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An Approximation Method of Calculating the Present Worth of Nonintegrable Cash Flow Patterns

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Abstract

An approximation method is provided for calculating the present worth of nonintegrable continuous cash flows that have common industrial economic applications. Two limiting cases of particular use in engineering screening analyses are given for each model. Practical examples are presented to illustrate the application of the cash flow models to manpower reductions due to computerized process control and to cash flows for a pollution-abatement facility.

1. INTRODUCTION

The evaluation of engineering projects requiring capital investments is becoming increasingly difficult in today's inflationary and resource-limited environment. For example, the recent Alaskan pipeline project was initially estimated to cost one billion dollars, but the actual cost prior to the first flow of oil through the pipeline was eight billion dollars. There were many reasons for this cost overrun but the point is clear - capital project evaluation is tough and getting tougher.

One of the major problems facing many companies and government agencies is that less real funds are available for new capital projects because of recent high inflation rates. In this environment, engineering economic project evaluations are becoming more important in both the private and public sectors. In the public sector, for example, government agencies and companies doing projects for these agencies are turning to the concept of Life Cycle Costs (LCC).

LCC for a capital project are the total of the expected future revenues and disbursements, or <u>cash flows</u>, over the entire project lifetime. LCC analysis is used to compare alternative projects by minimizing the present worth of the LCC. This analysis is similar to approaches used for many years by the private sector, such as discounted cash flow rate-of-return or present worth analysis.

Economic evaluations and sensitivity analyses of engineering projects are often very time-consuming because general equations for the present worth of the cash flow functions are usually unavailable, especially for nonintegrable cash flow functions. This paper presents formulas for determining the present worth of nonintegrable continuous cash flow profiles that might be encountered in industrial and government capital project evaluations.

In many economic analyses of engineering projects, the cash flows associated with the project can be approximated by a continuous function of time. We denote this continuous cash flow function by C(t); the expected cash flow at time t is C(t). Cash flows may include, for example, expenditures for research, development, testing, construction or training. For simplicity, we express the project startup time as 0 and the project lifetime as n. The results presented herein include expressions for PW, the present worth at time 0 of the continuous cash flows.

Since cash flows are continuous, PW is determined using continuous compounding of interest. If the nominal annual cost-of-capital is r, then e^{-rt} is the single-payment present worth factor applied to a cash flow occurring at time t. Consequently, PW is defined as

$$PW = \int_0^n C(t)e^{-rt}dt \qquad (1)$$

In some cases the integral in Equation 1 may be evaluated directly to obtain the algebraic solution for PW; several cash flow functions of this type are discussed in detail in a recent paper [1]. In other situations the function $C(t)e^{-rt}$ is not integrable and methods of approximation must be applied to determine PW; cash flow functions of this form are the focus here. For each of the two representative models examined in this paper, an example is given to illustrate application to actual industrial situations, and limiting cases for a long-term project life or a negligible interest rate are presented.

The reader may also be interested in two related papers, both recent: a generalized polynomial model for calculating the present worth of projects with discrete cash flows [2], and a model for combining interest and inflation rates for present-worth analysis in eleven industrialized countries [3].

II. DEVELOPMENT OF THE MODELS

For some types of cash flow functions C(t), the expression $C(t)e^{-rt}$ in Equation 1 is nonintegrable and numerical methods must be used to approximate the present worth PW of the cash flows. The approximation method presented here is referred to as the Trapezoidal Rule and is especially useful since it allows us to control the error [4]. This method was chosen over other methods, such as the Rectangle Rule or Simpson's Rule, because it offers a greater degree of accuracy or simplicity in many cases.

Consider any cash flow function C(t) with a continuous second derivative (all models presented here satisfy this condition). For an arbitrary positive integer m, the Trapezoidal Rule gives the following approximation for PW:

$$PW \doteq \frac{n}{2m}\left[C(0) + C(n)e^{-rn} + 2\sum_{k=1}^{m-1} C\left(\frac{kn}{m}\right)e^{-rkn/m}\right]. \quad (2)$$

The maximum error in using this approximation is

$$\text{Maximum Error} = \frac{n^3}{12m^2} \max_{0 \leq t \leq n} |G(t,r)|, \quad (3)$$

where

$$G(t,r) = \left[\frac{d^2}{dt^2}C(t) - 2r\frac{d}{dt}C(t) + r^2C(t)\right]e^{-rt}. \quad (4)$$

By choosing m sufficiently large, this error can be made as small as we like. The Trapezoidal Rule will be used in evaluating the following two models.

