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RINGS OF CONTINUOUS FUNCTIONS IN WHICH EVERY FINITELY GENERATED IDEAL IS PRINCIPAL(1)

BY

LEONARD GILLMAN AND MELVIN HENRIKSEN

An abstract ring in which all finitely generated ideals are principal will be called an **F**-*ring*. Let $C(X)$ denote the ring of all continuous real-valued functions defined on a completely regular (Hausdorff) space X. This paper is devoted to an investigation of those spaces X for which $C(X)$ is an **F**-ring.

In any such study, one of the problems that arises naturally is to determine the *algebraic* properties and implications that result from the fact that the given ring is a ring of functions. Investigation of this problem leads directly to two others: to determine how specified algebraic conditions on the ring are reflected in topological properties of the space, and, conversely, how specified topological conditions on the space are reflected in algebraic properties of the ring.

Our study is motivated in part by some purely algebraic questions concerning an arbitrary **F**-ring S—in particular, by some problems involving matrices over S. Continual application will be made of the results obtained in the preceding paper [4]. This paper will be referred to throughout the sequel as GH.

We wish to thank the referee for the extreme care with which he read both this and the preceding paper, and for making a number of valuable suggestions.

The outline of our present paper is as follows. In §1, we collect some preliminary definitions and results. §2 inaugurates the study of **F**-rings and **F**-spaces (i.e., those spaces X for which $C(X)$ is an **F**-ring).

The space of reals is not an **F**-space; in fact, a metric space is an **F**-space if and only if it is discrete. On the other hand, if X is any locally compact, σ-compact space (e.g., the reals), then $\beta X-X$ is an **F**-space. Examples of necessary and sufficient conditions for an arbitrary completely regular space to be an **F**-space are:

(i) for every $f\in C(X)$, there exists $k\in C(X)$ such that $f=k|f|$; (ii) for every maximal ideal M of $C(X)$, the intersection of all the prime ideals of $C(X)$ contained in M is a prime ideal.

In §§3 and 4, we study Hermite rings and elementary divisor rings(2).

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(2) For definition, see GH.

A necessary and sufficient condition that $C(X)$ be an Hermite ring is that for all $f, g \in C(X)$, there exist $k, l \in C(X)$ such that $f = k|f|$, $g = l|g|$, and $(k, l) = (1)$.

We also construct an **F**-ring that is not an Hermite ring, and an Hermite ring that is not an elementary divisor ring. To produce these examples, we translate the algebraic conditions on $C(X)$ into topological conditions on X, as indicated above. The construction of a ring having one algebraic property but not the other is then accomplished by finding a space that has the topological properties corresponding to the one, but not to the other.

In §§5 and 6, we investigate some further special classes of **F**-rings, including regular rings and adequate rings(2). Appendices (§§7 and 8) touch upon various related questions. A diagram is included to show the implications that exist among the principal classes of spaces that have been considered.

1. **Preliminary remarks.** Every topological space X considered herein is assumed to be a completely regular (Hausdorff) space(3). To avoid trivialities, we shall suppose that X contains at least two points. For every subset A of X, the closure of A (in X) is denoted by $\overline{A}$.

The ring of all continuous real-valued functions on X is denoted by $C(X)$, the subring of all bounded functions in $C(X)$ by $C^*(X)$.

DEFINITION 1.1. Let $f \in C(X)$; we define:

$$Z(f) = \{x \in X : f(x) = 0\};$$
$$P(f) = \{x \in X : f(x) > 0\};$$
$$N(f) = \{x \in X : f(x) < 0\}.$$

The set $Z(f)$ is called the *zero-set* of f, and is, of course, closed, while each of the sets $P(f)$, $N(f)$ is open.

Evidently, $f \in C(X)$ is a unit of $C(X)$ if and only if $Z(f) = \emptyset$. (This is not always the case, however, for $f \in C^*(X)$.) Hence for $f, g \in C(X)$, we have $(f, g) = (1)$ if and only if $Z(f) \cap Z(g) = \emptyset$ (as usual, $(f_1, \cdots, f_n)$ denotes the ideal generated by $f_1, \cdots, f_n$).

An ideal I of $C(X)$ is called *free* or *fixed* according as the set $\bigcap_{f \in I} Z(f)$ is empty or nonempty.

Every completely regular space X can be imbedded in a compact space βX, called the *Stone-Čech compactification* of X, and characterized by the following three properties [15; 2]: (a) βX is compact, (b) X is (homeomorphic with) a dense subspace of βX, and (c) every function in $C^*(X)$ has a (unique) continuous extension over all of βX. By (c), $C^*(X)$ is isomorphic with $C(\beta X)(= C^*(\beta X))$.

In discussions involving both a space X and its Stone-Čech compactifica-

(3) For references in topology, see [1; 10]; for algebra, see [14; 18]. For general background in rings of functions, see [8].

tion βX (with $X\neq\beta X$), we shall use the symbol A^β to denote the closure in βX of any subset A of βX (reserving $\overline{A}$ to denote the closure in X of a subset A of X).

LEMMA 1.2 (GELFAND-KOLMOGOROFF). *A subset M of $C(X)$ is a maximal ideal of $C(X)$ if and only if there is a unique point $p\in\beta X$ such that M coincides with the set*

$$M^p = \{f \in C(X): p \in Z(f)^\beta\}.$$

For a proof of this result, see [5, Theorem 1]. It is clear that the maximal ideal M^p is fixed or free according as $p\in X$ or $p\in\beta X-X$.

DEFINITION 1.3. Let p be any point of βX. The set of all $f\in C(X)$ for which there exists a neighborhood Ω of p such that $f(\Omega\cap X)=0$ is easily seen to constitute an ideal of $C(X)$; we denote this ideal by N^p. When $p\in X$, and when this fact deserves emphasis, we write N_p in place of N^p.

THEOREM 1.4. *Let X be a completely regular space, and let M^p be any maximal ideal of $C(X)$ (Lemma 1.2). Then the intersection of all the prime ideals of $C(X)$ that are contained in M^p is precisely the ideal N^p.*

Proof. Clearly, $Z(f^n)=Z(f)$ for every $f\in C(X)$ and every positive integer n. Hence if $f\notin N^p$, then $f^n\notin N^p$, whence by Zorn's lemma, there is a prime ideal containing N^p but not f (cf. [11, p. 105]). Therefore N^p is the intersection of all the prime ideals that contain it; and by [3, Lemma 3.2 and Theorem 3.3 ff.], these are precisely the prime ideals that are contained in M^p.

We close this section with several easy lemmas that will be found useful.

LEMMA 1.5. *Let X be any completely regular space. Then to every $f\in C(X)$, there correspond $f^*, f_0\in C(X)$, such that*

(i) $|f^*(x)|\leqq 1$ for all $x\in X$, and $f^*(x)=f(x)$ wherever $|f(x)|\leqq 1$,

(ii) f_0 is everywhere positive, and

(iii) $f=f^*f_0$—whence $f^*=(1/f_0)f$, so that f and f^* belong to the same ideals of $C(X)$.

Proof. Define $f^*(x)=f(x)$ if $|f(x)|\leqq 1$, $f^*(x)=1$ if $f(x)>1$, $f^*(x)=-1$ if $f(x)<-1$, $f_0(x)=1$ if $|f(x)|\leqq 1$, and $f_0(x)=|f(x)|$ if $|f(x)|>1$.

LEMMA 1.6. *Let X be any completely regular space, consider any function $\phi\in C(\beta X)$, and let f denote the restriction of ϕ to X. Then $P(f)^\beta=\overline{P}(f)^\beta=P(\phi)^\beta$ (and $N(f)^\beta=\overline{N}(f)^\beta=N(\phi)^\beta$).*

Proof. Since $P(f)\subset P(\phi)$, we have $\overline{P}(f)\subset P(f)^\beta\subset P(\phi)^\beta$. Hence $\overline{P}(f)^\beta\subset P(\phi)^\beta$. Conversely, let $p\in P(\phi)^\beta$. Then every neighborhood (in βX) of p meets $P(\phi)\cap X$ (since $P(\phi)$ is open, and X is dense in βX), hence contains points of $P(f)$. Therefore $P(\phi)^\beta\subset P(f)^\beta\subset\overline{P}(f)^\beta$.

LEMMA 1.7. *Let Y be a subspace of a completely regular space X such that every element of $C(Y)$ has a continuous extension to X. Then $C(Y)$ is a homomorphic image of $C(X)$.*

Proof. The mapping that takes every element of $C(X)$ to its restriction to Y is obviously a homomorphism of $C(X)$ into $C(Y)$. The postulated extension property implies that this homomorphism is onto.

The Tietze-Urysohn extension theorem shows that, in particular, the hypothesis of the lemma is fulfilled in case Y is compact, or in case X is normal and Y is closed. (For then Y is a closed subset of the normal space βX resp. X.)

LEMMA 1.8. *If a real-valued function is continuous on each of a finite number of closed subsets of a topological space, then it is continuous on their union.*

The proof of this well-known lemma is straightforward.

2. Rings in which every finitely generated ideal is principal.

DEFINITION 2.1. A commutative ring S with identity is called an **F**-*ring* if every finitely generated ideal of S is a principal ideal. A completely regular space X such that $C(X)$ is an **F**-ring is called an **F**-*space*(4).

In this section, we obtain several characterizations of **F**-spaces (Theorems 2.3, 2.5 and 2.6), and we construct some examples of these spaces.

It is not hard to see that every discrete space is an **F**-space.

Obviously, every homomorphic image of an **F**-ring is an **F**-ring. Hence, using Lemma 1.7, we have:

THEOREM 2.2. *Let Y be a subspace of an* **F**-*space X such that every element of $C(Y)$ has a continuous extension to X. Then Y is also an* **F**-*space.*

Two subsets A, B of a space X are said to be *completely separated* if there is a function $k \in C(X)$ such that $k(A)=0$ and $k(B)=1$ (whence also $\overline{A}$, $\overline{B}$ are completely separated). Čech [2] showed that completely separated subsets of X have disjoint closures in βX. Urysohn's classical theorem states that any two disjoint *closed* subsets of a *normal* space are completely separated.

Our next theorem, and others later, involve the function $|f|$. This function has the following algebraic significance: $|f|$ is the unique element g such that $g^2=f^2$, and $g+u^2$ is a unit for every unit u (uniqueness follows from the fact that any such g must be non-negative).

