Journal of Humanistic Mathematics

Volume 6 | Issue 2 July 2016

Patterns Formed by Coins

Andrey M. Mishchenko
Formlabs

Recommended Citation
Andrey M. Mishchenko, "Patterns Formed by Coins," *Journal of Humanistic Mathematics*, Volume 6 Issue 2 (July 2016), pages 96-113. DOI: 10.5642/jhummath.201602.08. Available at: https://scholarship.claremont.edu/jhm/vol6/iss2/8

Patterns Formed by Coins

Cover Page Footnote

The author was partially supported by NSF grants DMS-0456940, DMS-0555750, DMS-0801029, DMS-1101373.

Patterns Formed by Coins

Andrey M. Mishchenko

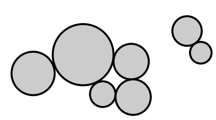
Formlabs, Somerville, Massachussetts, USA
www.mishchea.com
mishchea@gmail.com

Synopsis

This article is a gentle introduction to the mathematical area known as *circle packing*, the study of the kinds of patterns that can be formed by configurations of non-overlapping circles. The first half of the article is an exposition of the two most important facts about circle packings, (1) that essentially whatever pattern we ask for, we may always arrange circles in that pattern, and (2) that under simple conditions on the pattern, there is an essentially unique arrangement of circles in that pattern. In the second half of the article, we consider related questions, but where we allow the circles to overlap. The article is written with the idea that no mathematical background should be required to read it.

1. What is a circle packing?

Suppose we arrange some coins flat on a table. We get a picture that looks like this:



The coins may be of different sizes, and they may touch, but for now we insist that they do not overlap. Configurations such as these are traditionally known as *circle packings*:

Definition 1. *A* ***circle packing*** *is a collection of disks which don't overlap.*

We begin by exploring the patterns that can be formed by the disks in a circle packing:

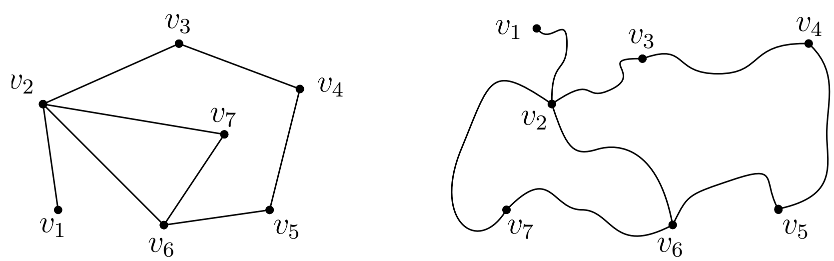
(A) (B)

Figure 1: Two circle packings which have "the same pattern" in some sense

To make our notion of a "pattern" more precise, we need to know what a *graph* is:

Definition 2. *A* ***graph*** *is a collection of* ***vertices****, which we usually draw as dots, and* **edges***, which are connections between the dots.*

The easiest way to represent a graph is via a drawing. However, we stress that a graph is an abstract object which is independent of any drawn representation of it.



(A) (B)

Figure 2: Two different representations of the same graph. The graph we have drawn here has 7 vertices. Here the vertices have been labeled $v_1, v_2, \ldots, v_7$. Then there is an edge between v_i and v_j in (A) whenever there is an edge between v_i and v_j in (B) and vice versa. For example, we do not consider that there is an edge between v_3 and v_5 in this graph, because we cannot get from v_3 to v_5 using edges without passing through other vertices.

Graphs are all around us. For example, the *handshake graph* is the graph that has a vertex for every person and an edge between two people if they have ever shaken hands. You may be more used to hearing graphs referred to as *networks*. For example, the *Facebook friend graph* is the graph that has a vertex for every Facebook account and an edge whenever the two accounts are friends on Facebook. Of course, the graphs in these two examples both have huge numbers of vertices, and it would be very hard to draw representations of them. Try the following easier task, to build your intuition:

Problem 1. *Draw a representation of the following graph:*

- *The vertices are the English words for the whole numbers from 1 to 6:* one, two, three, four, five, six.
- *Two vertices have an edge connecting them whenever the two corresponding words share a letter.*

