Journal of Humanistic Mathematics

Volume 6 | Issue 1 January 2016

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Recommended Citation

V. R. Satyam, "The Importance of Surprise in Mathematical Beauty," *Journal of Humanistic Mathematics*, Volume 6 Issue 1 (January 2016), pages 196-210. DOI: 10.5642/jhummath.201601.12. Available at: https://scholarship.claremont.edu/jhm/vol6/iss1/12

The Importance of Surprise in Mathematical Beauty

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Synopsis

Mathematicians, mathematics education researchers, and philosophers have written about mathematical beauty and many of the qualities commonly associated with it, such as simplicity, brevity, enlightenment, etc. One key theme that underlies many of these qualities is surprise or the unexpected. In this article, I discuss the integral role surprise plays in mathematical beauty. Through examples, I argue that simplicity alone is oftentimes not enough for a piece of mathematics to be considered beautiful, but rather it is *unexpected* simplicity that we seek. I propose, moreover, that surprise is necessary for enlightenment. The paper also reports results from an activity designed to elicit an appreciation of mathematical beauty from elementary preservice teachers; the majority reaction was a feeling of surprise. Understanding the relevance of surprise to mathematical beauty may offer us a feasible way to create opportunities for students to experience mathematical beauty.

Keywords: *mathematical beauty; preservice teachers; visual proofs*

There is a well-documented phenomenon of mathematicians writing and talking about beauty in mathematics [8, 9, 11, 16, 24], using terms like "beautiful" and "elegant" when talking about mathematics they admire [17, 18, 21]. People have attempted to unpack this phenomenon — what does it mean for mathematics to be beautiful? — to tease out some of the general qualities underlying mathematical beauty, with the understanding that a precise definition of mathematical beauty is impossible. Some commonly stated features of mathematical beauty include simplicity, brevity, inevitability, economy, enlightenment, and understanding, among many others [1, 9, 25].

It is not guaranteed, however, that mathematics that exhibits any one of these qualities will automatically be considered beautiful. Mathematicians are upfront about the ill-defined nature of mathematical beauty [12, 25]. One cannot decide if a piece of mathematics will be beautiful based on some set of metrics. Mathematical beauty cannot be defined, but people who study mathematics have a sense of beauty and can appreciate it [4].

A key theme that underlies many of the characteristics of mathematical beauty is surprise or unexpectedness [1, 3, 5, 18]. A piece of mathematics that is beautiful is often surprising or unexpected in some way. Some attention has been paid to the notion of surprise as part of mathematical beauty. The mathematician G. H. Hardy [9] highlighted the presence of unexpectedness in theorems he found beautiful. This has implications for schooling, in that there is a need for illuminating the affective side of mathematics to students learning mathematics [18]; surprise may be one way to expose students to mathematical beauty within their existing curriculum.

In this paper, I argue that surprise is an important part of mathematical beauty. In fact, it is oftentimes the combination of surprise and another valued characteristic that makes a piece of mathematics beautiful. First, I review some of the literature on mathematical beauty. Second, I argue that simplicity and brevity alone do not automatically make a piece of mathematics beautiful; instead, it is unexpected simplicity. Third, I claim that surprise is integral to the experience of enlightenment, another commonly stated feature of mathematical beauty. I provide examples of mathematics throughout to illustrate these situations. Fourth, I share results of an experience from my own teaching, about the reactions of elementary preservice teachers to an activity designed to elicit mathematical beauty. Finally, I discuss some implications of the importance of surprise for teaching and learning mathematics. My goal is to extend current understanding of the nature of mathematical beauty and to consider how we can provide opportunities to students to experience mathematical beauty, to motivate the learning of mathematics.

1. Background on Mathematical Beauty

In this section, I discuss some of the existing literature on mathematical beauty. First, I explore some of the qualities of mathematical beauty. Then, I discuss two general perspectives on mathematical beauty – mathematical beauty as an objective, intrinsic property of the mathematics itself or as subjective, dependent on time and context, and projected onto the mathematics by the observer [3, 21].

1.1. Mathematical Beauty and Aesthetics

A number of authors have explored what mathematicians mean when they call a piece of mathematics beautiful and their possible criteria for beauty. In the same way that it is difficult to define beauty, it is difficult to define mathematical beauty. Sinclair [18, 19] identified some of the characteristics of beauty to be visual appeal, connectedness, fruitfulness, apparent simplicity, and indeed, surprise. In *A Mathematician's Apology*, Hardy [9] claimed that theorems that are beautiful tend to exhibit a triumvirate of inevitability, economy and unexpectedness. Hardy also placed great importance on the seriousness of mathematics, that good mathematics should have significance, generality, and depth. Using a factor analysis, Inglis and Aberdein [10] investigated the structure of mathematical beauty; they found that mathematicians tended to evaluate proofs along four major dimensions: aesthetics, intricacy, utility, and precision.

