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Spirograph® Math

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1. INTRODUCTION

The Spirograph® has been a popular toy for many decades. A search on the world wide web will result in many sites that discuss the mathematics of a spirograph pattern, and your local toy store most likely has an assortment of spirographs, each with its own special feature. The basic spirograph set includes numerous circular discs and annuli. To construct a spirograph design, a pencil or pen is placed in one of a number of perforations on a disc, and a pattern is traced as the disc rolls around a fixed annulus. There are seemingly endless possibilities of designs to be made, including the designs in Figure 1. The construction and comparison of these designs is a rich source of mathematics, adaptable to many levels of students. In this paper we will explore what determines the pattern, and what patterns are possible.

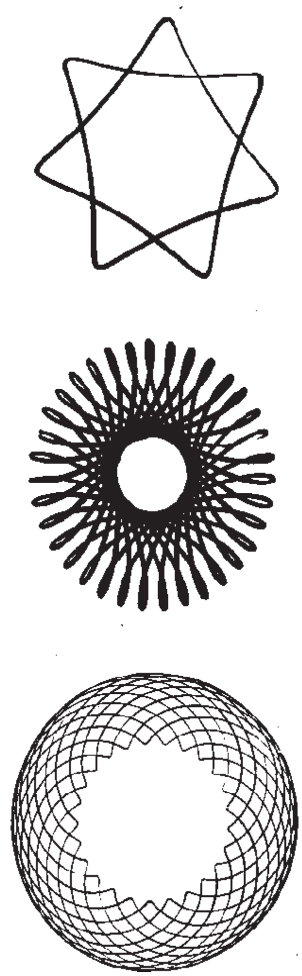

Figure 1
Examples of Spirograph Patterns

The designs made with a spirograph set have mathematical names. In approximately 1600 Gallileo Gallilei gave the name **cycloid** to the curve of the trace of a point on the circumference of a circle that rolls along a straight line. If the tracing point is in the interior of the circle, the resulting curve is called a **trochoid**. In comparison with the cycloid, the trochoid is more flat, as illustrated in Figure 2. In fact, when the tracing point is in the exact center of the disc, the pattern would be a straight line.

A spirograph is a generalization of Gallileo's cycloid. Instead of the circle rolling along a straight line, it rolls along another circle. The curve that is made from rolling a circle inside a fixed circle is called a **hypocycloid** when the tracing point is on the boundary of the circle, and a **hypotrochoid** when the point is on the interior of the circle. If the circle rolls along the outside of a fixed circle, it is called an **epicycloid** or **epitrochoid**.

2. PARAMETRIC EQUATIONS

As well as having mathematical names, these curves also can be described by parametric equations. Most calculus texts include the derivation of these equations in the exercises of the parametric equations section. Denote the radius of the fixed circle by a, and the radius of the rolling circle by b. The parametric equations are determined by the coordinates of the tracing point.

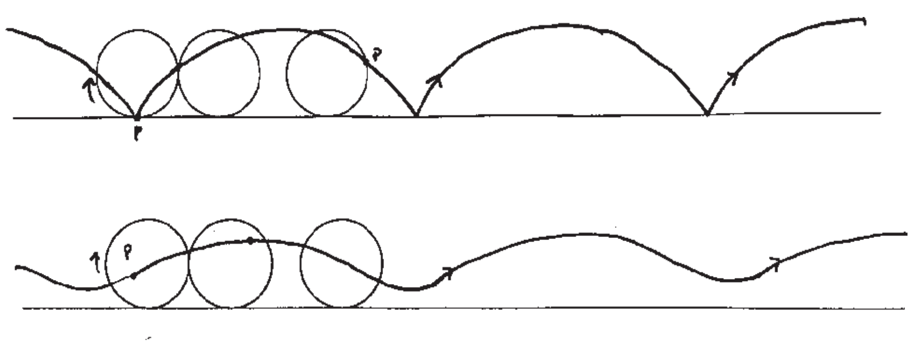

Figure 2
Cycloid and Trochoid

Hypocycloid:

$$x := (a-b)\cos(t) + b\cos\left(\frac{(a-b)t}{b}\right)$$

$$y := (a-b)\sin(t) - b\sin\left(\frac{(a-b)t}{b}\right)$$

The only difference between the hypocycloid and hypotrochoid is the position on the rolling circle of the tracing point. Thus the only change in the formula for the hypotrochoid is the multiplication factor on the second term. The factor is h, the distance from the center of the rolling circle to the tracing point.