The concept of a graph helps us formalize what we mean when we discuss the "pattern" formed by the disks of a circle packing. In particular, we may use a graph to capture the information of "how the disks of the packing meet":

Definition 3. *Suppose we have a circle packing $\mathcal{P}$ consisting of the disks $D_1, D_2, D_3, \ldots$. Then the **contact graph** of the packing $\mathcal{P}$ is the graph described as follows:*

- *The vertices are the disks $D_1, D_2, D_3, \ldots$ of the packing.*
- *We connect two vertices with an edge if, and only if, the corresponding disks touch.*

We explore some more examples:

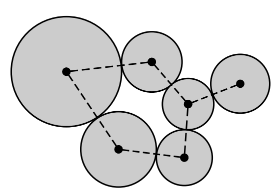
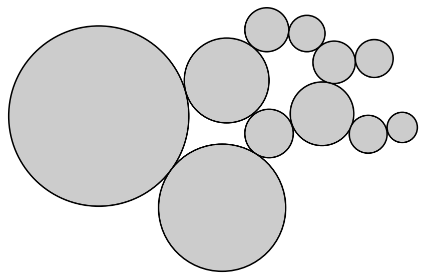
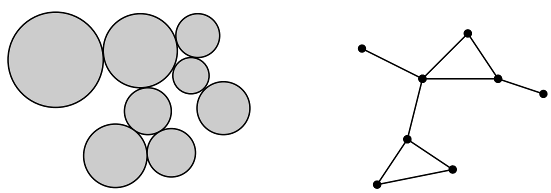


Figure 3: A circle packing and its contact graph. The edges of the graph are drawn as dashed lines. This is the easiest way to visualize the contact graph of a circle packing, and usually the easiest way to draw a representation of it.

Problem 2. *Draw the contact graph of the following circle packing:*



If you did Problem 2, you probably drew the contact graph directly on top of the packing, as we did in Figure 3, but it is not always necessary to do this:



(A) (B)

Figure 4: Another circle packing and its contact graph. Here we have drawn the contact graph separately from the packing, to again emphasize that the graph is an abstract object distinct from any drawn representation of it.

Problem 3. *Verify that the graph shown in Figure 4B is the contact graph of the circle packing shown in Figure 4A.*

Experience leads us to conclude:

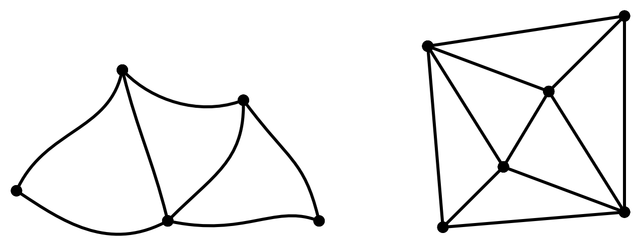
Observation. *If we start with a circle packing, it is not so hard to draw its contact graph.*

In other words, it is easy to "go from" a circle packing to its contact graph. Then we might ask, when can we go "in the other direction?" This is our segue into the next section.

2. Existence of circle packings

To build intuition, we start this section with an exercise:

Problem 4. *Draw circle packings having the following contact graphs:*



This leads us to consider the following natural question:

Existence Question. *If we start with a graph G, is there always a circle packing having G as its contact graph?*

For simplicity, we will consider only *finite* graphs, meaning graphs which have finitely many vertices and edges. We first observe that circle packings' contact graphs never have *loops*, and never have *repeated edges*:

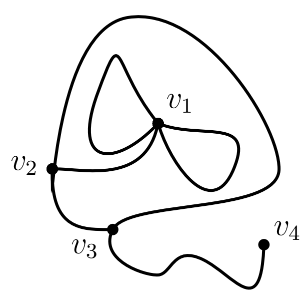


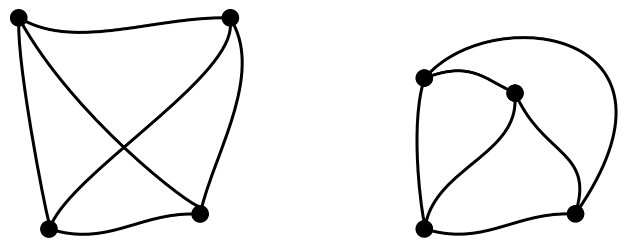
Figure 5: A graph which *does* have loops and repeated edges. The vertices v_2 and v_3 have two edges connecting them, an example of what is meant by a *repeated edge.* The vertex v_1 has two loops on it: a *loop* is an edge from a vertex to itself.