Examples of mathematical beauty exist not just in the realm of research mathematics but in school as well and across grade levels. Some examples include the theorem that there are only five Platonic solids, Gauss's formula for summing the integers 1 to n, and Cantor's diagonal argument as a technique to prove that the set of reals is uncountable [17]. Another way of intuitively grasping the notion of mathematical beauty is to look at what it is not. Mathematics that is lacking in beauty may be called "ugly." Some generally agreed upon characteristics is that a piece of mathematics is ugly if it is messy, vague, long, too narrow in focus, arbitrary, useless, or has other undesirable qualities. Long equations, long calculations, and facts that are isolated and useless for doing other mathematics may be considered ugly. For example, the process of finding the square root of a number is considered ugly because it involves doing a string of arithmetic operations and results in an approximate solution, not an exact one. Another example would be the fact that every number greater than 77 can be decomposed into a sum of two integers whose reciprocals sum to 1. While true, this fact does not shed much light onto surrounding mathematics. Proofs that lack beauty do serve a purpose however, providing motivation for others mathematicians to try to improve upon them.

Aesthetics is thus another word commonly used in conjunction with mathematical beauty. One may think of mathematical beauty as falling under the umbrella of an aesthetic phenomenon, of experiencing an emotional response to something. When mathematicians experience mathematical beauty, this correlates with activity in the same part of the brain associated with enjoying art [26]. There exist various definitions for what constitutes aesthetics, across disciplines. Some articulate aesthetics as a general human trait of treating certain things as "special," [5] while others conceptualize aesthetics as a yearning for unity and fit [3, 19]. I use aesthetics in the domain of mathematics to mean a sense of good fit [19], because it is consistent with how other mathematicians have written about mathematics as aesthetically pleasing. The term can be expanded even further, however, to mean the values a person brings when doing mathematics, especially in the case of mathematics students and schooling [19].

1.2. Two perspectives on mathematical beauty: Objective vs. subjective

There are philosophical differences in the "location" of the beauty, whether beauty is in the object itself or in the perceiver [3, 18, 19]. Rota [17] explained that many mathematicians see beauty as a property of the mathematics itself and an objective quality:

> The beauty of a piece of mathematics does not consist merely in subjective feelings experienced by an observing mathematician. The beauty of a theorem is a property of the theorem, on a par with its truth or falsehoodBoth the truth of a theorem and its beauty are equally objective qualities, equally observable characteristics of a piece of mathematics which are equally shared and agreed upon by the community of mathematicians. (page 175)

Rota argued that because many mathematicians agree on what mathematics is considered beautiful, then mathematical beauty must be objective on some level. Thus, the beauty of a mathematical object is independent of the observer. Many famous mathematicians have written about beauty from this objective stance, as living inside the mathematics itself [9, 16].

Even so, Rota said there are still subjective elements to mathematical beauty [17]. Wells [25] surveyed mathematicians and found widespread variation in what mathematicians find to be beautiful. Wells' findings lead us to this other point of view, that mathematical beauty is personal and projected onto the mathematics by the observer. Mathematical beauty "lives" inside the perceiver, not the mathematics itself [3, 18, 19]. As such, mathematical beauty is subjective and is dependent on time, context, interests, and one's own mathematical experience [25].

Note on my position

It would be impossible to talk about something like mathematical beauty without one's own perspective seeping in. My position is somewhere along the spectrum between these two perspectives. It is difficult for me personally to not think about beauty as an intrinsic property of the mathematics itself. That being said, I believe that mathematical beauty is personal to the observer; what one person finds to be beautiful in mathematics, another person may not. The sense of fit and satisfaction that is central to an aesthetic response is subjective.

For me to describe examples of mathematical beauty as though everyone agrees that they are universal examples of beauty is, admittedly, at odds with my position that mathematical beauty is personal. To reconcile this, I am presenting examples that I personally believe to be beautiful and analyze why. The purpose of the examples is more to draw out and ground discussion of their features rather than to state that they are universally beautiful mathematics.

In the following sections, I argue for why surprise is important to mathematical beauty: that it is unexpected simplicity that often makes a piece of mathematics beautiful and that surprise is integral to the experience of enlightenment, another oft-mentioned quality of mathematical beauty. By grounding the discussion in concrete examples, it may be easier to (a) draw out and observe the features of mathematical beauty, rather than discussing them abstractly and (b) see how surprise is what triggers the sense of beauty.

2. Simplicity is not enough for beauty; beauty comes from unexpected simplicity.

Simplicity is the criterion most commonly associated with mathematical beauty [10, 25]. The physicist Paul Dirac [4] had a "principle of simplicity" for judging whether a piece of physics could be true, and this has carried over to modern times. Sinclair [19] indeed identified apparent simplicity as one of many characteristics of mathematical beauty.

The relation of simplicity to beauty is not so "simple" itself; simplicity on its own is oftentimes not enough to guarantee that something is beautiful. For example, take the proof that the sum of two odd numbers is even:

Let a, b be odd numbers.
Let $a = 2m + 1$, $b = 2k + 1$, for integers m and k.

$$\begin{aligned} a + b &= (2m + 1) + (2k + 1) \\ &= 2m + 2k + 1 + 1 \\ &= 2(m + k + 1) \end{aligned}$$

Thus, $a + b$ is an even number.

The proof is both simple and short (in that it only requires algebra), but it is safe to say that most communities of mathematicians would not consider this to be a beautiful proof. A statement that is obvious is not always considered beautiful; this is corroborated by the oftentimes negative connotations of words like *obvious* and *trivial* in many mathematical communities. Inglis and Aberdein [10] even argued that simplicity is independent of beauty, because a proof could be seen as ugly and dull due to its extreme simplicity.

The conversation on beauty shifts slightly, however, when we look at a visual proof for another number fact, for why the sum of consecutive odd integers is always a square number, specifically that the sum of odd integers from 1 to $2n - 1$ equals n^2; see Figure 1.

One can prove algebraically that the sum of odd integers from 1 to $2n-1$ will equal the square of n. This picture shows instead that by representing the odd integers as dots, one can carefully arrange each subsequent odd number in such a way that the sum of odd numbers will always be a square number, for any n.

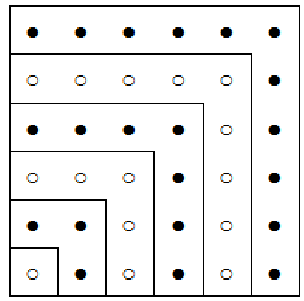

Figure 1: A visual proof that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

This visual proof illustrates the power of unexpected simplicity. What are the elements that lead to this visual proof being aesthetically pleasing on some level? First, it is surprising that odd numbers can be arranged in such a way, "filling in" the border of a square. Second, there is a sense of fit and a eureka moment triggered by how each subsequent odd number neatly wraps around the existing sum of odd numbers. For every dot at the corner, there will be three dots adjacent to it in the next "layer." Each subsequent layer looks like they snap together, creating a square with side one unit larger than before.

As another example of beauty dependent on surprise, let us consider Euler's identity

$$e^{i\pi} + 1 = 0$$

which is often cited as one of the most beautiful pieces of mathematics [21, 25, 26]. One given reason for its beauty is that it relates five important mathematical constants — e, i, π, 1, and 0 — together in one equation. The identity fulfills the ideals of simplicity, brevity, connections, etc. Wells [25], however, found considerable disagreement over whether Euler's identity was beautiful. Some mathematicians rated it as not very beautiful at all, saying that because it was now obvious, it had become a trite example and not as beautiful as it once was. This element of surprise affects what one finds beautiful over time. This situation also illustrates how surprise mediates how the word "obvious" is used in mathematical communities, positively or negatively. A mathematical statement that is expected is called "obvious," with a negative connotation of being dull or ugly. Meanwhile, a statement that is unexpectedly obvious can in fact do the opposite, evoking an aesthetic response for that sense of fit and order where there previously was none.

A term that better captures more aspects of the phenomenon would be *unexpected simplicity* [15, 25], a reduction in the complexity and a new insight. Narins [12] stated that, “a good definition of mathematical beauty would be simplicity that leads to insight” (page 51). When simplicity unexpectedly appears in this situation, this tends to evoke an aesthetic response.