THEOREM 2.3. *For every completely regular space X, the following statements are equivalent.*

(a) *X is an* **F**-*space, i.e., every finitely generated ideal of the ring $C(X)$ is a principal ideal.*

(4) Our terminology seems convenient, even though some of our terms are used elsewhere with different meanings.

(a*) *βX is an* **F**-*space, i.e., every finitely generated ideal of the ring $C(\beta X)$ (or $C^*(X)$) is a principal ideal.*

(b) *For all $f, g \in C(X)$, the ideal (f, g) is the principal ideal $(|f|+|g|)$.*

(c) *For all $f \in C(X)$, the sets $P(f)$, $N(f)$ (or $\overline{P}(f)$, $\overline{N}(f)$) are completely separated.*

(d) *For all $f \in C(X)$, f is a multiple of $|f|$, i.e., $f=k|f|$ for some $k \in C(X)$ (whence $|f|=kf$).*

(e) *For all $f \in C(X)$, the ideal $(f, |f|)$ is principal.*

Proof. We first outline the proof, which is somewhat involved. We divide it into three parts.

In **I**, we establish the cycle of implications (c)→(d)→(e)→(c). (Incidentally, these implications are "local," i.e., they hold for any one function f.)

In the course of the remainder of the proof, we shall also have to deal with the following auxiliary propositions, concerning the space βX.

(b*) For all $\phi, \psi \in C(\beta X)$ (or $C^*(X)$), the ideal (ϕ, ψ) is the principal ideal $(|\phi|+|\psi|)$.

(c*) For all $\phi \in C(\beta X)$, the sets $P(\phi)$, $N(\phi)$ are completely separated.

(d*) For all $\phi \in C(\beta X)$, ϕ is a multiple of $|\phi|$.

Applying **I** to the space βX, we obtain the result that (c*) implies (d*).

In **II**, we establish the chain of implications (b*)→(b)→(a)→(c)→(c*). Applying this to the space βX, we obtain the chain (b*)→(a*)→(c*). (The parenthetical remarks in the statements (a*) and (b*) are justified by the fact that $C(\beta X)$ and $C^*(X)$ are isomorphic.)

Finally, in **III**, we establish the implication (c*)→(b*). This completes the two chains of **II** into cycles. On now combining all our results, we obtain the theorem.

I. (c) *implies* (d). By hypothesis, there is a function $k \in C(X)$ that is 1 everywhere on $P(f)$, and -1 on $N(f)$. Hence $f=k|f|$. (Likewise, (d) implies (c).)

(d) *implies* (e). Trivial.

(e) *implies* (c). By hypothesis, there is a $d \in C(X)$ such that $(f, |f|)=(d)$. Write $f=gd$, $|f|=hd$, and $d=sf+t|f|$. Then $d=(sg+th)d$. Therefore, since d has no zeros on $P(f) \cup N(f)$, we have $sg+th=1$ thereon. Next, $g=h$ on $P(f)$, and $g=-h$ on $N(f)$. Hence if we put

$$a_1 = sg + tg, \qquad a_2 = sh + th,$$
$$b_1 = sg - tg, \qquad b_2 = -sh + th,$$

then we have

$$a_1a_2 = 1 \text{ on } P(f), \qquad a_1a_2 \leqq 0 \text{ on } N(f),$$

$$b_1b_2 \leqq 0 \text{ on } P(f), \qquad b_1b_2 = 1 \text{ on } N(f).$$

Define

$$k = \max\{a_1a_2, 0\} - \max\{b_1b_2, 0\};$$

then $k=1$ on $P(f)$ and $k=-1$ on $N(f)$. Therefore the sets $P(f)$, $N(f)$ are completely separated.

II. (b*) *implies* (b). Consider any $f, g \in C(X)$. By Lemma 1.5, there exist $f^*, g^* \in C^*(X)$, and everywhere positive functions $f_0, g_0 \in C(X)$, such that $f=f^*f_0$, $g=g^*g_0$. By hypothesis, the ideal (f^*, g^*)—or $(|f^*|, |g^*|)$—of $C^*(X)$, is generated by the element $|f^*|+|g^*|$ of $C^*(X)$. Evidently, the ideal (f, g) of $C(X)$ is generated by this same element $(\in C(X))$. To show that this ideal is generated by the element $|f|+|g|$, it suffices to show that the elements $|f|+|g|$, $|f^*|+|g^*|$ are multiples of one another (in $C(X)$). Let $m \in C^*(X)$ satisfy $|f^*| = m(|f^*|+|g^*|)$. We may certainly suppose that $0 \leqq m \leqq 1$ everywhere. Then the element $u=f_0m+g_0(1-m)$ is everywhere positive, hence is a unit of $C(X)$. The observation that $|f|+|g| = u(|f^*| + |g^*|)$ now completes the proof.

(b) *implies* (a). Trivial.

(a) *implies* (c). Trivially, (a) implies (e), and from **I**, (e) implies (c).

(c) *implies* (c*). Consider any function $\phi \in C(\beta X)$. Let f denote its restriction to X. By hypothesis, the sets $P(f)$, $N(f)$ are completely separated. Now as remarked before, completely separated subsets of X have completely separated closures in βX. Hence, by Lemma 1.6, the sets $P(\phi)^\beta$, $N(\phi)^\beta$ are completely separated, q.e.d.

III. (c*) *implies* (b*). Consider any two functions $\phi, \psi \in C(\beta X)$; we are to show that $(\phi, \psi) = (|\phi|+|\psi|)$. Now as previously observed, our hypothesis (c*) implies the condition (d*). From this latter, it is clear that $(\phi, \psi) = (|\phi|, |\psi|)$. We may accordingly assume throughout the remainder of the proof that both ϕ and ψ are non-negative.

Define $\theta = \phi + \psi$. Then $\theta \in (\phi, \psi)$, so $(\theta) \subset (\phi, \psi)$. It remains, then, to show that $(\theta) \supset (\phi, \psi)$. To this end, it suffices to construct a $\phi_1 \in C(\beta X)$ such that $\phi = \phi_1\theta$ (for then $\psi = (1-\phi_1)\theta$). Since $\phi \geqq 0$ and $\psi \geqq 0$, we have $\theta \geqq 0$, and $\theta(x) = 0$ if and only if $\phi(x) = \psi(x) = 0$.

Define

$$\phi_1 = \frac{\phi}{\theta} \quad \text{on} \quad P(\theta). \tag{1}$$

Then ϕ_1 is continuous on $P(\theta)$. We shall first extend ϕ_1 to all of $P(\theta)^\beta$.

Consider any fixed $p \in P(\theta)^\beta - P(\theta)$. Then $\theta(p) = 0$. For every real r, define a function $\mu_r \in C(\beta X)$ by:

$$\mu_r(x) = \phi(x) - r\theta(x). \tag{2}$$

Obviously, if $r > s$, then $\mu_r(x) \leqq \mu_s(x)$ for every $x \in \beta X$ (since $\theta(x) \geqq 0$). Furthermore, $\mu_r(p) = 0$ for every real r.

For all $x \in \beta X$, we have $\mu_0(x) = \phi(x) \geqq 0$; and for all $x \in P(\theta)$, and every real $\epsilon > 0$, we have $\mu_{1+\epsilon}(x) \leqq -\epsilon\theta(x) < 0$. Therefore, since every neighborhood of p meets $P(\theta)$, we may put

$$(3) \quad \phi_1(p) = \sup \{r: \mu_r(x) \geqq 0 \text{ throughout some neighborhood of } p\} \qquad (p \in P(\theta)^\beta - P(\theta)).$$

The function ϕ_1 is now defined on all of $P(\theta)^\beta$.

To establish continuity of ϕ_1 on $P(\theta)^\beta$, it suffices to establish its continuity at any point $p \in P(\theta)^\beta - P(\theta)$. Write $\alpha = \phi_1(p)$. By (3), for every $r > \alpha$, and for every neighborhood U of p, there is an $x \in U$ such that $\mu_r(x) < 0$. Since the hypothesis (c*) applies to the function $\mu_r \in C(\beta X)$, the sets $P(\mu_r)^\beta$, $N(\mu_r)^\beta$ are disjoint. Consequently, since $\mu_r(p) = 0$, there is a neighborhood V of p such that $\mu_r(x) \leqq 0$ for all $x \in V$.

On the other hand, by (3), for every $s < \alpha$, there is a neighborhood W of p such that $\mu_s(x) \geqq 0$ for all $x \in W$. Thus, for every $\epsilon > 0$, there is a neighborhood U of p such that

$$\mu_{\alpha+\epsilon}(x) \leqq 0 \leqq \mu_{\alpha-\epsilon}(x) \qquad \text{for all } x \in U.$$

With the substitution (2), this reads:

$$\phi(x) - (\alpha + \epsilon)\theta(x) \leqq 0 \leqq \phi(x) - (\alpha - \epsilon)\theta(x) \quad \text{for all } x \in U.$$

If we further restrict x to lie in $P(\theta)$, then, on applying (1), this last reduces to:

$$| \phi_1(p) - \phi_1(x) | \leqq \epsilon \qquad \text{for all } x \in U \cap P(\theta).$$

From this, it follows further that $|\phi_1(p) - \phi_1(q)| \leqq 2\epsilon$ for all $q \in U - P(\theta)$. We now conclude that ϕ_1 is continuous on $P(\theta)^\beta$. Obviously, $\phi = \phi_1\theta$ thereon.

Finally, since βX is normal and $P(\theta)^\beta$ is closed, ϕ_1 can be extended continuously over all of βX. Since $\theta \geqq 0$, we have $\beta X - P(\theta)^\beta \subset Z(\theta) \subset Z(\phi)$. Therefore $\phi = \phi_1\theta$ everywhere on βX. This completes the proof of the theorem.

COROLLARY 2.4. *Any point of an* **F**-*space at which the first axiom of countability holds is an isolated point.*

Proof. If the first axiom of countability holds at a nonisolated point y of a space X, there is a denumerable subspace $Y = \{y_1, y_2, \cdots, y\}$ of X in which y is the only limit point. Define $f \in C(Y)$ by: $f(y_n) = (-1)^n/n$, $f(y) = 0$. Then $y \in \overline{P}(f) \cap \overline{N}(f)$. Therefore Y is not an **F**-space. But Y is compact. Hence, by Theorem 2.2 (see the remarks following Lemma 1.7), X is not an **F**-space.

In particular, a metric space is an **F**-space if and only if it is discrete.

