Paul's Dilemma: Is This a Polyhedron?

Bethany Noblitt
Northern Kentucky University

Shelly Harkness
University of Cincinnati

Follow this and additional works at: <https://scholarship.claremont.edu/jhm>

Part of the [Education Commons](#), and the [Physical Sciences and Mathematics Commons](#)

Recommended Citation

Bethany Noblitt & Shelly Harkness, "Paul's Dilemma: Is This a Polyhedron?," *Journal of Humanistic Mathematics*, Volume 7 Issue 2 (July 2017), pages 72-89. DOI: 10.5642/jhummath.201702.05. Available at: <https://scholarship.claremont.edu/jhm/vol7/iss2/5>

©2017 by the authors. This work is licensed under a Creative Commons License.

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | <http://scholarship.claremont.edu/jhm/>

The editorial staff of JHM works hard to make sure the scholarship disseminated in JHM is accurate and upholds professional ethical guidelines. However the views and opinions expressed in each published manuscript belong exclusively to the individual contributor(s). The publisher and the editors do not endorse or accept responsibility for them. See <https://scholarship.claremont.edu/jhm/policies.html> for more information.

Paul's Dilemma: Is This a Polyhedron?

Bethany Noblitt

Department of Mathematics and Statistics, Northern Kentucky University, USA
noblittb@nku.edu

Shelly Sheats Harkness

Teachers College, University of Cincinnati, Cincinnati, Ohio, USA
shelly.harkness@uc.edu

Abstract

Teachers play the *believing game* when they honor students' mathematical thinking, even when it means they must suspend their own mathematical thinking momentarily [2, 3]. The study reported here tells the story of what happened in a university mathematics classroom when one student did not think that a particular figure satisfied the definition of a polyhedron and the instructor chose to play the believing game. The result was a very rich discussion, where both students and the authors grappled with their own mathematical understanding. One author served as the instructor of the course and the other author was an observer, taking field notes and video recordings that provided evidence.

Definitions of geometry terms are important in order for students to classify figures. When considering a set of figures, teachers want their students to be able to distinguish between triangles and pyramids and to identify which figures are rectangles and which figures are squares. But definitions are important for another reason. They can prompt rich mathematical discussions in the classroom. The narrative we portray in this paper occurred when a student described a particular three-dimensional figure and asked if it was a polyhedron. Simply asking if a figure satisfies the definition of a polyhedron is not in itself a high-level task sufficient to facilitate meaningful discussion. What can an instructor do to encourage meaningful discussion of a mathematical definition? The mathematical discussion described here was the result of taking advantage of a teachable moment created by a perfect storm

consisting of a difficult definition, a nonstandard example, and an instructor willing to participate in both the believing and the doubting games.

Bethany was teaching the course, Mathematics for Elementary and Middle School Teachers, and Shelly was taking field notes as she videotaped the classroom discourse. We were collecting data to answer the following research question: How does a teacher play the *believing game* [2, 3] in a mathematics classroom? Elbow [2, 3] contended that we can improve our attempt to understand by using two opposing processes: methodological belief and methodological doubt. In terms of the art of teaching, believing is an endeavor to find virtues and strengths, no matter how unlikely students' ideas, solutions or answers might seem and doubting is an attempt to find flaws or contradictions. