cdots$ be the sequence of primes in increasing order. If there is a largest prime p_k, the series $\sum 1/p_n$ converges. Euler had proved that the series $\sum 1/p_n$ diverges, hence that there is no largest prime. Erdős found a new proof that $\sum 1/p_n$ diverges, described by the editors as a proof "of compelling beauty" [1, page 5]. A central idea of that proof can be used to prove the infinity of primes, without reference to the series $\sum 1/p_n$. Here are a couple of preliminaries (a definition and a lemma) needed for the proof.

Definition. An integer $n > 1$ is *square-free* if and only if it has no square factor greater than 1.

Notice that an integer greater than 1 is square-free if and only if no prime occurs more than once in its prime factorisation. For example,

$$18 = 2 \times 3 \times 3 \text{ is not square-free.}$$

$$30 = 2 \times 3 \times 5 \text{ is square-free.}$$

Lemma. For every $n \geq 1$, $n = rs^2$ where r is 1 or square-free and s is positive.

Proof. *Let $n \geq 1$. 1^2 divides n, and if k^2 divides n, $k^2 \leq n$. So there must be a maximum positive square, call it s^2, that divides n. So for some integer r, $n = rs^2$.*

Suppose r is greater than 1 and not square-free. Then there is a prime p that occurs more than once in the prime factorisation of r, so that for some integer j, $r = jpp$. But then $n = j(ps)^2$ and so s^2 would not be the largest square that divides n, contradicting the definition of s^2. So r is 1 or square-free.

Theorem. $\pi(n)$, the number of primes less than or equal to n, increases without bound as n increases.

Proof. *Let $N \geq 1$. Let $rs^2 \leq N$, with r square-free or $r = 1$.*

(A) First we seek an upper bound for the number of possibilities for r. If two numbers have exactly the same primes in their prime factorisations, at least one of those primes must occur more times in the prime factorisation of one of the numbers than in the prime factorisation of the other. So one of those primes occurs at least twice in the prime factorisation of one of the numbers. So no two square-free numbers have exactly the same primes in their prime factorisations.

This means that we can define a one-to-one function f on the set

$$\{r \mid r = 1 \text{ OR } 1 < r \leq N \text{ AND } r \text{ is square-free}\}$$

as follows:

$$f(r) = \begin{cases} \emptyset & \text{if } r = 1; \\ \{\text{primes in the prime factorisation of } r\} & \text{if } 1 < r \leq N \text{ AND } r \text{ is square-free.} \end{cases}$$

Counting the empty set $\emptyset$ as a set of primes less than or equal to N, there are $2^{\pi(N)}$ of sets of primes less than or equal to N. As the mapping f is one-to-one into the set of sets of primes less than or equal to N, there are at most $2^{\pi(N)}$ possibilities for r.

(B) Now we seek an upper bound for the number of possibilities for s, where s is a number such that $rs^2 \leq N$. Note that if $s > \sqrt{N}$ then $s^2 > N$. So $s \leq \sqrt{N}$. In other words, there are most $\sqrt{N}$ possibilities for s.

(C) Now we make use of these upper bounds, $2^{\pi(N)}$ for r and $\sqrt{N}$ for s. The number of numbers $rs^2 \leq N$ is at most $2^{\pi(N)} \cdot \sqrt{N}$

As every positive number can be expressed in the form rs^2 (by the lemma), *the number of numbers less than or equal to N of that form is N. So*

$$N \leq 2^{\pi(N)} \cdot \sqrt{N}, \text{ which after dividing by } \sqrt{N} \text{ gives } \sqrt{N} \leq 2^{\pi(N)}.$$

But $\sqrt{n}$ increases without bound, as n increases. So $\pi(n)$ increases without bound, as n increases.

This is not exactly the Erdős proof, the proof judged to have compelling beauty. But it is quite close to the Erdős proof and has much of its aesthetic value. The main idea and structure of the proof are clear, and its argument is not an obvious way to go but is imaginative, even ingenious. Moreover, it is short without relying on heavy machinery. But it does not explain its conclusion.

Are beautiful but non-explanatory proofs rare? One might be inclined to think so if most of the aesthetically appealing proofs one has come across are explanatory proofs. But unless you are a research mathematician the proofs you have come across will constitute an unrepresentative sample. At the level of research mathematics a significant proportion of beautiful proofs may be non-explanatory. The claim that beautiful proofs tend to be explanatory cannot reasonably be accepted without scrutinising a large sample of beautiful proofs unbiased to the elementary. As far as I am aware, this has not happened.

5. Conclusions

Mathematical proofs, despite their abstractness, are proper candidates for aesthetic evaluation and some of them can be correctly judged to be beautiful. It is widely accepted that some proofs not only prove their conclusions but also explain them. Often these virtues of a proof, being beautiful and being explanatory, are found together; it is natural to generalise and think that proofs that have one of them without the other are rare exceptions. But many explanatory proofs are too basic to be beautiful; so there is good reason to doubt that explanatory proofs tend to be beautiful, given a strong reading of "tend to". In the other direction, we lack good reason both to doubt and to believe that beautiful proofs tend to be explanatory. The experience of most of us is too limited to give us a representative sample of published proofs from which to draw such general conclusions.

Those are the conclusions argued for in this paper. Permit me one final remark. If our interest is in sources of pleasure in mathematical experience, where that can be the experience of school children, university students, teachers as well as researchers, our focus on the beautiful is harmfully narrow. For educational purposes what delights a primary school child is no less important than what gives pleasure to a professional mathematician. This entails that we should not overlook proofs and other elements of mathematics which have an aesthetic value short of beauty. But also, there are non-aesthetic sources of pleasure, such as surprise resulting from the violation of expectations. These other sources too are worthy of attention.

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