THEOREM 2.5. *A completely regular space X is an* **F**-*space if and only if, for every maximal ideal M of $C(X)$, the intersection of all the prime ideals contained in M is a prime ideal—in other words, if and only if, for every point*

$p \in \beta X$, the ideal N^p of $C(X)$ (Definition 1.3) is a prime ideal.

The equivalence of the two formulations is a consequence of Theorem 1.4.

Proof. Obviously, an ideal I of $C(X)$ is prime if and only if $I \cap C^*(X)$ is a prime ideal of $C^*(X)$ (Lemma 1.5). Accordingly, in view of Theorem 2.3 (a, a*), there is no loss of generality in supposing that the space X is compact.

Assume, first, that X is not an **F**-space. Then there exists an $f \in C(X)$ such that the sets $\overline{P}(f)$, $\overline{N}(f)$ are not completely separated (Theorem 2.3(a, c)). Since X is normal, this means that the two sets are not disjoint. There accordingly exists a point $p \in Z(f)$ such that f changes sign on every neighborhood of p. Define $g = \max \{f, 0\}$, $h = \min \{f, 0\}$. Then $g \notin N^p$ and $h \notin N^p$, while $gh = 0 \in N^p$. Therefore the ideal N^p is not prime.

Conversely, suppose that there is a point $p \in X$ for which the ideal N^p is not prime. Then there exist $g, h \in C(X)$, and a neighborhood U of p, such that gh vanishes identically on U, while neither g nor h vanishes identically on any neighborhood of p. Hence if V is any neighborhood of p that is contained in U, there must exist $x, y \in V$ such that $g(x) \neq 0$, $h(y) \neq 0$. But then $h(x) = g(y) = 0$. The function $f = |g| - |h|$ $(\in C(X))$ therefore changes sign on V. Thus the sets $\overline{P}(f)$, $\overline{N}(f)$ are not disjoint (hence not completely separated). Therefore X is not an **F**-space (Theorem 2.3(a, c)).

THEOREM 2.6. *A completely regular space X is an* **F**-*space if and only if, for every zero-set Z of X, every function $\theta \in C^*(X - Z)$ has a continuous extension $h \in C^*(X)$.*

Proof. Suppose that the extension property holds, and consider any function f in $C(X)$. Define $\theta \in C^*(X - Z(f))$ as follows: $\theta(P(f)) = 1$, $\theta(N(f)) = 0$. The continuous extension of θ over all of X separates $P(f)$ from $N(f)$. Hence X is an **F**-space (Theorem 2.3(a, c)).

Conversely, let X be an **F**-space, and consider any zero-set Z of X. Say $Z = Z(f)$ $(f \in C(X))$. Since $Z(|f|) = Z(f)$, we may assume that f is non-negative. First, let θ be any *non-negative* function in $C^*(X - Z)$. Define g on X as follows: $g = f\theta$ on $X - Z$, $g(Z) = 0$. Since θ is *bounded*, g is continuous, i.e., $g \in C(X)$. Since both f and g are non-negative, we have $(f, g) = (f + g)$ (Theorem 2.3 (a, b)). Let $f_1 \in C(X)$ satisfy $f = f_1(f + g)$. Since $f + g = f(1 + \theta)$ on $X - Z$, we have $f_1 = 1/(1 + \theta)$ thereon. Since θ is bounded (on $X - Z$), f_1 is bounded away from zero on $X - Z$; say $f_1(x) \geqq \delta > 0$ for all $x \in X - Z$. Then the function $f_2 = \max \{f_1, \delta\}$ coincides with f_1 on $X - Z$, and is bounded away from zero everywhere on X. The function $h = (1/f_2) - 1$ is then a continuous extension of θ over X. In the general case, with θ an arbitrary element of $C^*(X - Z)$, we define $\theta_1 = \max \{\theta, 0\}$ and $\theta_2 = -\min \{\theta, 0\}$. Then θ_i has a continuous extension h_i over X, as just described, whence $h_1 - h_2$ is a continuous extension of θ over X, as required.

As a corollary, we observe that if Z is any zero-set of an **F**-space X, then

$X-Z$ is also an **F**-space. For $C^*(X)$ is an **F**-ring (Theorem 2.3(a, a*)); and the extension property of Theorem 2.6 implies that $C^*(X-Z)$ is a homomorphic image of $C^*(X)$. Hence $C^*(X-Z)$ is also an **F**-ring, whence, by Theorem 2.3 again, $X-Z$ is an **F**-space. (Cf. Theorem 2.2.)

We now introduce an extensive class of **F**-spaces. Here (and later) we shall make use of the well-known fact (which can be established without difficulty [10, p. 163]) that a (Hausdorff) space X is locally compact([5]) if and only if X is open in βX.

THEOREM 2.7. *For every locally compact, σ-compact*([6]) *space X, $\beta X-X$ is a compact* **F**-*space.*

Proof. Since X is locally compact, $\beta X-X$ is closed in the compact space βX, and is therefore compact. Since X is σ-compact, $\beta X-X$ is a closed G_δ in the normal space βX; therefore $\beta X-X$ is a zero-set of βX (see [2])—say $\beta X-X=Z(\tau)(\tau\in C(\beta X))$. We may suppose that $0\leqq\tau(p)\leqq 1$ for all $p\in\beta X$. Let t denote the restriction of τ to X. Then t vanishes nowhere, so $s=1/t$ is in $C(X)$.

Define $S_n=\{x\in X: n\leqq s(x)\leqq n+1\}$ $(n=1, 2, \cdots)$. We notice that each S_n is compact: for since τ vanishes precisely on $\beta X-X$, S_n is the closed subset $\tau^{-1}([1/(n+1), 1/n])$ of the compact space βX. Evidently, $\bigcup_n S_n=X$.

Consider any function $F\in C(\beta X-X)$; we are to show that the sets $P(F)$, $N(F)$ are completely separated (Theorem 2.3(a, c)). Let ϕ denote any continuous extension of F over all of βX, and denote the restriction of ϕ to X by f. For each $n=1, 2, \cdots$, define a function $e_n\in C(X)$ as follows:

$$e_n(x) = \begin{cases} 1/n \text{ if } f(x) \geqq 1/n, \\ f(x) \text{ if } |f(x)| \leqq 1/n, \\ -1/n \text{ if } f(x) \leqq -1/n; \end{cases} \tag{4}$$

and define e according to:

$$e(x) = (n+1-s(x))e_n(x) + (s(x)-n)e_{n+1}(x) \qquad (x \in S_n)$$

$(n=1, 2, \cdots)$. One may easily verify, with the aid of Lemma 1.8, that e is continuous everywhere on X. Since $|e_n(x)|\leqq 1/n$ on S_n, we see that $|e(x)|\leqq 1/n$ on S_n. Thus $e\in C^*(X)$. We also have:

$$\begin{cases} 0 < e_{n+1}(x) \leqq e(x) \leqq e_n(x) \quad \text{on} \quad S_n \cap P(f), \\ 0 = e_{n+1}(x) = e(x) = e_n(x) \quad \text{on} \quad S_n \cap Z(f), \\ 0 > e_{n+1}(x) \geqq e(x) \geqq e_n(x) \quad \text{on} \quad S_n \cap N(f). \end{cases} \tag{5}$$

We shall show first that the continuous extension ϵ of e over all of βX

([5]) Every locally compact (Hausdorff) space is completely regular.

([6]) A space is *σ-compact* if it is expressible as the union of denumerably many compact spaces.

vanishes everywhere on $\beta X - X$. Indeed, let p be any point of $\beta X - X$, and let n be any positive integer. There is a neighborhood Ω_n (in βX) of p that misses the closed set $\bigcup_{k=1}^{n} S_k$. Since $|e(x)| \geqq 1/n$ only on this set, we have $|e(x)| < 1/n$ on $\Omega_n \cap X$. It follows that we must have $\epsilon(p) = 0$.

Now define $f' = f - e$, and denote the continuous extension of f' over all of βX by ϕ'. Since X is dense in βX, the given functional relation is preserved in the extension, that is, we have $\phi' = \phi - \epsilon$. Since ϵ vanishes everywhere on $\beta X - X$, ϕ' coincides with ϕ thereon; thus ϕ' coincides on $\beta X - X$ with the originally given function $F \in C(\beta X - X)$.

We now show that the sets $P(f')$, $N(f')$ are completely separated. This will complete the proof. For then, by the theorem of Čech quoted at the beginning of this section, their closures in βX are completely separated. Thus, by Lemma 1.6, the sets $P(\phi')^\beta$, $N(\phi')^\beta$ are completely separated. Hence, *a fortiori*, so are the sets $P(F)$, $N(F)$, q.e.d.

Write $k_n = ne_n$ $(n = 1, 2, \cdots)$, and define k by

$$k(x) = (n + 1 - s(x))k_{n+1}(x) + (s(x) - n)k_{n+2}(x) \qquad (x \in S_n)$$

$(n = 1, 2, \cdots)$. One may easily verify, with the aid of Lemma 1.8, that k is continuous on all of X. Now let x be any point of $P(f')$. Then $f(x) > e(x)$.

Let m be such that $x \in S_m$. By (5), either $f(x) > e_{m+1}(x)$ or $f(x) > e_m(x)$. Now by (4), for any n, $f(x) > e_n(x)$ implies that $f(x) > 1/n$. Hence, in either of the preceding cases, we find that $f(x) > 1/(m+1)$. Thus, again by (4), we have $e_{m+1}(x) = 1/(m+1)$, and $e_{m+2}(x) = 1/(m+2)$. Therefore $k_{m+1}(x) = k_{m+2}(x) = 1$. Hence $k(x) = 1$. Thus $k = 1$ on $P(f')$. Similarly, $k = -1$ on $N(f')$. Therefore k separates these two sets, as required.

We have observed that every discrete space is an **F**-space. Now, with the help of the preceding theorem, we can construct a *connected* **F**-space.

EXAMPLE 2.8. *A compact connected* **F**-*space*. Let R^+ denote the space of non-negative reals. By the preceding theorem, $\beta R^+ - R^+$ is a compact **F**-space. We shall show that it is connected. Suppose the contrary, and let $F \in C(\beta R^+ - R^+)$ assume each of the values 0 and 1, but no other values. Let ϕ denote any continuous extension of F over all of βR^+. Then ϕ assumes values arbitrarily near to 0, and values arbitrarily near to 1, at arbitrarily large $x \in R^+$. Since R^+ is connected, ϕ assumes the value $1/2$ at arbitrarily large $x \in R^+$. Therefore $\phi(p) = 1/2$ for at least one $p \in \beta R^+ - R^+$, a contradiction.