A graph without loops and repeated edges is called *simple.* We observe:

Existence Condition 1. *If G is the contact graph of a circle packing, then G is simple.*

Next, we make a more subtle observation:

Fact. *Not every graph can be drawn without its edges crossing each other.*



(A) (B)

Figure 6: Two representations of the same graph, one planar and one not. The graph represented here has four vertices. In (A), two edges of our graph cross. This does not happen in (B). The drawing in (A) is called a *planar representation* for this graph.

The idea of *edges crossing* is illustrated in the following example. A graph is called *planar* if it can be drawn (in the plane) without edge crossings. (You can think of "the plane" as an infinite flat piece of paper, that goes on forever in all directions.) To convince yourself that not all graphs are planar, try the following exercise.

Problem 5. *Try to draw a representation of the following graph, so that the edges don't cross each other:*

- *It has five vertices, and*
- every pair *of vertices has an edge connecting them.*

In other words, draw five dots on a piece of paper, and try to draw a connection between every pair of the dots so that these connections don't cross. (It is impossible, but try to convince yourself of this, rather than taking it on faith.)

On the other hand, suppose that the graph G is the contact graph of some circle packing. Then we may draw a representation of G as we did in Figure 3, by putting the vertices corresponding to our disks at the centers of the disks, and connecting the vertices of touching disks by straight line segments. We conclude:

Existence Condition 2. *If G is the contact graph of a circle packing, then G is planar.*

Thus for example, there is no circle packing whose graph is the one described in Problem 5.

Before moving on, do the following quick exercise. We will use your answer soon.

Problem 6. *Draw a simple planar graph different from any of the ones we have seen so far.*

An amazing fact is that Existence Conditions 1 and 2 are the *only conditions* needed to guarantee the existence of a circle packing in whatever pattern you like:

Existence Theorem. *If G is a simple planar graph, then there is some circle packing having G as its contact graph.*

Thus, without seeing your answer to Problem 6, I know that it is possible to do the following:

Problem 7. *Draw a circle packing whose contact graph is the one you drew in Problem 6.*

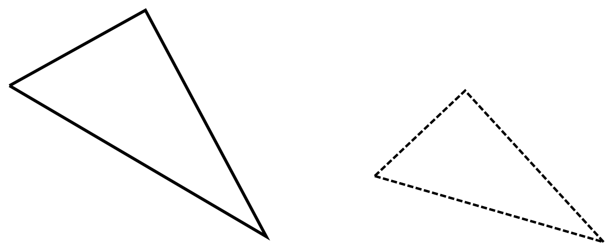
The Existence Question for circle packings was first asked by Paul Koebe, and he answered it himself in 1936, by proving the above existence theorem [2]. The proof uses *complex analysis*, the theory of calculus on the complex numbers $\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R}\}$. It turns out that the field of circle packing is a kind of discrete analog of complex analysis. This is a major motivating factor for the study of circle packings.

3. Rigidity of circle packings

Different circle packings can have the same contact graph, for example, the packings in Figure 1 on p. 97. In this section, we explore the question of whether there is a contact graph so that *only one* circle packing has that contact graph.

Of course, if we start with a circle packing and, imagining that our disks are glued together wherever they touch, pick all of the disks up, move them around, and then set them back down, we will get a new circle packing having the same contact graph. This leads us to the notion of *similar packings*.

Recall from school geometry that two triangles are called *similar* if one is a copy of the first, but scaled up or down and moved around, such as these two below:



We can define two packings to be similar in an analogous way:

Definition 4. *Two circle packings are called* ***similar*** *if it is possible to get from one to the other by combining the following four operations:*

- *sliding,*
- *rotating,*
- *reflecting ("flipping"), and*
- *scaling ("stretching").*

Remember that the operations are always applied to a *whole packing at once*, not to individual disks one at a time! We consider similar packings to be "essentially the same":

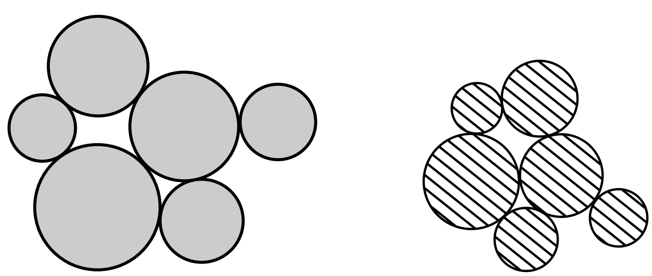


Figure 7: Two circle packings which are "essentially the same"

The reason for this is that similar packings *always* have the same contact graph.

On the other hand, there may still be some circle packing $\mathcal{P}$ such that *every* packing which has the same contact graph as $\mathcal{P}$ turns out to be similar to $\mathcal{P}$. In other words:

Uniqueness Question. *When is there an* essentially *unique circle packing having some given contact graph?*

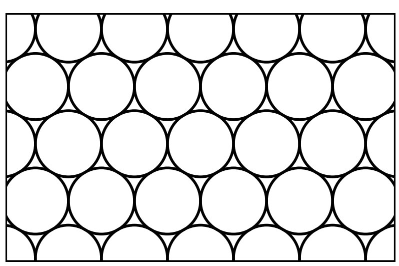
For every circle packing we have seen so far, it is possible to find an essentially different packing, meaning a packing which is *not* similar to the original one, which has the same contact graph. Convince yourself:

Problem 8. *Look back, and pick one of the circle packings we have seen so far, call it $\mathcal{P}$. Draw a new circle packing which has the same contact graph as your chosen packing $\mathcal{P}$, but which is not similar to $\mathcal{P}$.*

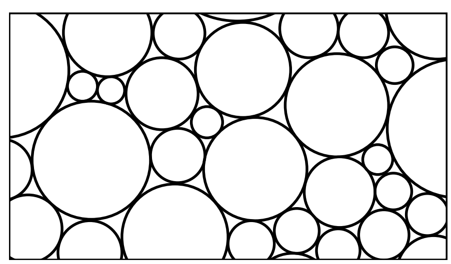
In fact, it is not too hard to believe the following:

Fact. *If a circle packing $\mathcal{P}$ is made up of finitely many disks, then there are circle packings having the same contact graph as $\mathcal{P}$, but which are not similar to $\mathcal{P}$.*

So, we turn our attention to *infinite packings*, packings which have infinitely many disks:



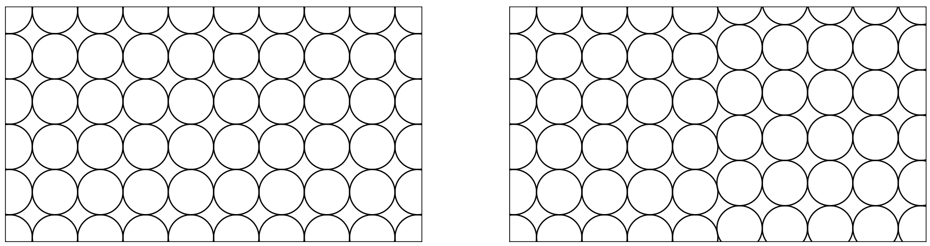
(A)



(B)

Figure 8: Pieces of infinite circle packings. Of course, it is impossible to draw the entirety of a packing that goes on forever. However, sometimes, it is still possible to completely describe the packing. For example, the packing shown in (A) is the infinite packing by disks which all have the same size, so that each disk has exactly 6 neighbors. This is known as the *penny packing*. We leave what happens in (B) to the imagination.

We might be tempted to believe that if a packing "goes on forever in all directions," then it is essentially the only packing having its contact graph. Unfortunately, this does not turn out to be true. For example:



(A) (B)

Figure 9: Two infinite packings which are not similar, but which have the same contact graph.