Hypotrochoid:

$$x := (a-b)\cos(t) + h\cos\left(\frac{(a-b)t}{b}\right)$$

$$y := (a-b)\sin(t) - h\sin\left(\frac{(a-b)t}{b}\right)$$

Equations for the epicycloid and epitrochoid are similar to those above. The main difference is the multiplication factor on the first term. For the hypotrochoid the factor is the difference of radii, *(a-b)*, and this is the distance from the origin to the center of the rolling circle. Since the rolling circle is placed outside the fixed circle for the epitrochoid, the distance from the origin to the center of the rolling circle is the sum of their radii, (a+b).

Epicycloid:

$$x := (a+b)\cos(t) - b\cos\left(\frac{(a+b)t}{b}\right)$$

$$y := (a+b)\sin(t) - b\sin\left(\frac{(a+b)t}{b}\right)$$

Epitrochoid:

$$x := (a+b)\cos(t) - h\cos\left(\frac{(a+b)t}{b}\right)$$

$$y := (a+b)\sin(t) - h\sin\left(\frac{(a+b)t}{b}\right)$$

With the above equations one can use either a graphing calculator or computer software such as Maple to draw spirograph designs without a spirograph set. One can also check the validity of the parametric equations by comparing the computer version with the spirograph version of a particular design. Technically, every spirograph is a hypotrochoid or epitrochoid, since the perforation farthest from the center on any disc is in the interior of the disc.

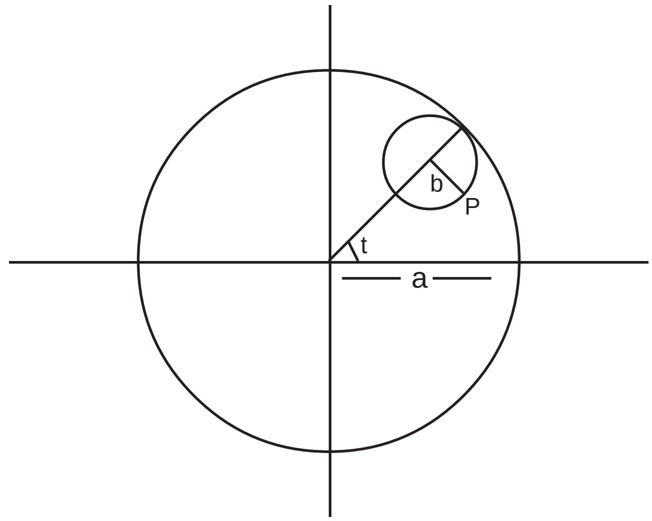

Figure 3
Construction of a Hypocycloid

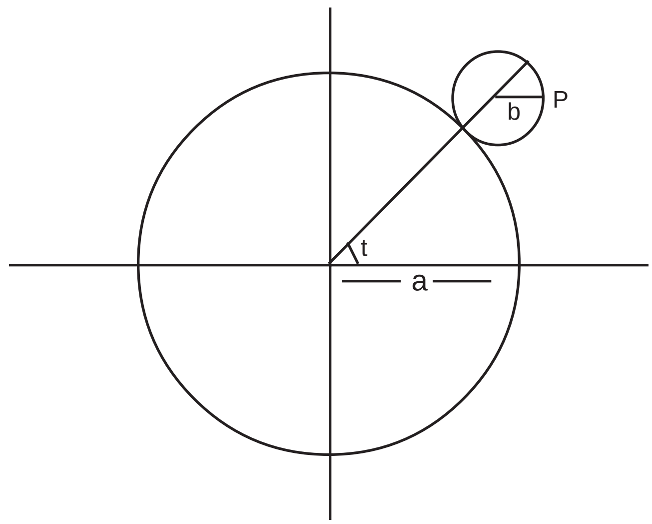

Figure 4
Construction of an Epicycloid

3. EFFECT OF *a*, *b*, AND *h*

From the equations, one sees that three measurements affect the pattern: the radius a of the fixed circle, the radius b of the disc, and the offset h. First investigate the effect of h on the pattern.

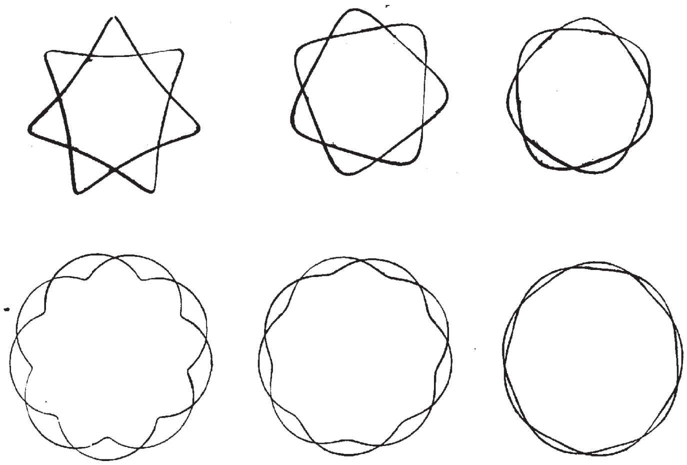

Figure 5
105/30 with offset perforations 1, 6, 8
144/32 with offset perforations 1, 6, 13