3. Hermite rings and T-spaces. A completely regular space X is called a **T**-space if the ring $C(X)$ is an Hermite ring (for definition, see GH, Theorem 2). Hence every **T**-space is an **F**-space (GH, Theorem 2 ff.). In this section, we obtain necessary and sufficient conditions that a space be a **T**-space, we construct a connected **T**-space, and we present an example of an **F**-space that is *not* a **T**-space. This last yields the algebraic result that *the condition that all finitely generated ideals be principal is not sufficient to insure that a com-*

mutative ring with identity be an Hermite ring. While this result is not surprising, it is new as far as we know.

Clearly, every homomorphic image of an Hermite ring is an Hermite ring. Hence, using Lemma 1.7, we have:

THEOREM 3.1. *Let Y be a subspace of a* **T**-*space X such that every element of $C(Y)$ has a continuous extension to X. Then Y is also a* **T**-*space.*

The main theorem on **T**-spaces is:

THEOREM 3.2. *For every completely regular space X, the following statements are equivalent.*

(a) *X is a* **T**-*space ($C(X)$ is an Hermite ring), i.e., for all $f, g \in C(X)$, there exist $f_1, g_1, h \in C(X)$ such that $f = f_1 h$, $g = g_1 h$, and $(f_1, g_1) = (1)$ (see GH, Theorem 3).*

(a*) *βX is a* **T**-*space ($C^*(X)$ is an Hermite ring).*

(b) *For all $f, g \in C(X)$, there exist $k, l \in C(X)$ such that $f = k|f|$, $g = l|g|$, and $(k, l) = (1)$.*

Proof. We establish the cycle of implications (a*)→(a)→(b)→(a*).

(a*) *implies* (a). Note first that the parenthetical statement in (a*) is justified by the fact that $C(\beta X)$ and $C^*(X)$ are isomorphic.

Consider any $f, g \in C(X)$. By Lemma 1.5, there exist $f^*, g^* \in C^*(X)$, and units f_0, g_0 of $C(X)$, such that $f = f^* f_0$ and $g = g^* g_0$. By hypothesis, there exist $f', g', h, s', t' \in C^*(X)$ such that $f^* = f'h$, $g^* = g'h$, and $s'f' + t'g' = 1$. Define $f_1 = f'f_0$, $g_1 = g'g_0$, $s = s'/f_0$, $t = t'/g_0$. Then $f = f_1 h$, $g = g_1 h$, and $sf_1 + tg_1 = 1$, as required.

(a) *implies* (b). Consider any $f, g \in C(X)$. Since X is a **T**-space, it is an **F**-space; therefore $(f, g) = (|f| + |g|)$ (Theorem 2.3(a, b)). Let f_1, g_1, h be as in (a); in virtue of GH, Lemma 4, we may suppose that $h = |f| + |g|$. In particular, then, $h \geqq 0$, so we have $P(f) \subset P(f_1)$, and $N(f) \subset N(f_1)$. Moreover, $|f_1(x)| \leqq 1$ wherever $f(x) \neq 0$, so by Lemma 1.5, we may assume that $|f_1(x)| \leqq 1$ everywhere.

The sets $P(f_1)$, $N(f_1)$ are completely separated (Theorem 2.3(a, c)); let $s \in C(X)$ be such that $s(P(f_1)) = 1$, $s(N(f_1)) = 0$. Let $m \in C(X)$ satisfy $f = m|f|$ (Theorem 2.3(a, d)). Now define

$$k = s \max \{m, f_1\} + (1 - s) \min \{m, f_1\}.$$

Then $k \in C(X)$, $f = k|f|$, and $Z(k) \subset Z(f_1)$. Similarly, define $l \in C(X)$ such that $g = l|g|$ and $Z(l) \subset Z(g_1)$. Since $Z(f_1) \cap Z(g_1) = \varnothing$, we have $Z(k) \cap Z(l) = \varnothing$.

(b) *implies* (a*). Given $\phi, \psi \in C(\beta X)$, we are to find $\phi_1, \psi_1, \theta \in C(\beta X)$ such that $\phi = \phi_1 \theta$, $\psi = \psi_1 \theta$, and $(\phi_1, \psi_1) = (1)$. Let f, g denote the restrictions of ϕ, ψ, resp., to X. We shall find $f_1, g_1, h \in C^*(X)$ as in (a), and, in addition, such that $|f_1| + |g_1|$ is bounded away from zero; their continuous extensions ϕ_1, ψ_1, θ to βX will then be as required.

Let k, l be as in (b). By (b) again, there exist $s, t \in C(X)$ such that $k = s|k|$, $l = t|l|$, and $(s, t) = (1)$. We may evidently suppose that s and t are bounded. Clearly, $f = s|f|$, $g = t|g|$.

Next, let $h = |f| + |g|$. Then there is an $f' \in C(X)$ such that $|f| = f'h$ (Theorem 2.3(b, d)). We may assume that f' is bounded. Now define

$$u = \frac{|s|}{|s| + |t|}(1 - |t| + 2|t|f').$$

Then $|f| = uh$. To see this, note that $|s| = 1$ where $f \neq 0$, $|t| = 1$ where $g \neq 0$, $f' = 0$ where $f = 0$ but $g \neq 0$, and $f' = 1$ where $g = 0$ but $f \neq 0$. With these substitutions, we find that $u = f'$ wherever $h \neq 0$. It follows that $|f| = uh$ everywhere.

Observe, further, that $u = 0$ where $s = 0$, that $u = 1$ where $t = 0$, and that $u \in C^*(X)$.

Now let $p, q \in C^*(X)$ satisfy $s = p|s|$, $t = q|t|$. Then $f = p|f|$, $g = q|g|$. Define $f_1 = pu$, $g_1 = q(1 - u)$. We shall first verify that $|f_1| + |g_1|$ is bounded away from zero. Where $|p| < 1$, we have $s = 0$; hence $u = 0$; also, $t \neq 0$, so $|q| = 1$; therefore $|g_1| = 1$. And where $|u| < 1$, we have $t \neq 0$, so $|q| = 1$; hence where $|u| \leqq 1/2$, we have $|g_1| \geqq 1/2$. It follows that $|g_1| \geqq 1/2$ wherever $|f_1| \leqq 1/2$. Thus $|f_1| + |g_1| \geqq 1/2$ everywhere.

Finally, we have $f_1 h = puh = p|f| = f$, and $g_1 h = q(h - uh) = q(h - |f|) = q|g| = g$. This completes the proof of the theorem.

EXAMPLE 3.3. *A compact connected* **T**-*space*. The space $\beta R^+ - R^+$ of Example 2.8 is a compact and connected **F**-space. We shall show that it is a **T**-space. Let $F, G \in C(\beta R^+ - R^+)$. By Theorem 3.2, it suffices to find K, $L \in C(\beta R^+ - R^+)$ such that $F = K|F|$, $G = L|G|$, and $(K, L) = (1)$.

The proof of Theorem 2.7 shows how to construct a function $f \in C^*(R^+)$ (called f' there) whose continuous extension to βR^+ coincides with F on $\beta R^+ - R^+$, and such that $\overline{P}(f) \cap \overline{N}(f) = \varnothing$. If f never changes sign on R^+, then F never changes sign on $\beta R^+ - R^+$. Then $F = K|F|$ for $K = \pm 1$. Since $\beta R^+ - R^+$ is an **F**-space, there is an L such that $G = L|G|$. Then $(K, L) = (1)$.

Henceforth, then, we shall assume that f does change sign on R^+. Let us designate as an *f-interval* any closed interval $\subset Z(f)$ one of whose end points is in $\overline{P}(f)$ and the other in $\overline{N}(f)$. Since $\overline{P}(f) \cap \overline{N}(f) = \varnothing$, there is an f-interval between any two points at which f has opposite signs. Only finitely many f-intervals are contained in any bounded set, for a limit point of f-intervals would be in both $\overline{P}(f)$ and $\overline{N}(f)$.

Correspondingly, we find a g, and define its g-intervals. As above, we shall assume that g changes sign on R^+.

We now define functions $k, l \in C^*(R^+)$. First, with every f-interval $I = [a, b]$, we associate a subinterval $I' = [a', b']$, as follows. If I is entirely contained in some g-interval, we take I' to be the middle third of I; if not, we take it to be the middle third of some subinterval of I that is disjoint from every g-interval. We define $k(a')=\pm 1$, and $k(b')=\mp 1$, according as $a\in\overline{P}(f)$ or $\overline{N}(f)$, and we take k to be linear on I'. Having thus defined k on I' for every f-interval I, we extend k continuously so that $|k|=1$ on the remainder of R^+. Then $k=1$ on $P(f)$ and -1 on $N(f)$. Hence $\kappa=1$ on $P(f)^\beta$ and -1 on $N(f)^\beta$, where κ denotes the continuous extension of k over βR^+. Therefore, denoting the restriction of κ to βR^+-R^+ by K, we see that $F=K|F|$.

Next, any g-interval J contains only finitely many f-intervals. We choose as J' the middle third of some subinterval of J on which $|k|=1$ (cf. the construction of k). Then we define l to be 1 and -1 at the end points of J', and linear on J', and then ± 1 elsewhere, analogously to the definition of k. We extend l to λ on βR^+, restrict λ to L on βR^+-R^+, and we have $G=L|G|$. By our construction, there is no point of R^+ at which both $|k|$ and $|l|$ are $\leqq 1/2$. Therefore there is no point of βR^+-R^+ at which both $|K|$ and $|L|$ are $\leqq 1/2$. Hence $(K, L)=(1)$. This completes the proof that βR^+-R^+ is a **T**-space.

EXAMPLE 3.4. *An* **F**-*ring that is not an Hermite ring*. Let X denote the strip of the euclidean plane consisting of all points (x, y) for which $x\geqq 0$ and $|y|\leqq 1$. We shall show that $\beta X-X$ is an **F**-space, but not a **T**-space.

In fact, we know from Theorem 2.7 that $\beta X-X$ is a compact **F**-space. To show that it is not a **T**-space, we shall find functions F and $G\in C(\beta X-X)$ such that, for all K and $L\in C(\beta X-X)$ satisfying $F=K|F|$, $G=L|G|$, we have $(K, L)\neq(1)$ (Theorem 3.2).