A. Reciprocal model

The cash flow for this model is of the form

$$C(t) = \frac{1}{(t+f)^d},$$

where d is some positive integer and f is a positive constant. Figure 1 shows the graph of this cash flow function.

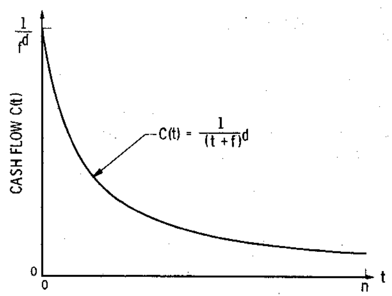

Figure 1. Reciprocal model cash flow.

Consider d = 1. Then C(t) = 1/(t + f) and using Equation 4, we find

$$G(t,r) = \frac{r^2(t+f)^2 + 2r(t+f) + 2}{(t+f)^3 e^{rt}} > 0.$$

It can easily be verified that G(t,r) is decreasing in t for all positive t. Thus,

$$\text{Maximum Error} = \frac{n^3}{12m^2} \max_{0 \leq t \leq n} |G(t,r)| \quad (5)$$

$$= \frac{n^3}{12m^2} G(0,r) = \frac{n^3(r^2f^2 + 2rf + 2)}{12m^2f^3}.$$

Using Equation 2, the present worth formula for d = 1 is

$$PW = \int_0^n \frac{e^{-rt}}{t+f}\,dt \doteq \frac{n}{2m}\left[\frac{1}{f} + \frac{e^{-rn}}{n+f} + 2m\sum_{k=1}^{m-1}\frac{e^{-rkn/m}}{nk+mf}\right], \quad (6)$$

with a maximum error determined by Equation 5.

An alternative method of finding PW for the Reciprocal Model with d = 1 is to use the infinite alternating series expansion:

$$PW = \int_0^n \frac{e^{-rt}}{t+f}\,dt$$

$$= e^{rf}\left[\ln(1+\frac{n}{f}) - \frac{r(n+f)-rf}{1!} + \frac{r^2(n+f)^2 - r^2f^2}{2 \cdot 2!} - \frac{r^3(n+f)^3 - r^3f^3}{3 \cdot 3!} + \ldots\right].$$

However, no simple formula is available for determining the error introduced by truncating this series.

If d ≥ 2 in the Reciprocal Model, iterative integration by parts produces

$$PW = \frac{(-r)^{d-1}}{(d-1)!}\left\{\sum_{j=0}^{d-2}\frac{j!}{(-r)^{j+1}}\left[\frac{1}{f^{j+1}} - \frac{e^{-rn}}{(n+f)^{j+1}}\right] + \int_0^n \frac{e^{-rt}}{t+f}\,dt\right\}, \quad (7)$$

reducing the problem to the case d = 1. Equation 6 is then applied to determine PW.

<u>Example 1</u>. As a result of anticipated computerized process control, the manpower costs (in thousands of dollars) for a fifteen year petrochemical project are expected to vary in accordance with a cash flow curve of the type shown in Figure 1. A preliminary economic analysis has shown that the cash flow function C(t) = $9300/(t + 14.142)^2$ can be expected to accurately predict the manpower costs. Given a 10% cost-of-capital, what is the present worth of the cash flows within an error of $30,000?

The cash flow function can be depicted by the Reciprocal Model with parameters f = 14.142 and d = 2 (and leading coefficient 9300). Using Equations 1 and 7, we have

$$PW = 9300 \int_0^{15} \frac{e^{-0.1t}}{(t+14.142)^2}\,dt$$

$$= 586 - 930 \int_0^{15} \frac{e^{-0.1t}}{t+14.142}\,dt.$$

Our next step is to determine the integer m from Equation 6 that will assure a maximum error of \$30,000. Using Equation 5 with n = 15, we have

$$\text{Maximum Error} = \frac{6315}{m^2} \leq 30\ (\times\ \$1000).$$

The smallest positive integer satisfying this relation is m = 15. Thus, Equation 6 is used with m = 15 to find

$$\int_0^{15} \frac{e^{-0.1t}}{t+14.142}\,dt \doteq 0.415.$$

Hence,

$$PW = [586 - 930(0.415)]\ (\times\ \$1000) = \$200{,}000,$$

with a maximum error of (6315/225) (x \$1,000) = \$28,000.