The patterns in Figure 5 are indexed with the perforation number on the actual spirograph disc. All the perforations are numbered, with #1 being closest to the circumference and the bigger numbers increasingly closer to the center of the disc. If one chooses a perforation close to the center of the disc, the figure looks more like a circle. If a perforation is at the exact center of the disc, then the offset h is zero, and the equations result in a circle.

In the derivation of the formulas, the radii of the circles were the crucial measurements. In a spirograph set, each circle is numbered according to the number of teeth it has on its circumference. As it turns out, there are 30 teeth for each millimeter of the radius. Notice that the ratio of radii will be the same as the ratio of number of teeth. The figures are labeled according to the number of teeth.

After drawing several patterns, one might want to predict what the resulting designs will be. This is easy using the reduced fraction of number of teeth. In Figure 6 look for a relationship among the designs and the numerator and denominator.

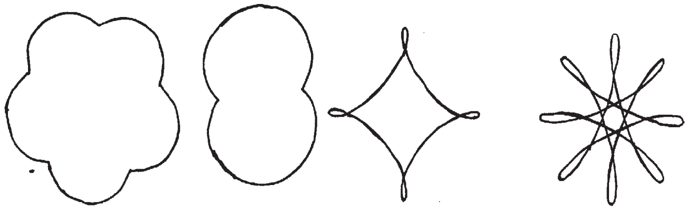

Figure 6
150/30 = 5/1; 144/71 = 2/1; 96/72 = 4/3; 96/60 = 8/5

As one can see, the reduced numerator is the number of cusps. A cusp is made each time the rolling circle completes a revolution. To compute the number of cusps, imagine cutting each circle and laying the resulting straight lines along side each other, as in Figure 7. Paste in copies of each line until their ends match. The number of revolutions that the rolling circle makes is the number of copies of the rolling circle. The resulting equation $s*a=r*b$, describes the situation. Since r and s are the smallest number of copies needed, r/s is the smallest reduced fraction of a/b. Therefore r is indeed the reduced numerator.

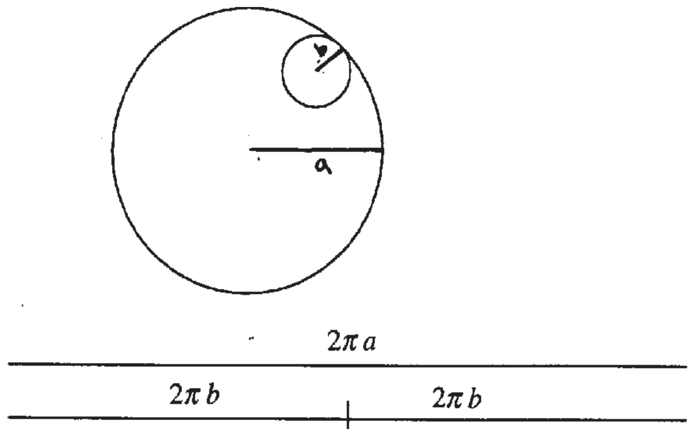

Figure 7
Cut and Flattened Circles

In the above equation s is the reduced denominator and is the number of revolutions of the fixed circle needed for the trace to meet its starting point. This number is called the **period** of the design.

By using the reduced fractions one can determine what kind of figures are possible. In addition to the period and number of cusps, one might recognize a difference in the behavior near a cusp. In some designs the pattern crosses itself before a cusp to form a **node**, and in other designs there is no such crossing. For hypocycloids, when $2b>a$, there is a node for each cusp. For the hypotrochoids, the offset h affects the nodes.

4. SYMMETRY GROUPS

One can also ask what kind of symmetry these designs have. The symmetries of any two-dimensional design are either rotations or reflections, and they form a group with the operation composition. If the designs have both rotational and reflectional symmetry, then the symmetries form a **dihedral** group. A design with

n rotations and n reflections has symmetry group D_n. If the designs have only rotational symmetry, then the symmetries form a **cyclic** group. A design with n rotations has symmetry group C_n. In Figure 8, the seven axes of reflection are included.