We define F and G as follows. Let $f, g\in C^*(X)$ be given by: $f(x, y)=y$, $g(x, y)=\cos \pi x$. Let ϕ, ψ denote the continuous extensions of f, g, respectively, over all of βX. Then F, G are taken to be the restrictions to $\beta X-X$ of ϕ, ψ, respectively.

Let A denote the subset $\{(x, 1): x\geqq 0\}$ of X, and B the subset $\{(x, -1): x\geqq 0\}$. Since $f=1$ on A, we have $F=1$ on $A^\beta-X$. Likewise, $F=-1$ on $B^\beta-X$. Let K be any element of $C(\beta X-X)$ such that $F=K|F|$. Then $K=1$ on $P(F)$; therefore $K=1$ on $A^\beta-X$. Likewise, $K=-1$ on $B^\beta-X$.

Since $\beta X-X$ is closed, K has a continuous extension κ defined over all of βX. Let k denote the restriction of κ to X. Then k must approach 1 on the set A, as $x\rightarrow\infty$. Likewise, $k\rightarrow -1$ on B. Hence there is a number $x_0\geqq 0$ such that $k(x, 1)\geqq 1/2$, and $k(x, -1)\leqq -1/2$, for all $x\geqq x_0$.

Next, let V_x denote the vertical line segment at x: $V_x=\{(x, y): |y|\leqq 1\}$. Let L be any function in $C(\beta X-X)$ such that $G=L|G|$, let λ be any continuous extension of L over all of βX, and let l denote the restriction of λ to X. As above, we see that there is an integer $n_0\geqq x_0$ such that, for all $n\geqq n_0$, we have $l(p)\geqq 1/2$ for all $p\in V_{2n}$, and $l(q)\leqq -1/2$ for all $q\in V_{2n+1}$.

A simple topological argument shows that for every $n\geqq n_0$, the rectangle cut off from X by the lines V_{2n}, V_{2n+1} contains a common zero of k and l(7).

(7) The result may be inferred from [9, p. 43, B].

Hence the set $Z = Z(k) \cap Z(l)$ contains points (x, y) with arbitrarily large x. Therefore $Z(K) \cap Z(L)$ contains the nonempty set $Z^\beta - X$. Thus, $(K, L) \neq (1)$.

4. Elementary divisor rings and D-spaces. A completely regular space X is called a $\mathbf{D}'$-*space* if the ring $C(X)$ satisfies the condition $\mathbf{D}'$ of GH, Theorem 6; X is called a $\mathbf{D}$-*space* if $C(X)$ is an elementary divisor ring (for definition, see GH, Theorem 2). Hence X is a $\mathbf{D}$-space if and only if it is both a $\mathbf{T}$-space and a $\mathbf{D}'$-space (GH, Theorem 6).

In the present section, we obtain a necessary condition that a space X be a $\mathbf{D}'$-space, we construct a connected $\mathbf{D}$-space, and we present an example of a $\mathbf{T}$-space that is *not* a $\mathbf{D}$-space. This last yields the algebraic result that *not every Hermite ring is an elementary divisor ring*. Again, as far as we know, this result is new.

Clearly, every homomorphic image of an elementary divisor ring is an elementary divisor ring. Hence, using Lemma 1.7, we have:

THEOREM 4.1. *Let Y be a subspace of a* $\mathbf{D}$-*space X such that every element of $C(Y)$ has a continuous extension to X. Then Y is also a* $\mathbf{D}$-*space.*

We do not know whether $C(X)$ an elementary divisor ring implies $C^*(X)$ an elementary divisor ring, or conversely.

The condition $\mathbf{D}'$ seems of little significance in itself, without $\mathbf{T}$. Nevertheless, we shall find a use for the result, now to be established, that βR^+ is a $\mathbf{D}'$-space (even though, obviously, it is not even an $\mathbf{F}$-space).

LEMMA 4.2. *The space βR^+ (where R^+ denotes the non-negative reals) is a* $\mathbf{D}'$-*space.*

Proof. Let $\phi, \psi, \theta \in C(\beta R^+)$, with $(\phi, \psi, \theta) = (1)$. By compactness, these functions are bounded in absolute value, say by 1; also, there is a number $\delta > 0$ such that $|\phi| + \psi^2 + \theta^2 \geqq 3\delta$. Denote the restrictions to R^+ by f, g, h, resp. Let

$$S = \{x \in R^+ : |f(x)| \leqq \delta\};$$

then $g^2 + h^2 \geqq 2\delta$ on S. Cover S with an open set U such that $|f(x)| \leqq 2\delta$ everywhere on U. Express U as the union of disjoint open intervals (the components of U). By adding to the set U any point that is a common end point of two such intervals, we secure the condition that the components (of the enlarged set) have disjoint closures. Let V denote the union of those components that meet S(8). Only finitely many components of V can be contained in any bounded set: for a limit point of components of V would be in the closed set S, hence in V, and the component containing this point would meet other components of V.

We shall first define two auxiliary functions, s' and t', on $\overline{V}$. For every

(8) If S is empty, then ϕ is a unit, and the whole problem is trivial.

component (a, b) of V[9], we proceed as follows. Since V is open, there exist a', b' such that $a<a'<b'<b$, and with $|f|>\delta$ on (a, a') and on (b', b). We shall work first with $[a', b]$. Define $s'=g$ on $[a', b']$. Define s' on $[b', b]$ so that it is continuous there, and subject to the following.

(i) If $g\equiv 0$ on $[b', b]$, define $s'(b)=1$, and let s be arbitrary on (b', b), subject to continuity on $[b', b]$, and the condition $|s'|\leqq 1$.

(ii) If there is a $b_1\in(b', b)$ for which $g(b_1)>0$, choose $b_2\in(b_1, b]$ such that $g>0$ everywhere on $[b_1, b_2]$. Then construct s' on $[b', b]$ so that $s'=g$ on $[b', b_1]$, $g(x)\leqq s'(x)\leqq 1$ for all $x\in[b_1, b_2]$, and $s'\equiv 1$ on $[b_2, b]$.

(iii) If neither of these possibilities occurs, construct s' so that $s'(b)=-1$, and with $g(x)\geqq s'(x)\geqq -1$ for all $x\in[b', b]$.

Define $t'=h$ on $[a', b]$. We shall show that $|s'f|+|s'g+t'h|\geqq\delta$ on $[a', b]$.

On $[a', b']$, we have $s'g+t'h=g^2+h^2\geqq\delta$. On $[b', b]$, we consider the three cases. In (i), we have $s'g+t'h=g^2+h^2\geqq\delta$. In (ii), we consider the three subintervals. On $[b', b_1]$, we have $s'g+t'h=g^2+h^2\geqq\delta$; on $[b_1, b_2]$, we have $s'g+t'h\geqq g^2+h^2\geqq\delta$; on $[b_2, b]$, we have $|s'f|=|f|\geqq\delta$. Case (iii) is similar to the second subcase of (ii).

Now, on $[a, a']$, we define $t'=h$, and we define s' in a manner analogous to its definition on $[b', b]$. We then have $|s'f|+|s'g+t'h|\geqq\delta$ everywhere on $[a, b]$.

We are now prepared to define s and t. Choose any component (a, b) of V. Define $s=s'$ and $t=t'$ on $[a, b]$. In case no component of V follows (a, b), define $s\equiv s'(b)$ (and, say, $t\equiv t'(b)$) on (b, ∞); then $|sf|=|f|\geqq\delta$ there.

Otherwise, let (c, d) be the next following component of V. Define $s=\pm s'$ and $t=\pm t'$ on $[c, d]$, according as $s'(c)=\pm s'(b)$. Define $s\equiv s'(b)$ on $[b, c]$, and let t be arbitrary on $[b, c]$, subject to continuity and the condition $|t|\leqq 1$. Then on $[b, c]$, we have $|sf|=|f|\geqq\delta$.

It is now clear how to define $s, t\in C^*(R^+)$ so that $|sf|+|sg+th|\geqq\delta$ everywhere on R^+. Their continuous extensions σ, τ to βR^+ therefore satisfy $Z(\sigma\phi)\cap Z(\sigma\psi+\tau\theta)=\emptyset$, i.e., $(\sigma\phi, \sigma\psi+\tau\theta)=(1)$, q.e.d.

We remark that a simplification of the foregoing proof shows that R^+ itself is also a $\mathbf{D}'$-space. And both proofs generalize to arbitrary linearly ordered spaces, although there they are more complicated.

Let a, b, c be elements of a commutative ring S with identity; by the symbol $((a, b, c))$, we shall mean that there exist $p, q\in S$ such that $(pa, pb+qc)=(1)$. Thus X is a $\mathbf{D}'$-space if and only if $((f, g, h))$ holds for all $f, g, h\in C(X)$ for which $(f, g, h)=(1)$.

LEMMA 4.3. *Let $f, g, h\in C(X)$, with $(f, g, h)=(1)$. Suppose that there exist a connected subset Z_f of $Z(f)$, and a connected subset Z_h of $Z(h)$, whose intersection meets both $P(g)$ and $N(g)$. Then $((f, g, h))$ fails (whence X is not a $\mathbf{D}'$-space).*

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[9] In case $a=0$ or $b=\infty$, some obvious modifications must be made in the proof.

Proof. Let $x \in Z_f \cap Z_h \cap P(g)$, and $y \in Z_f \cap Z_h \cap N(g)$. Consider any $s, t \in C(X)$. If $(s, h) \neq (1)$, then $(sf, sg+th) \neq (1)$. If $(s, h) = (1)$, then s is of one sign on the connected subset Z_h of $Z(h)$; then $sg+th$ has opposite signs at x and y, hence has a zero on the connected subset Z_f of $Z(f)$, whence again $(sf, sg+th) \neq (1)$.

DEFINITION 4.4. A completely regular space X is called a **C**-*space* if the intersection of any two closed connected subsets of X is connected(10).

LEMMA 4.5. *Every normal* $\mathbf{D}'$-*space is a* **C**-*space.*

Proof. Let X be a normal space that is not a **C**-space. Then there exist two closed connected sets, Z_f and Z_h, whose intersection is not connected. Write $Z_f \cap Z_h = A \cup B$, where A, B are disjoint nonempty closed subsets of $Z_f \cap Z_h$, hence closed subsets of X. Since X is normal, there are open sets $U_A \supset A$, and $U_B \supset B$, whose closures are disjoint. Put $U = U_A \cup U_B$. The closed sets $Z_f - U$, $Z_h - U$ are disjoint, hence are contained in disjoint open sets V_f, V_h, resp. There exist $f, g, h \in C(X)$ such that $f(Z_f) = 0$ and $f(X - V_f - U) = 1$, $h(Z_h) = 0$ and $h(X - V_h - U) = 1$, $g(\overline{U}_A) = 1$ and $g(\overline{U}_B) = -1$. Then f, g, h satisfy the hypotheses of Lemma 4.3, whence it follows that X is not a $\mathbf{D}'$-space.

Obviously, every linearly ordered space is a **C**-space (as well as a normal $\mathbf{D}'$-space). The converse of Lemma 4.5 is false, however, as is shown by the example of the following noncompact subset of the plane: the union of the sequence of segments $\{(x, -1/n) : |x| \leqq 1\}$ $(n = 1, 2, \cdots)$ with the semicircle $y = (1 - x^2)^{1/2}$. (The reasoning is like that in Lemma 4.3.) We conjecture but have been unable to prove that every *compact* **C**-space is a $\mathbf{D}'$-space.

The proofs of the next two lemmas are easy and are therefore omitted.

LEMMA 4.6. *Let a, b, c be elements of a commutative ring with identity. Then* $((a, b, c))$ *if and only if* $((c, b, a))$.

LEMMA 4.7. *For any* $f, g, h \in C(X)$, *the following are mutually equivalent:* $((f, g, h))$, $((|f|, g, h))$, $((f, g, |h|))$, $((|f|, g, |h|))$.

LEMMA 4.8. *Every compact subspace of a normal* $\mathbf{D}'$-*space is a* $\mathbf{D}'$-*space.*

Proof. Let Y be a compact subspace of a normal $\mathbf{D}'$-space X, and let $F, G, H \in C(Y)$, with $(F, G, H) = (1)$. We are to show that $((F, G, H))$. By Lemma 4.7, we may assume that $F \geqq 0$. Let g, h be arbitrary continuous extensions to X of G, H, resp., and let F' denote the restriction of F to $Z(G) \cap Z(H)$. There is a $\delta > 0$ such that $F \geqq \delta$ on $Z(G) \cap Z(H)$. Therefore F' can be continuously extended to a function ϕ, defined on all of $Z(g) \cap Z(h)$, and $\geqq \delta$ there. Now define f' on the set $Y \cup (Z(g) \cap Z(h))$, by setting $f' = F$ on Y, and $f' = \phi$ on $Z(g) \cap Z(h)$. Evidently, f' is well-defined and continuous. Finally, let f be any continuous extension of f' over all of X. Obviously, $(f, g, h) = (1)$.

(10) Cf. the concept of *unicoherence* [14].

Therefore, by hypothesis, we have $((f, g, h))$. Restriction to Y yields $((F, G, H))$.

EXAMPLE 4.9. *A compact connected* **D**-*space.* The space $\beta R^+ - R^+$ of Example 3.3 is a compact connected **T**-space. Since βR^+ is a **D**′-space (Lemma 4.2), so is $\beta R^+ - R^+$ (Lemma 4.8). Therefore $\beta R^+ - R^+$ is a **D**-space.

Applying Lemma 4.5, we obtain:

COROLLARY 4.10. *The intersection of any two closed connected subsets of the connected space $\beta R^+ - R^+$ is a connected set.*

EXAMPLE 4.11. *An Hermite ring that is not an elementary divisor ring.* We may regard R^+ as the subset $\{(x, 0): x \geqq 0\}$ of the plane. Let

$$S^+ = \{(x, \sin \pi x): x \geqq 0\},$$

and define $X = R^+ \cup S^+$. (Notice that X is not a **C**-space.) We shall show that $\beta X - X$ is a **T**-space, but not a **D**-space.

First of all, by Theorem 2.7, $\beta X - X$ is an **F**-space. The fact that it is also a **T**-space may be established by an evident extension of the argument given in Example 3.3 to show that $\beta R^+ - R^+$ is a **T**-space. To show, finally, that $\beta X - X$ is not a **D**′-space, it is sufficient, by Lemma 4.5, to show that it is not a **C**-space. We shall make use of Čech's theorem that if Y is a closed subset of a normal space X, then the closure of Y in βX is identical with βY[2]. Thus, $(R^+)^\beta$ is identical with βR^+. Therefore the set $Z_F = (R^+)^\beta - R^+$ is closed and connected (Example 2.8). Likewise, $Z_H = (S^+)^\beta - S^+$ is closed and connected.

Now consider the function $g \in C^*(X)$ defined by $g(x, y) = \cos \pi x$, and let ψ denote the continuous extension of g to βX. Since $g(n, 0) = (-1)^n$ $(n = 1, 2, \cdots)$, ψ assumes both the values 1 and -1 on $Z_F \cap Z_H$. Now the set of all $p \in R^+$ for which $|g(p)| \leqq 1/2$ is a closed subset of the normal space X that is disjoint from the closed set S^+; therefore these two sets have disjoint closures in βX. Hence ψ has no zeros on $Z_F \cap Z_H$. Thus $Z_F \cap Z_H$ is not connected. Therefore $\beta X - X$ is not a **C**-space, q.e.d.

5. U-rings and U-spaces.

DEFINITION 5.1. Let X be a completely regular space. The ring $C(X)$ (resp. $C^*(X)$) is called a **U**-*ring* if for every $f \in C(X)$ (resp. $C^*(X)$), f and $|f|$ are associates, i.e., there is a *unit* u of $C(X)$ (resp. $C^*(X)$) such that $f = u|f|$ (whence $|f| = uf$). If $C(X)$ is a **U**-ring, then X is called a **U**-*space.*

As already pointed out, u is a unit of $C(X)$ if and only if $Z(u)$ is empty. Thus every discrete space is a **U**-space. Obviously, every **U**-space is an **F**-space (Theorem 2.3(a, d)).

THEOREM 5.2. *For any completely regular space X, $C(X)$ is a* **U**-*ring if and only if $C^*(X)$ is a* **U**-*ring. Equivalently, X is a* **U**-*space if and only if βX is a* **U**-*space.*

The equivalence of the formulations is a consequence of the isomorphism between $C(\beta X)$ and $C^*(X)$.

Proof. Suppose, first, that $C^*(X)$ is a **U**-ring, and consider any $f \in C(X)$. Let f^*, f_0 be as in Lemma 1.5. By hypothesis, $f^* = u|f^*|$ for some u that is a unit of $C^*(X)$, hence a unit of $C(X)$. Since $f = f^* f_0$, and $f_0 > 0$, we have $f = u|f|$.

Conversely, suppose that $C(X)$ is a **U**-ring, and consider any $\phi \in C^*(X) \subset C(X)$. By hypothesis, $\phi = u|\phi|$ for some unit u of $C(X)$. Define $v \in C^*(X)$ by: $v(P(u)) = 1$, $v(N(u)) = -1$. Then v is a unit of $C^*(X)$, and $\phi = v|\phi|$.

THEOREM 5.3. *Every* **U**-*space is a* **D**-*space* (*hence a* **T**-*space and an* **F**-*space*).

Proof. We must show that every **U**-ring is an Hermite ring that satisfies the condition **D**$'$ of GH, Theorem 6. Consider any 1 by 2 matrix $[f\,g]$. Applying **U**, we see from Theorem 2.3(b, d) that $(|f|, |g|) = (|f| + |g|)$. Let u, v be units satisfying $|f| = uf$ and $|g| = vg$, let g_1 satisfy $|g| = g_1(|f| + |g|)$, and define

$$P = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & -g_1 \\ 0 & 1 \end{bmatrix}.$$

Then PQR is nonsingular, and $[f\,g]PQR = [|f| + |g| \;\; 0]$. Thus every 1 by 2 matrix can be diagonalized. Therefore $C(X)$ is an Hermite ring (GH, Theorem 2 ff.). Now consider any f, g, h with $(f, g, h) = (1)$. By **U**, we have $|g| = sg$, $|h| = th$, where s and t are suitable units. Obviously, $(sf, sg + th) = (1)$. Therefore **D**$'$ holds.

The following lemma is well known. We omit the proof, which is straightforward.

LEMMA 5.4. *For any completely regular space X, the following statements are equivalent.*

(a) *Any two completely separated subsets of X are contained in disjoint open and closed sets.*

(b) *βX is zero-dimensional*(11).

THEOREM 5.5. *An* **F**-*space X is a* **U**-*space if and only if βX is zero-dimensional.*

Proof. Recall that X is an **F**-space if and only if βX is an **F**-space (Theorem 2.3(a, a*)).

Assume, first, that βX is a zero-dimensional **F**-space, and consider any $f \in C(X)$. By Theorem 2.3(c), the sets $P(f)$, $N(f)$ are completely separated. By Lemma 5.4, there is an open and closed subset V of X that contains $P(f)$ but is disjoint from $N(f)$. Let u satisfy: $u(V) = 1$, $u(X - V) = -1$. Then u is a unit of $C(X)$, and $f = u|f|$. Hence X is a **U**-space.

Conversely, assume that X is a **U**-space, and let A, B be any two com-

(11) I.e., has a base of open and closed sets.

pletely separated subsets of X. There is an $f\in C(X)$ such that $f(A)=1$, $f(B)=-1$. Since X is a **U**-space, there is a unit u of $C(X)$ such that $f=u|f|$. Then $P(u)$ is an open and closed subset of X that contains A but is disjoint from B. Therefore, by Lemma 5.4, βX is zero-dimensional.

It follows that every **U**-space is zero-dimensional, hence disconnected. Therefore we have:

EXAMPLE 5.6. *A* **D**-*space that is not a* **U**-*space.* The **D**-space of Example 4.9 is connected, hence cannot be a **U**-space.

We close this section with two algebraic characterizations of **U**-spaces.

THEOREM 5.7. *For every completely regular space X, the following statements are equivalent.*

(a) *X is a* **U**-*space.*

(b) *For any f, $g\in C(X)$, there exist f_1, g_1, h, $t\in C(X)$ such that $f=f_1h$, $g=g_1h$, and f_1+tg_1 is a unit.*

(c) (i) *X is an* **F**-*space, and*

(ii) *for any f, $g\in C(X)$, with $(f, g)=(1)$, there exists $t\in C(X)$ such that $f+tg$ is a unit.*

Proof. We shall show, in turn, that each of the first two properties is equivalent to the third.