Problem 9. *Describe the contact graph of the packings in Figure 9. Can you find a third packing which has the same contact graph, but which is not similar to either of (A) and (B)?*

In Figure 9, we "perturbed" some disks of the packing, leaving others where they were, to get a new, fundamentally different packing having the same contact graph. On the other hand, it does not seem so easy to perturb in a similar way the disks of either packing in Figure 8. This is because of a special property of the contact graph:

Definition 5. *Suppose that $\mathcal{P}$ is an infinite circle packing. Represent its contact graph G by putting a vertex at every circle's center, and connecting vertices of touching circles by straight segments. If the graph G then "cuts the plane into triangles," we say that the packing $\mathcal{P}$ is* ***triangulated***.

In Figure 10 we see how the contact graph of the penny packing "triangulates" the plane.

Problem 10. *Verify that the packing in Figure 8 appears to be triangulated as well. (Of course, we don't really know what happens outside of the rectangular region we have drawn.)*

The following amazing fact turns out to be true:

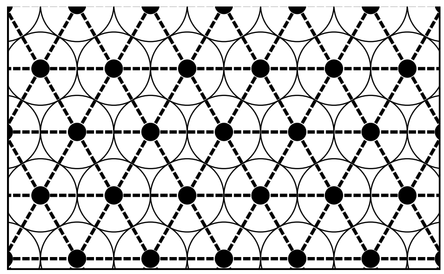


Figure 10: The contact graph of the penny packing. We see that the contact graph, drawn in the way we described, cuts the plane into triangles.

Uniqueness Theorem. *If two circle packings have the same contact graph and are both triangulated, then they are similar. In other words, if a packing $\mathcal{P}$ is triangulated and has contact graph G, then $\mathcal{P}$ is essentially the* only *circle packing having contact graph G.*

An alternative way to phrase this theorem is to say that triangulated circle packings are *rigid*, that is, we cannot modify a triangulated circle packing without changing its contact graph, except by rotating, scaling, etc. Compare this to the situation in Figure 9, where it *is* possible to modify the packings significantly while keeping the contact graph the same. Furthermore, considering the example of Figure 9, the following is not hard to believe:

Fact. *Suppose a circle packing $\mathcal{P}$ is not triangulated. Then there are circle packings with the same contact graph as $\mathcal{P}$, but which are not similar to $\mathcal{P}$.*

In other words, the condition that $\mathcal{P}$ is triangulated is *necessary* for $\mathcal{P}$ to be rigid. The Uniqueness Theorem says that this condition is also *sufficient*. Thus the uniqueness guaranteed by the Uniqueness Theorem is the best that we could possibly hope for.

As we mentioned earlier, Koebe initiated the study of circle packings in 1936, and proved the Existence Theorem for circle packings. He also managed to prove a different kind of uniqueness theorem for circle packings: instead of considering packings in the plane, as we are, he considered packings on the surface of a sphere. The truth or falsehood of the above uniqueness theorem was unknown for a long time after Koebe, until a proof was finally found in 1991 by Oded Schramm [5].

4. Overlapping disks

We next ask ourselves, what happens if we allow the disks to overlap? It will be helpful to have some vocabulary available to us:

Definition 6. *A* ***disk configuration*** *is any collection of disks, so that no disk in the collection is completely contained inside of another.*

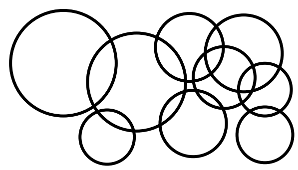


Figure 11: Our first example of a disk configuration.