Notice that all of the designs thus far have both reflectional and rotational symmetry, so their symmetry groups are dihedral groups. The numerators of the reduced teeth ratios determine which groups are possible. There can be many different designs that represent the same group, illustrating the difference between equal and isomorphic groups. That is, groups can have the same structure but be represented differently. All of the designs in Figure 9 have symmetry group D_5.

What kind of groups occur in a combination of designs? The greatest common denominator of the two numerators give the subscript of the dihedral group. If t is the greatest common denominator of m and n, the number of cusps in each design, then the combination of designs can be divided into t equal pieces. The t pieces will consist of m/t cusps from the first design and n/t cusps from the second design. Thus the symmetry group will be D_t. In Figure 10, the symmetry groups are D_{12}, D_6 and D_5 respectively.

If the reduced fraction of teeth is a whole number, then another kind of design is made by tracing one loop for each perforation in the rolling disc. Such designs have cyclic symmetry groups (see Figure 11). The table of reduced fractions gives that the possible cyclic groups made in this way are C_2, C_3, C_4, C_5 and C_6.

It is also possible to get different symmetry groups from one design by coloring segments formed by cusps. If a design has symmetry group D_n, then there is a coloration that will result in D_m for each m that divides n, and a coloration that will result in C_m for each m that divides n, except $n/2$ and n.

To demonstrate this, take a design of dihedral group D_n. To get D_m, choose 2 colors. With the first color, fill in m equidistant segments. Color the rest of the segments with the second color. D_m can be generated by a rotation of $(360/m)^\circ$ and a reflection across the diametrical axis through one of the cusps with the first color. To get C_m, choose r colors where $n = rm$ and color m consecutive segments with each of the colors,

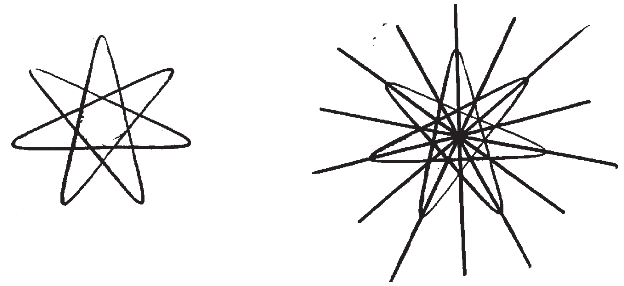

Figure 8
Axes of Reflection on 105/45 = 7/3

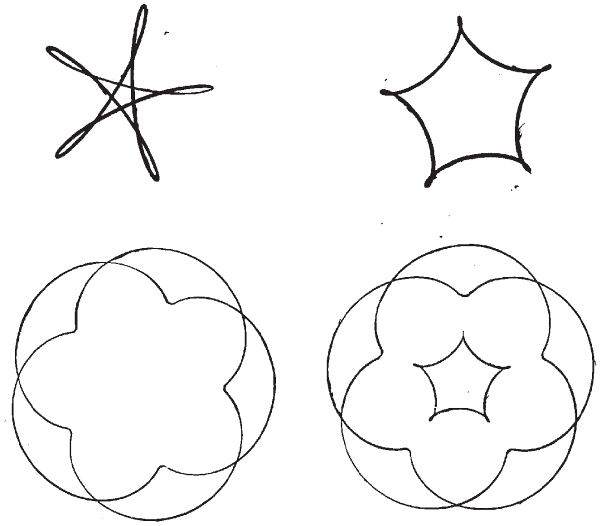

Figure 9
Designs with Symmetry Group D_5
105/63 = 5/3; 105/84 = 5/4; 150/60 = 5/2