(a) *implies* (c). By Theorem 5.3, (a) implies (i). To establish (ii), consider any f, g with $(f, g)=(1)$. By (a), we have $f=u|f|$, $g=v|g|$, where u (and v) are suitable units. Then $uf+vg=|f|+|g|$, which is a unit, whence $f+(v/u)g$ is a unit, as required.

(c) *implies* (a). Let A, B be completely separated subsets of X, and let $f\in C(X)$ be such that $f(A)=1$, $f(B)=-1$. If $g=1-f^2$, then $g(A)=g(B)=0$, and $(f, g)=(1)$. By (ii), there is a $t\in C(X)$ such that $f+tg$ is a unit, i.e., $f+tg$ vanishes nowhere on X. Then $P(f+tg)$ is an open and closed subset of X that contains A but is disjoint from B. Thus, by Lemma 5.4, βX is zero-dimensional. Hence, by (i) and Theorem 5.5, X is a **U**-space.

(b) *implies* (c). Obviously, (b) implies (i). Now consider any f, g with $(f, g)=(1)$. By (b), there exist f_1, g_1, h, t such that $f=f_1h$, $g=g_1h$, and f_1+tg_1 is a unit u. Then $f+tg=uh$, from which it follows that $(f, g)=(h)$, and therefore that h is a unit. Hence $f+tg$ is a unit. This establishes (ii).

(c) *implies* (b). By a previous part of the proof, (c) implies that X is a **U**-space; hence, by Theorem 5.3, X is a **T**-space, i.e., $C(X)$ is an Hermite ring. Consider any f, $g\in C(X)$. By GH, Theorem 3, there exist f_1, g_1, h such that $f=f_1h$, $g=g_1h$, and $(f_1, g_1)=(1)$. By (ii), there is a t with f_1+tg_1 a unit. Thus (b) holds.

6. Regular rings and P-spaces.

DEFINITION 6.1. A commutative ring S with identity is called a **P**-*ring* if every (nonzero, proper) prime ideal of S is a maximal ideal. A completely regular space X such that $C(X)$ is a **P**-ring is called a **P**-*space.*

A point $p\in X$ is called a **P**-*point* of X if for all $f\in C(X)$, $f(p)=0$ implies that f vanishes on an entire neighborhood of p.

The following theorem was proved by the authors in [3, Lemma 3.2 and Theorem 5.3 ff.].

THEOREM 6.2. *For every completely regular space X, the following statements are equivalent.*

(a) *X is a **P**-space, i.e., every prime ideal of $C(X)$ is maximal.*

(b) *Every prime fixed ideal of $C(X)$ is maximal.*

(c) *For every $p\in\beta X$, the ideal N^p of $C(X)$ (Definition 1.3) is maximal.*

(d) *For every $p\in X$, the ideal N_p of $C(X)$ is maximal.*

(e) *Every point of X is a **P**-point of X, i.e., every zero-set of X is open.*

(f) *Every countable intersection of open subsets of X is open.*

(g) *Every ideal of $C(X)$ is an intersection of maximal ideals.*

(h) *$C(X)$ is a regular ring (GH, Definition 9).*

Obviously, every discrete space is a **P**-space. Several examples of non-discrete **P**-spaces—in fact, of **P**-spaces containing no isolated points whatsoever—are given in [3].

The fact that every **P**-space is an **F**-space can be seen in many ways: e.g., every regular ring is an **F**-ring—in fact, an elementary divisor ring (GH, Remark 12). But the following theorem tells more.

THEOREM 6.3. (a) *Every **P**-space is a **U**-space (hence a **D**-, **T**- and **F**-space).*

(b) *There exist **U**-spaces that are not **P**-spaces. In particular, if X is an infinite **P**-space (e.g., an infinite discrete space), then βX is a **U**-space but not a **P**-space.*

Proof. (a) is immediate from (e) of the preceding theorem. As for (b), if X is a **P**-space, then by (a), it is a **U**-space, and βX is also a **U**-space (Theorem 5.2). But every compact **P**-space is finite [3, Corollary 5.4], so if X is infinite, then βX cannot be a **P**-space.

There exist non-normal **P**-spaces [3, Theorem 7.7], hence non-normal **U**-spaces.

An interesting comparison between **F**-spaces and **P**-spaces is afforded by Theorems 2.5 and 6.2(c): for an **F**-space, every ideal N^p is prime; for a **P**-space, every N^p is maximal.

Using Theorem 2.3(c), it is not hard to see that in order for a *linearly ordered* space X to be an **F**-space, it is necessary and sufficient that no point of X be the limit of an ω-sequence, either increasing or decreasing. But this is precisely the condition that X be a **P**-space (cf. [3, Corollary 7.2]). So we have:

THEOREM 6.4. *A linearly ordered space is an **F**-space if and only if it is a **P**-space.*

Another algebraic characterization of **P**-spaces is given by the following theorem.

THEOREM 6.5. *A completely regular space X is a* **P**-*space if and only if $C(X)$ is an adequate ring* (*GH*, *Definition* 7).

Proof. If X is a **P**-space, then $C(X)$ is regular (Theorem 6.2(h)), hence adequate (GH, Theorem 11). (One may also give a direct proof for $C(X)$: given f and g, define $f_1=1$ and $h=0$ on $Z(f)\cap Z(g)$, and $f_1=f$ and $h=1$ elsewhere; then $f=f_1h$, $(f_1, g)=(1)$, and no nonunit divisor h' of h is relatively prime to g.)

Conversely, suppose that X is not a **P**-space. Then there exist a function $g\in C(X)$ and a point $p\in Z(g)$ such that g vanishes on no entire neighborhood of p (Theorem 6.2(e)). Choose any point $q\neq p$, let U be a neighborhood of p such that $q\notin \overline{U}$, and construct $f\in C(X)$ such that $f(\overline{U})=0$ and $f(q)=1$. (Then $f\neq 0$.) Now consider any f_1, $h\in C(X)$ with the properties (i) and (ii) of condition **A** (GH, Definition 7), i.e., such that $f=f_1h$ and $(f_1, g)=(1)$. We shall show that (iii) of condition **A** must fail. We have $f_1(p)\neq 0$ (since $g(p)=0$). Hence there is a neighborhood V of p, with $V\subset U$, such that f_1 vanishes nowhere on V. Then $h(V)=0$ (since $f=f_1h$). By definition of g, there is a $y\in V$ for which $g(y)\neq 0$. Then there is a neighborhood W of y, with $W\subset V$, such that g vanishes nowhere on W. Now construct $h'\in C(X)$ such that $h'(y)=0$, $h'(X-W)=1$. Clearly, $h=hh'$. Therefore h' is a nonunit divisor of h. But, obviously, $(h', g)=(1)$. Hence condition **A** fails, so $C(X)$ is not adequate.

Referring to Theorem 6.2(a, h), we have:

COROLLARY 6.6. *For a ring $C(X)$, the following algebraic conditions are equivalent: the ring is adequate, it is regular, every prime ideal is maximal.*

COROLLARY 6.7. *Not every elementary divisor ring is adequate.*

Proof. By Theorem 6.3(b), there exist **D**-spaces that are not **P**-spaces.

COROLLARY 6.8. *Every adequate ring $C(X)$ is an elementary divisor ring* (*cf. GH*, *Theorem* 8).

Proof. By Theorem 6.3(a), every **P**-space is a **D**-space.

APPENDICES

7. Some further examples of rings. Examples readily come to mind to show that not all the above implications that hold for a ring $C(X)$ carry over to arbitrary commutative rings with identity.

Every regular ring (commutative, with identity) is adequate (GH, Theorem 11), and is a **P**-ring (since, clearly, every homomorphic image of a regular ring is regular, and every regular integral domain is a field). On the other hand, the ring of integers is an adequate **P**-ring that is not regular.

The ring of entire functions is adequate [6], but neither regular (since it is not a field), nor **P** [7, Theorem 1(a)].

Examples of **P**-rings that are not even **F**-rings are familiar from the theory of algebraic integers—e.g., the ring of all $a+b(-5)^{1/2}$, where a and b are rational integers (see, e.g., [13, Theorem 8.7 and proof of Theorem 7.11]).

The following example, in a slightly different connection, may also be of interest. Consider any **F**-ring $C(X)$ that is not an Hermite ring (Example 3.4). Then, *a fortiori*, $C(X)$ is not an elementary divisor ring. Yet every homomorphic image $C(X)/P$, where P is a prime ideal, is an elementary divisor ring. For $C(X)/P$, as a homomorphic image of an **F**-ring, is an **F**-ring, and $C(X)/P$ contains a unique maximal ideal [3, Corollary 3.4]; and it is easily seen that any such ring is adequate. Being an integral domain, $C(X)/P$ is therefore an elementary divisor ring [6].

8. **Some further topological spaces.** The principal spaces discussed thus far are, in decreasing order of generality, **F**, **T**, **D**, **U**, **P**. We conclude our paper by comparing these with some further classes of topological spaces (for which, however, we have no algebraic counterparts).

DEFINITION 8.1. A completely regular space X is called a **P**′*-space* if for all $f \in C(X)$, and all $p \in Z(f)$, there is a deleted neighborhood U' of p such that either $f(U')=0$ or $f(U')>0$ or $f(U')<0$.

Obviously, every **P**-space is a **P**′-space (Theorem 6.2(e)).

Particularizing, $p \in Z(f)$ is a **P**′*-point of* f if a deleted neighborhood U' exists as above, and p is a **P**′*-point of* X if p is a **P**′-point of every f for which $p \in Z(f)$. Thus X is a **P**′-space if and only if every point of X is a **P**′-point of X.

Let $f, g \in C(X)$. Define $f \leqq g$ to mean (as heretofore) that $f(x) \leqq g(x)$ for all $x \in X$. Then $C(X)$ becomes a partially ordered set, and, in fact, a lattice.

An arbitrary lattice is said to be *conditionally complete* if every nonempty subset that has an upper (resp. lower) bound has a least upper (resp. greatest lower) bound, *σ-complete* if the corresponding conditions hold for *countable* subsets. M. H. Stone [16; 17] and H. Nakano [12] have investigated relations between topological properties of a space X and completeness properties of the lattice $C(X)$.

DEFINITION 8.2. A completely regular space X is said to be *extremally disconnected*(12) if any of the following equivalent conditions holds: the closure of every open set is open, any two disjoint open sets have disjoint closures, $C(X)$ is a conditionally complete lattice.