B. Logarithmic model

We consider next a continuous cash flow function of the form

$$C(t) = \ln(t+1).$$

Figure 2 represents the general shape of this logarithmic function.

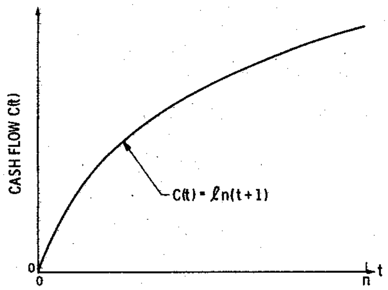

Figure 2. Logarithmic cash flow.

Using Equation 4, we find

$$G(t,r) = \left[\frac{-1}{(t+1)^2} - \frac{2r}{t+1} + r^2 \ln(t+1)\right] e^{-rt}.$$

This function has a unique critical point at t_0, which is uniquely determined by the equation

$$3r^2(t_0+1)^2 + 3r(t_0+1) + 2 = r^3(t_0+1)^3\ln(t_0+1)\ (r \neq 0) \qquad (8)$$

The point t_0 specifies an absolute maximum of G(t,r) on the positive t-axis. Since $G(0,r) = -1-2r < 0$ and $G(t,r) > 0$ for $t > t_0$, we have

$$\text{Maximum Error} = \frac{n^3}{12m^2} \max\{(1+2r), |G[\min(t_0,n),r]|\} \qquad (9)$$

by Equation 3. Using Equation 8, the corresponding present worth formula is

$$PW \doteq \frac{n}{2m}\left[\ln(n+1)e^{-rn} + 2\sum_{k=1}^{m-1} \ln\left(\frac{kn}{m}+1\right)e^{-rkn/m}\right]. \qquad (10)$$

Example 2. The operating costs (in thousands of dollars) of a government-mandated pollution abatement facility are expected to increase over the next twenty-five years according to the cash flow function C(t) = 900 ln(t + 1). Given a nominal interest rate of 5% (the company uses this rate for mandated pollution abatement facilities) and a maximum error margin of \$4,500,000, what is the present worth of these operating costs?

The solution to Equation 8 with r = 0.05 was found by trial-and-error to be $t_0 = 31.34$. Hence, $\min(t_0,n) = 25$ and

$$G[\min(t_0,n),r] = G(25,0.05) = 0.0008 < 1.1 = 1 + 2r.$$

Consequently, Equation 9 yields

$$\text{Maximum Error} = 900\,\frac{(25)^3(1.1)}{12m^2} = \frac{1{,}290{,}000}{m^2} \leq 4500(\times\$1000)$$

The smallest integer satisfying this relation is m = 17. Thus, we use m = 17 in Equation 10 to find

$$PW = 900(30.2)(\text{X } \$1000) = \$27{,}200{,}000,$$

with a maximum error of

$$\frac{1{,}290{,}000}{(17)^2}\ (\text{X } \$1000) = \$4{,}460{,}000.$$

An equivalent formulation of the present worth for the Logarithmic Model is

$$PW = \int_0^n \ln(t+1)e^{-rt}dt = \frac{1}{r}\left[PW^* - e^{-rn}\ln(n+1)\right], \qquad (11)$$

where PW* is the present worth of the Reciprocal Model with analogous r and n and f = d = 1. Although use of Equation 11 makes the trial-and-error calculation of t_0 in Equation 8 unnecessary, it increases the error inherent in Equation 6 by introducing the multiplicative factor 1/r (usually significantly large).

III. LIMITING CASES

The continuous models presented in this paper have two special cases of engineering interest. The first case concerns the limiting behavior of the present worth PW as the project life n becomes infinite. This behavior is important when evaluating projects such as dams and football stadiums that have lifetimes in excess of, say, forty years. Also, infinite lifetimes are often assumed for permanent chemical plant and refinery offsite facilities such as those providing drinking or cooling water. The second case occurs when the interest rate approaches zero. A zero-interest model can be used, for example, when doing quick engineering calculations with interest and inflation rates assumed to nearly cancel [5].