When discussing disk configurations, it will be helpful to have a notion of *overlap angle*:

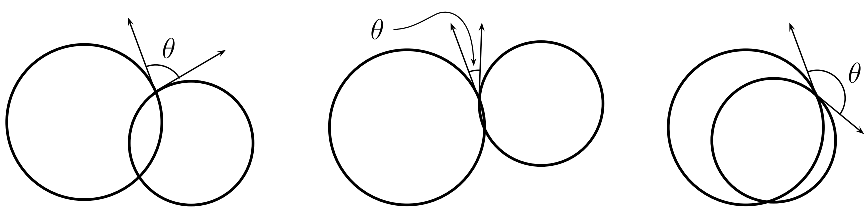


Figure 12: The overlap angles between several pairs of disks. In each case, we have chosen a point where the boundary circles meet, and drawn the tangent rays to the circles pointing outside of the disks. Then the *overlap angle* θ is the angle between these rays. We note two facts: (1) the overlap angle gets bigger as the disks "move closer together," or gets smaller as the disks "move apart," and (2) the overlap angle is the same at both points of intersection of the boundary circles.

The ***contact graph*** of a disk configuration is defined the same way as the ***contact graph*** of a circle packing. In the case of disk configurations, however, we label each edge of the contact graph with the overlap angle between those two disks. See Figure 13.

Try the following problem to get a handle on these new concepts.

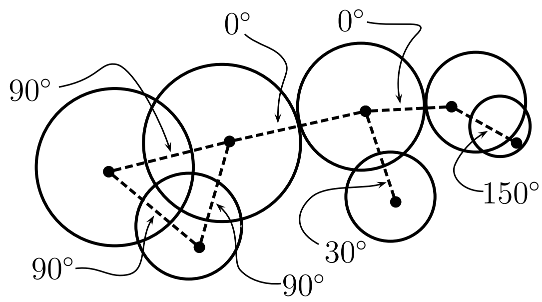
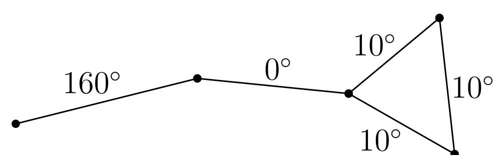


Figure 13: The contact graph and overlap angles of one disk configuration.

Problem 11. *Draw a disk configuration having the following contact graph and (approximate) overlap angles:*



In our new setting, we consider that the "pattern" of a disk configuration consists of both its contact graph and its overlap angles. We ask ourselves the following natural question:

Question 1. *Given a graph G whose edges are labeled with some angles, when does there exist a disk configuration having G as its contact graph with those overlap angles?*

This question turns out to be much harder than the existence question for circle packings. For our first example, consider what happens in Figure 14.

Also, not all contact graphs and overlap angles have disk configurations. See Figure 15 for an example.

This might lead us to conclude that no disk configuration has a contact graph containing, as a sub-graph, the graph pictured in Figure 15 including edge labels. However, this conclusion does not turn out to be true. Consider the example of Figure 16.

At this point, we give up on the question of existence and summarize our findings on this question in the following observation.

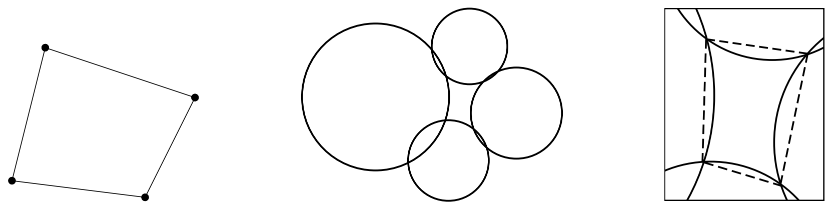


Figure 14: Four disks in a closed chain. Suppose four disks have the contact graph shown in (A). An example of four such disks is shown in (B). If we zoom in on the hole formed by the four disks, we see the picture in (C). The angles inside of the dashed quadrilateral are clearly bigger than the corresponding overlap angles between the disks. On the other hand, the sum of the angles inside of a quadrilateral is always exactly 360°. So if four disks have the contact graph shown in (A), then the sum of their overlap angles *must* be less than 360°.