3. Surprise is integral to enlightenment

Another facet of mathematical beauty is *enlightenment*, the sense or meaning of why something is true. Mathematicians have written about the experience of enlightenment in relation to doing mathematics. Leibniz commented negatively about mathematics that is not enlightening (as cited in [1]). Rota [17] said, “We say that a proof is beautiful when it gives away the secret of the theorem, when it leads us to perceive the inevitability of the statement being proved” (page 182).

Mathematicians like to think of mathematical beauty as a moment of instantaneous enlightenment, like a lightbulb turning on [17]. Hadamard [8] talked about discovery as a flash of insight, using the metaphor of light illuminating the darkness. Rota [18] described this kind of light-bulb moment as an ideal that mathematicians yearn for and think of when talking about beauty, even if in vain:

> We think back to instances of mathematical beauty as if they had been perceived by an instantaneous realization, in a moment of truth, like a light-bulb suddenly being lit. All the effort that went in understanding the proof of a beautiful theorem, all the background material that is needed if the statement is to make any sense, all the difficulties we met in following an intricate sequence of logical inferences, all these features disappear once we become aware of the beauty of a mathematical theorem, and what will remain in our memory of our process of learning is the image of an instant flash of insight, of a sudden light in the darkness. We would like mathematical beauty to consist of such a sudden flash; mathematical beauty should be appreciated with the instantaneousness of a light bulb being lit (page 179).

Yet this kind of instantaneous understanding does not happen often; enlightenment may happen as a slow process of incremental understanding over time. Gradual understanding can come from repeated attempts to crack a problem or from revisiting some mathematical concept after some time. Rota claimed that in order for someone to appreciate mathematical beauty, they must be familiar with the theory needed to understand it, and this takes much time and effort [17]. Instantaneous enlightenment is rare, and it is unfortunate that students believe that they should be able to see why something is true instantly in order to appreciate mathematics [17]. In fact, Poincaré [16] believed that any truly hard problem required multiple stages to solve: actively working on a problem and noticing features and applying strategies and techniques, but also engaging in unconscious work, where the person is not actively working or thinking about the problem but their mind is conducting some sort of further analysis as a background process. When the answer pops back into the person's mind, it is reminiscent of a flash of insight or instantaneous enlightenment, even though there was a good deal of work that went into it. When we revisit a problem in the long term and see more insights and connections to other mathematics than we did before, this series of small surprises may also lead to a sense of enlightenment. Rota's position was that mathematical beauty had more to do with enlightenment than other qualities, claiming that the entire concept of mathematical beauty was how mathematicians avoid having to talk about or deal with the messy, vague phenomenon of enlightenment [17].

When enlightenment occurs, surprise is a necessary part of it. The visual proof in Figure 1 provides a geometric intuition for why the sum of odd numbers is always a square number, for why that relationship exists. While an algebraic proof would also manage to prove that the result is true, the visual proof is a lot more powerful. There is a difference between an argument that proves a statement is true by a series of manipulations versus one that illuminates the mathematical mechanisms behind the concept. We may say then that this visual proof provides a moment of enlightenment, in understanding why something is true, by providing all the information at once in a novel manner.

Another example of mathematical beauty would be Bill Thurston's solution to Smale's paradox, on how to turn a sphere inside out in three-dimensional space. This method is depicted in the video "Outside In" [13].

What features of mathematical beauty are present here? One would be hard-pressed to claim this method is simple or short; there are numerous steps involved in turning a sphere inside out and each of them appear to be complicated movements. Yet Thurston's method provokes an aesthetic response. What is it that makes this method beautiful? For me, it is a complete sense of surprise that triggers curiosity, a "How did they do that?" reaction. Even though it is not clear how or why this method works, the unexpectedness drives a person to want to know why, to seek further enlightenment. There is also a sense of the mysterious, which is associated with surprise [25]. Einstein stated, "The most beautiful thing we can experience is the mysterious. It is the source of all true art and science" (as cited in [25, page 39]).

Here, I have shown two examples of how surprise is integral to enlightenment. Proofs may provide enlightenment for results that were already known. In addition, surprise also instigates a curiosity to find out more, a quest for further enlightenment.

4. Students' reactions to mathematical beauty

We now turn to mathematics education. One goal of mathematics education is that students should appreciate mathematics [6, 19]. However, this affective side of learning mathematics, one facet being that mathematics is satisfying in and of itself, has been relatively ignored and is an under-researched area [18, 19]. U.S. policy documents do not reference the affective side of learning mathematics [23].