A. Infinite Project Life

Denote by PW_∞ the asymptotic level that the present worth of the cash flows approaches as $n \to \infty$. Since

$$PW_\infty = \int_0^\infty C(t)e^{-rt}dt,$$

this asymptotic level is the Laplace transform $L\{C(t)\}$ of the cash flow function C(t). Expressions for PW_∞ for the nonintegrable models discussed in this paper involve the exponential integral function

$$Ei(x) = \int_{-\infty}^{x} \frac{e^v}{v}\,dv,$$

values of which are readily obtainable [6]. Expressions for PW_∞ are given in Table 1.

TABLE 1

Limiting cases of the present worth of the cash flow profile

Model	PW_∞ (Infinite Project Life)	PW_0 (Zero Interest Rate)
Reciprocal $C(t) = \frac{1}{(t+f)^d}$	$-e^{rf}Ei(-rf)$, $d=1$ $\frac{(-r)^{d-1}}{(d-1)!}\left\{\sum_{j=0}^{d-2}\frac{j!}{(-rf)^{j+1}} - e^{rf}Ei(-rf)\right\}$, $d \geq 2$	$\frac{d}{f^{d+1}} - \frac{d}{(n+f)^{d+1}}$
Logarithmic $C(t) = \ln(t+1)$	$-e^{r}Ei(-r)/r$	$(n+1)\ln(n+1)-n$

B. Zero Interest Rate

When the nominal annual interest rate r is zero, the present worth PW_0 of the cash flows is found by re-evaluating the integral from Equation 1. Thus,

$$PW_0 = \int_0^n C(t)dt,$$

which always exists [7]. The expressions for PW_0 for each model are presented in Table 1.

Example 3. Consider the same situation as in Example 1. What are the present worth limits 1) as $n \to \infty$, and 2) as $r \to 0$?

The situation is described by the Reciprocal Model with parameters n = 15, r = 0.10, f = 14.142, d = 2, and cash flow function $C(t) = 9300/(t + 14.142)^2$ (x $1,000). First, we consult a set of mathematical tables [6] to find

$$Ei(-rf) = Ei(-1.4142) = -0.114.$$

Referring to the appropriate entry in Table 1 for an infinite project life, we have

$$PW_\infty = 9300\left[\frac{1}{14.142} + (0.1)e^{1.4142}Ei(-1.4142)\right]$$
$$= 221.535\ (\times\ \$1000) = \$221{,}535,$$

and for a zero interest rate, we have

$$PW_0 = 9300\left[\frac{2}{(14.142)^3} - \frac{2}{(29.142)^3}\right](\times\ \$1000) = \$5825.$$

IV. SUMMARY

We have constructed mathematical models for determining the present worth of two types of representative cash flow profiles often encountered in industrial and governmental economic evaluations, and have illustrated the application of these models in several examples. Figures 1 and 2 are graphical representations of these cash flow profiles. We have presented one numerical technique for evaluating the present worth PW of these (nonintegrable) cash flows (future work might involve more refined techniques, such as Gaussian quadrature).

Limiting cases of the models were also examined, and it was found that in the case of an infinite project life or a negligible interest rate the calculations for the present worth of the cash flows were simplified.

Once the project lifetime and a model approximating the expected future cash flow profile are determined, a simple trial-and-error calculation will yield the discounted cash flow rate of return. When used in discounting the cash flows, this rate gives a zero present worth, and thus is the solution to the equation PW = 0. A project under consideration is usually evaluated by comparing its DCF with that of other projects or the company's existing return-on-investment.

NOMENCLATURE

C(t)	general continuous cash flow function
d	model parameter for Reciprocal Model
Ei(x)	exponential integral function
f	model parameter for Reciprocal Model
G(t,r)	Trapezoidal Rule maximum error shorthand function
L	Laplace transform operator
LCC	Life Cycle Costs

m	Trapezoidal Rule variable
n	project lifetime
PW	present worth
PW_∞	$\lim_{n \to \infty} PW$
PW_o	$\lim_{r \to 0} PW$
r	nominal annual cost-of-capital
t	time variable
t_o	critical point for Logarithmic Model

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