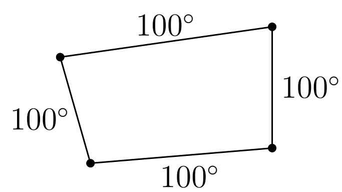
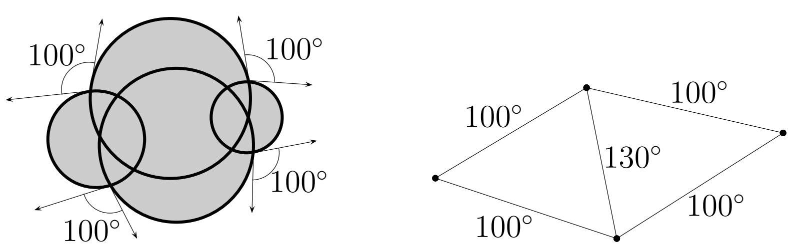


Figure 15: An impossible contact graph given the indicated overlap angles, impossible because the sum of the indicated overlap angles is greater than 360°.



(A) (B)

Figure 16: Another disk configuration, its contact graph, and overlap angles

Observation. *Given a graph G with edges labeled with angles, it is hard to determine whether there is any disk configuration which has G as its contact graph, with those overlap angles.*

5. Rigidity of thin disk configurations

Even though we can't easily figure out whether a given pattern (a contact graph with edges labeled with angles) can actually be realized by a disk configuration, we can still ask what properties a disk configuration should have to guarantee that it is the *only one* having its particular pattern:

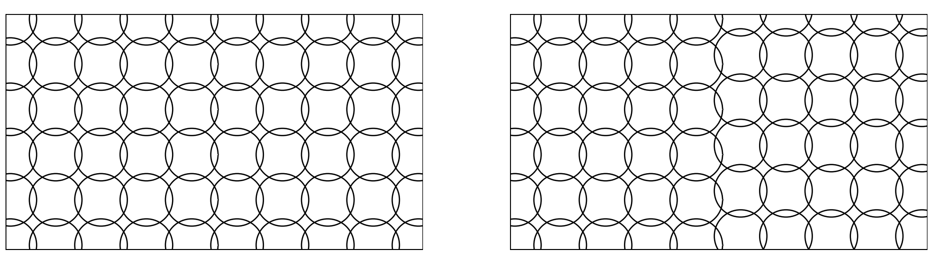
Question 2. *Suppose we start with a disk configuration. What conditions guarantee that there is essentially no other disk configuration having the same contact graph and overlap angles?*

What it means for two disk configurations to be *similar* is the same as what it meant for two circle packings to be similar. Then, as before, the following is not too hard to believe:

Fact. *Suppose that the disk configuration $\mathcal{C}$ consists of only finitely many disks. Then there are disk configurations with the same "pattern" as $\mathcal{C}$, which are not similar to $\mathcal{C}$.*

Problem 12. *Draw a disk configuration which has the same contact graph and overlap angles as the configuration depicted in Figure 13, but which is not similar to it.*

Also, as with circle packings, there are infinite disk configurations which have the same contact graph and overlap angles, but which are not similar. Here is an example:



(A) (B)

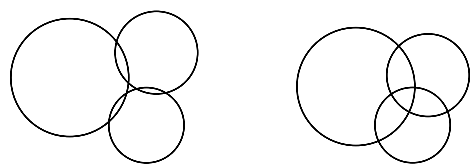
Figure 17: Two infinite disk configurations which are not similar, but which have the same contact graph and overlap angles.

As before, it will help to consider *triangulated* disk configurations: a disk configuration is *triangulated* if its contact graph "cuts the plane into triangles," in the same way as for circle packings. Then the following theorem turns out to be true:

Uniqueness Theorem. *If two thin triangulated disk configurations have the same pattern (contact graph with overlap angles), then the two disk configurations are similar.*

We still need to define the extra adjective *thin* in the statement of the above theorem:

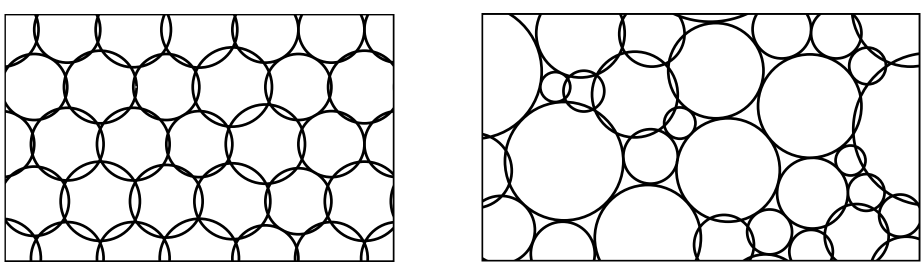
Definition 7. *A disk configuration is* ***thin*** *if no three of its disks meet at any point.*



(A) (B)

Figure 18: A disk configuration which is thin and one which is not. The configuration in (B) is not thin, because there are points which are in all three of the disks at once. This does not happen in (A), so the configuration in (A) is thin.