The equivalence of the first two properties is elementary; these properties were first investigated by Stone. For the proof of their equivalence with conditional completeness, see [17] or [12].

THEOREM 8.3. *For any completely regular space X, the lattice $C(X)$ is σ-complete if and only if, for every $f \in C(X)$, the set $\overline{P}(f)$ is open.*

The proof may be obtained from the proof of Stone [17, Theorem 15]

(12) This term was introduced by Hewitt.

by relaxing his requirement of normality of X to complete regularity, and compensating for this by considering only sets of the form $P(f)$ ($f\in C(X)$), rather than arbitrary F_σ's.

Equivalently, the sets $\overline{P}(f)$, $\overline{N}(f)$, interior of $Z(f)$ are mutually disjoint, and each is open and closed.

THEOREM 8.4. *If X is a* **P**'*-space, then $C(X)$ is a σ-complete lattice. In turn, if the latter condition obtains, then X is a* **U**-*space (hence a zero-dimensional* **D**-*space).*

The proof follows easily from Theorem 8.3.

We shall now consider some examples.

8.5. Let N denote the denumerable discrete space $\{e_1, e_2, \cdots\}$, let e be any fixed point of βN, and define E to be the subspace $N\cup\{e\}$ of βN. This notation will be retained throughout the remainder of our discussion. We shall refer to e as the *β-point* of E. We remark on the topology of E. Every point e_n is, of course, isolated. Deleted neighborhoods of e constitute a maximal family $(Z_s)_{s\in S}$ having the finite intersection property, each Z_s being infinite, and with $\bigcap_{s\in S} Z_s=\varnothing$ (see Lemma 1.2 and [8, Theorem 36]). There is no countable base of neighborhoods at e [2].

EXAMPLE 8.6. *An extremally disconnected* **P**'*-space that is not a* **P**-*space.* E is such a space. Obviously, it is extremally disconnected. Every e_n is isolated; and e is a **P**'-point of E because for any f, exactly one of $\overline{P}(f)$, $\overline{N}(f)$, $Z(f)$ is a neighborhood of e. But e is not a **P**-point of the function g defined by: $g(e_n)=1/n$, $g(e)=0$.

8.7. We denote by L the space of all ordinals $\leqq\omega_1$ (the smallest nondenumerable ordinal), under the following topology: neighborhoods of ω_1 are as in the interval topology, while every other point is isolated. It is well known that (in our terminology) L is a **P**-space.

J. R. Isbell has proved (written communication) that every extremally disconnected **P**-space is discrete, provided only that the cardinal number of the space is nonmeasurable(13). Here is a simple example of a nondiscrete **P**-space:

EXAMPLE 8.8. *A* **P**-*space (hence a* **P**'*-space) that is not extremally disconnected*: two copies of the space L, with their limit points (there is one in each) identified. (Any linearly ordered **P**-space having at least one point that is a limit from both sides will also serve; see [3, §7] for examples.)

EXAMPLE 8.9. *An extremally disconnected space that is not a* **P**'*-space (hence not a* **P**-*space)*: βX, for any infinite discrete X, is such a space. It is

(13) A cardinal m is *measurable* if a nontrivial, countably additive, two-valued measure can be defined on the set of all subsets of a set of power m. Most cardinals encountered in practice are known to be nonmeasurable, and no example of a measurable cardinal is known. See [3, p. 352] for discussion and references. (*Added in proof.*) For Isbell's result, see Tôhoku Math. J. (2) vol. 7 (1955) pp. 1–8.

extremally disconnected, since X is discrete [8, Theorem 25]. Suppose that it is a $\mathbf{P}'$-space. Since $\beta X - X$ is closed, every function in $C(\beta X - X)$ can be extended continuously over all of βX. Therefore $\beta X - X$ is also a $\mathbf{P}'$-space. But $\beta X - X$ cannot be a $\mathbf{P}$-space, as every compact $\mathbf{P}$-space is finite [3, Corollary 5.4]. So there is a point $p \in \beta X - X$ that is a $\mathbf{P}'$-point, but not a $\mathbf{P}$-point, of $\beta X - X$. Then p is an isolated zero of some function, and is therefore a G_δ-set. But this contradicts Čech's result that for any completely regular space X, every nonempty closed G_δ of $\beta X - X$ is infinite(14).

EXAMPLE 8.10. *A* $\mathbf{U}$*-space* X *for which* $C(X)$ *is not* σ*-complete.* Such a space can be constructed by identifying the β-point e of E (8.5) with a nonisolated point of any (nondiscrete) $\mathbf{P}$-space. E.g., let X be obtained by identifying e with the point ω_1 of L (8.7): every point $x \neq e$ is isolated, while a neighborhood of e is the union of a neighborhood of e in E with a neighborhood of e in L. It is easily verified that X is a $\mathbf{U}$-space. But the set $N \subset E$ is a $P(f)$, and its closure, E, is not open; therefore $C(X)$ is not σ-complete (Theorem 8.3).

EXAMPLE 8.11. *A space* X *that is neither a* $\mathbf{P}'$*-space nor extremally disconnected, but for which* $C(X)$ *is* σ*-complete.* The space $X = L \times E$ (8.5, 8.7) has these properties. If g is defined by: $g(\alpha, e_n) = 1/n$ and $g(\alpha, e) = 0$, for all $\alpha \leqq \omega_1$ and all $n < \omega$, then $g \in C(X)$, and the point (ω_1, e) is not a $\mathbf{P}'$-point of g. The set

$$\{(\xi, p): \xi < \omega_1, \xi \text{ even}; p \in E\}$$

is an open set whose closure is not open, whence X is not extremally disconnected. Finally, let $f \in C(X)$, and consider any point $(\alpha, p) \in \overline{P}(f)$. It is easily seen that $\overline{P}(f)$ contains a neighborhood of (α, p) in case $p \neq e$. If $(\alpha, e) \in \overline{P}(f)$, there is a deleted neighborhood U' of e (in the space E) such that $f(\alpha, q) > 0$ for all $q \in U'$; this implies the result for $\alpha < \omega_1$. Finally, if $(\alpha, p) = (\omega_1, e)$, then, since ω_1 is not cofinal with ω, there is a $\gamma < \omega_1$ such that $f(\delta, q) > 0$ for all $\delta > \gamma$ and all $q \in U'$, which implies the result in this case. It follows that $\overline{P}(f)$ is open. Hence, by Theorem 8.3, $C(X)$ is σ-complete.

Finally, we consider a more extensive class of spaces.

DEFINITION 8.12. A completely regular space X is called an $\mathbf{F}'$-space if for all $f \in C(X)$, the sets $\overline{P}(f)$, $\overline{N}(f)$ are disjoint.

Thus, every $\mathbf{F}$-space is an $\mathbf{F}'$-space (Theorem 2.3(c)), and, for *normal* spaces, the concepts coincide.

Examination of the proof of Theorem 2.5 leads to:

THEOREM 8.13. *A completely regular space* X *is an* $\mathbf{F}'$*-space if and only if, for every point* $p \in X$*, the ideal* N_p *of* $C(X)$ *is a prime ideal.*

The two theorems may be compared thus. Let $N(M)$ denote the inter-

(14) Čech proved [2, p. 835] that every such set is of power $\geqq c$; Hewitt [8, Theorem 49] strengthens this to $\geqq 2^c$ (and gives an example in which the equality holds).

section of all the prime ideals contained in the maximal ideal M. Then X is an $\mathbf{F}'$-space if and only if, for every maximal *fixed* ideal M of $C(X)$, the ideal $N(M)$ is prime, while X is an $\mathbf{F}$-space if and only if, for *every* maximal ideal M of $C(X)$, free or fixed, the ideal $N(M)$ is prime. (In either case, the replacement of *prime* by *maximal* characterizes X as a $\mathbf{P}$-space; see Theorem 6.2(c, d).) It should be noted, however, that *fixed* and *free* are not *algebraic* concepts.

EXAMPLE 8.14. *A (nonnormal)* $\mathbf{F}'$*-space that is not an* $\mathbf{F}$*-space.* Define L' to be the space of all ordinals $\leqq \omega_2$, with each $\gamma < \omega_2$ an isolated point, and neighborhoods of ω_2 as in the interval topology. Define $Y = L' \times L - \{(\omega_2, \omega_1)\}$, where L is as in 8.7. Next, introduce new distinct points $d_{\alpha,n}$ $(\alpha < \omega_1,\ n < \omega)$; and for each $\alpha < \omega_1$, write $d_\alpha = (\omega_2, \alpha)$, and let $D_\alpha = d_{\alpha,0}, d_{\alpha,1}, \cdots, d_\alpha$ be a copy of E, with d_α its β-point (8.5). Finally, define $X = \bigcup_\alpha D_\alpha \cup Y$, with the following topology: neighborhoods of points other than the d_α's are the same as originally, while for each α, a neighborhood of d_α is the union of a neighborhood of d_α in Y with a neighborhood of d_α in D_α.

It is easily seen that X is an $\mathbf{F}'$-space. Now define $f \in C(X)$ as follows: $f(d_{\alpha,n}) = \pm 1/n$ according as α is even or odd, and $f(Y) = 0$. Then $\overline{P}(f) = \bigcup_\alpha D_{2\alpha}$, $\overline{N}(f) = \bigcup_\alpha D_{2\alpha+1}$. Let A, B be any two disjoint open sets such that $A \supset \overline{P}(f)$ and $B \supset \overline{N}(f)$. By a familiar cofinality argument, their closures have a common intersection with the set $\{(\gamma, \omega_1) : \gamma < \omega_2\}$. Therefore $\overline{P}(f)$, $\overline{N}(f)$ are not completely separated. Thus X is not an $\mathbf{F}$-space.

8.15. The following diagram shows the implications among the principal spaces considered in this paper. As we have seen, none of these implications can be reversed.

$$
\begin{tikzcd}
 & \mathbf{P} \arrow[r] & \mathbf{P}' \arrow[dr] & & & \mathbf{D} \arrow[r] & \mathbf{T} \arrow[r] & \mathbf{F} \arrow[r] & \mathbf{F}' \\
\text{discrete} \arrow[ur] \arrow[drr] & & & \sigma\text{-complete} \arrow[r] & \mathbf{U} \arrow[ur] \arrow[dr] & & & & \\
 & & \text{extr. disconn.} \arrow[ur] & & & \text{zero-dim.} & & &
\end{tikzcd}
$$

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