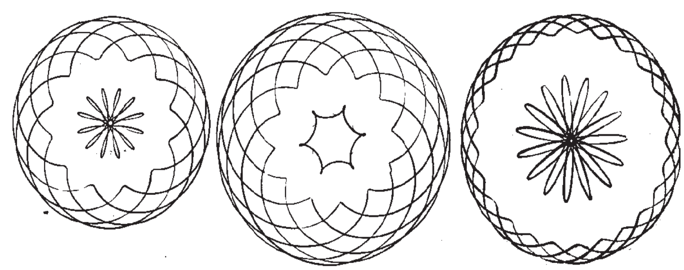

Figure 10
144/60, 96/56; 144/84, 96/80; 150/32, 105/48

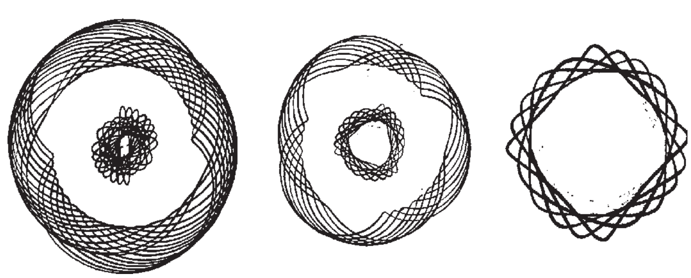

Figure 11
C_2, C_3 and C_4
96/48, 144/72; 96/32, 105/45; 95/24

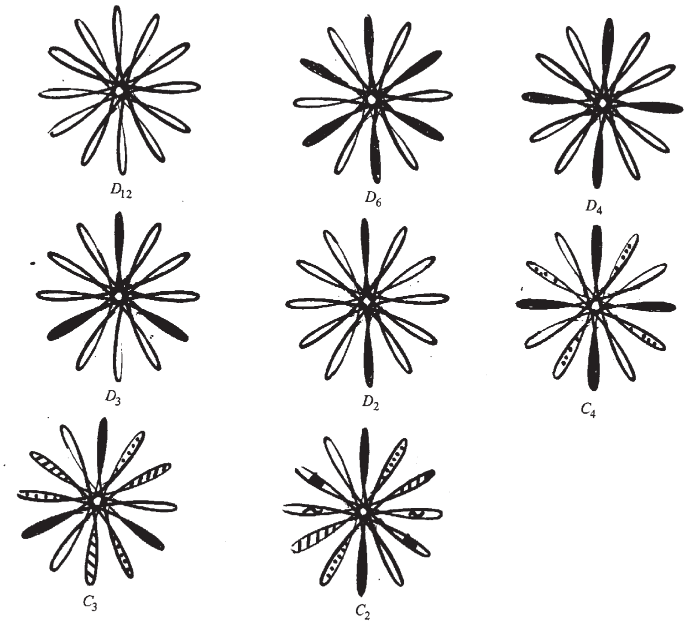

Figure 12
Colorations of 96/56 = 12/7

repeat in the same order until the whole design is colored. This coloration eliminates the reflections, but the m rotations of $(360/m)°$ remain. Notice that when $m = n/2$ or $m = n$, this pattern results in D_m instead of C_m.

While this type of coloring exercise is helpful to the group theory student, it is also meaningful at a more elementary level for a student studying symmetry.

There are other ways to generalize the hypotrochoid and epitrochoid. A spirograph set also contains a rounded square that can be used in place of an annulus. Figure 13 contains two designs made with a fixed rounded square and a circular rolling disc. For such designs, one can determine the kinds of possible patterns, the parametric equations, relationships between the reduced fraction of teeth and the number of cusps and period, and the possible symmetry groups.

Using a previous argument, the number of cusps will be the numerator and the period will be the denominator of the reduced fraction of teeth. Drawing a few designs demonstrates that the symmetry groups will not be D_n, where n is the number of cusps. The designs are patterned after a square, and the largest symmetry group possible is D_4. If the number of cusps is divisible by 4, the design can be divided into 4 equal parts and the symmetry group will be D_4. Likewise, if the number of cusps is divisible by 2 (but not 4), then the symmetry group will be D_2. If the number of cusps is odd, then the symmetry group is D_1, which consists of the identity and one reflection.

Another generalization is to let a and b be any positive real numbers. We have only looked at the designs one can make from a spirograph set, but these formulas work regardless of what kind of numbers a and b are. If they represent irrational numbers, then there won't be a period. Other types of designs occur when b is greater than a. One can get a feel for what designs are possible by going to one of the many Spirograph web sites. One such site can be found at the address:

http://www.wordsmith.org/-anu/java/spirograph.html#def

5. PROJECT QUESTIONS

1. Determine the polar equations for the hypotrochoid and epitrochoid.
2. Determine the parametric equations for a circle rolling inside and outside of a rounded square.
3. Determine conditions of a, b,and h so that singularities occur on a hypotrochoid.

6. REFERENCES

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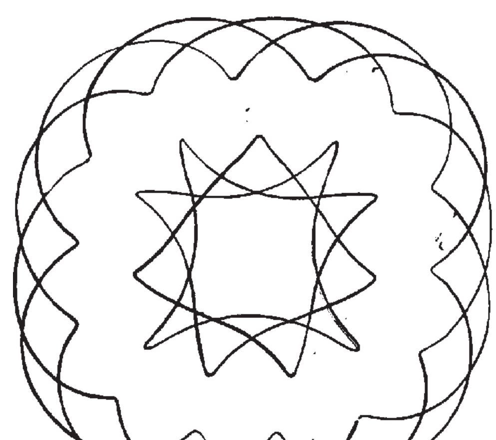

Figure 13
Design Using a Fixed Rounded Square
176/48 = 11/3, 224/48 = 14/3