We give two examples of disk configurations to which the above uniqueness theorem applies:



(A) (B)

Figure 19: Pieces of thin triangulated disk configurations.

We also give an example of a disk configuration which is not covered by the above theorem:

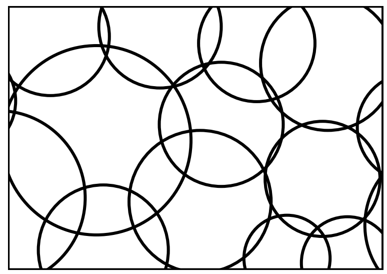


Figure 20: A triangulated, but non-thin, disk configuration.

Another way of stating the uniqueness theorem above is to say that thin triangulated disk configurations are *rigid*, meaning that we cannot modify them while keeping the same contact graph and overlap angles, except by applying rotations, scaling, etc.

As before, consideration of Figure 17 makes the following believable:

Fact. *Suppose that $\mathcal{C}$ is a thin disk configuration which is not triangulated. Then there are disk configurations with the same pattern as $\mathcal{C}$, which are not similar to $\mathcal{C}$.*

Thus the uniqueness guaranteed by the above theorem is the best uniqueness we could hope for, at least for thin disk configurations.

The uniqueness theorem for triangulated thin disk configurations was proved as part of [3]. The "thinness" condition is used strongly in the proof, but it seems likely that it is not necessary, so we can speculate:

Conjecture. *The uniqueness theorem for thin triangulated disk configurations is still true if we completely get rid of the thinness requirement from the statement.*

In 1999 Z.-X. He, in [1], proved the following:

Theorem. *The above uniqueness theorem is still true if we get rid of the thinness requirement from the statement, provided that none of the overlap angles is bigger than* $90°$.

However, the restriction that overlap angles stay below $90°$ is a strong one, and He never published a version of this theorem where he managed to eliminate this restriction.

6. Closing remarks and further references

You might wonder if circle packing has any real-world applications. To describe examples in any detail would require considerably more room than we have here. However, for instance, some people are trying to use circle packings as a tool in medical imaging, for example to map regions of the brain, c.f. [7, Section 23.4].

Stephenson has written an article [6] and a book [7] on circle packing, with many drawings and examples. They should both be readable for an advanced undergraduate math major. For more advanced readers, a nice survey of the research area as of 2011 is contained in the first half of [4], with a focus on the contributions of Oded Schramm. There is also a survey of the area including many references in Chapter 2 of [3].

References

[1] Zheng-Xi He. Rigidity of infinite disk patterns. *Annals of Mathematics*, 149(1):1–33, 1999.

[2] Paul Koebe. Kontaktprobleme der konformen abbildung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 88:141–164, 1936.

[3] Andrey Mishchenko. *Rigidity of thin disk configurations*. Ph.d. dissertation, University of Michigan, Ann Arbor, 2012, available at http://hdl.handle.net/2027.42/95930, accessed on July 7, 2016.

[4] Steffen Rohde. Oded Schramm: from circle packing to SLE. *Annals of Probability*, 39(5):1621–1667, 2011.

[5] Oded Schramm. Rigidity of infinite (circle) packings. *Journal of the American Mathematical Society*, 4(1):127–149, 1991.

[6] Kenneth Stephenson. Circle Packing: A Mathematical Tale. *Notices of the American Mathematical Society*, 50(11):1376–1388, 2003.

[7] Kenneth Stephenson. *Introduction to circle packing: the theory of discrete analytic functions*. Cambridge University Press, 2005.