To test how students reacted to mathematical beauty, I conducted a small survey-design experiment with a group of elementary preservice teachers ($N = 11$). Students took part in an activity designed to elicit mathematical beauty, dipping 3D platonic solids into bubble solution. First, students wrote down predictions for what the bubble films on the platonic solids would look like. Then, students used gumdrops and toothpicks to create the platonic solids, tied a piece of string to the top, and dipped these frames into a bucket of bubble solution. As they carefully pulled the gum drop-toothpick frame out of the solution, they observed the resulting bubble film. Most students had conjectured that the bubble surface would form along the outside faces but instead, bubble films formed along the minimal surface of each figure.

For example, in the middle of the cube solid platonic, a tiny cube or square formed. There were many audible "ooos" and "ahhs" throughout the class.

Students then wrote down three adjectives they would use to describe their reaction(s) to the activity. Grammatical variations of the same root word, i.e. "interesting" and "interested," were combined. A couple students wrote two or four adjectives instead. Table 1 shows the adjectives that students reported.

Adjective	**Frequency of students**
surprised	9
intrigued	5
interested	4
confused	3
excited	3
amazed	2
curious	2
pleased	1
shocked	1
impressed	1
captivated	1
fascinated	1

Table 1: **Students' Self-Reported Adjectives.** $N = 11$ students reported multiple adjectives to describe their reaction(s) to observing bubble surfaces of platonic solids.

Across all the students, twelve different adjectives were used. Nine out of the eleven participants used the word "surprised" to describe their reaction. Even though the sample size is small, for nearly all of the students to use the word surprise is curious. One student used "shocked" in the same vein as surprise. These results suggest that surprise was the most prevalent reaction, among the elementary preservice teachers.

What does this small experiment mean for the teaching and learning of mathematics? One, surprise is an accessible emotion to many, judging by the evidence that many of the students experienced it. Appreciation for some of the other characteristics of mathematics — such as a feeling of harmony or order — may be difficult to instill in students. This is even putting aside the question of whether, as educators, we should be instilling students with predetermined values of mathematical beauty or let them decide for themselves. Because of its accessibility, surprise may be an avenue through which to reach out the students about mathematical beauty.

Second, surprise and confusion go hand in hand. Three students wrote that they felt confused by the activity; all three of these students also said they were surprised, however. Confusion can be a positive or a negative emotion. If positive, confusion can act as a perturbation, leading the student to want to know why. This is one role of aesthetics in mathematics, to guide inquiry [18], independent of the significance of the problem. Too much confusion can have negative effects, however, and discourage students.

There are some caveats to this study. Students worked in small groups, so even though they reported independently, their reactions and word choice could be influenced by the rest of the group. However, this simple activity does show that surprise is a strong emotion that can arise naturally in the classroom, without much prodding.

5. Discussion

I have argued and illustrated that surprise is an integral part of experiencing mathematical beauty and moreover, that it is an emotion that can be brought out in the classroom. The importance of surprise has implications for teaching and learning, specifically in eliciting eureka or "aha" moments in learning, which may be founded on the element of surprise. Teaching in such a way where there are no "aha" moments effectively robs students of opportunities to experience surprise and thus, mathematical beauty.

This also highlights the importance of instructional sequencing, in choosing how to present material to support student learning. Sometimes presenting content in the most logical way, which may be aesthetically pleasing to the teacher, may not allow for students to experience that feeling of surprise and satisfaction in the classroom. Careful sequencing of examples and problems to elicit "aha" moments could be feasible with nearly any mathematics content, turning ordinary mathematical results into things to wonder over [22]. This suggests that creating opportunities for students to begin developing a sense of mathematical beauty within existing curricula and content may be possible.

The role of surprise also highlights the importance of conjecturing in the classroom. Surprise is only possible when there is some pre-existing expectation about what is going to happen. Sinclair [20] described this careful balancing act teachers do: "It's important to note that the moment of surprise and juxtaposition he offered was only possible because of the previous, extended experience of equations working-out (leading to solutions). It would be impossible to find aesthetic pleasure in constant change and surprise" (page 34). Moments of surprise can pique students' interests, whereas constant surprise can turn into confusion and turn students off. Nonetheless, the role of surprise in mathematical beauty brings hope that creating opportunities for students to experience mathematical beauty is possible.

Acknowledgements

I would like to thank those who provided insight and feedback at various stages of this work, specifically Sarah Klanderman, José Manuel Martínez-Hinestroza, Dr. Tonya Bartell, and my adviser, Dr. John P. Smith III. I would also like to acknowledge the reviewers for their highly regarded comments.

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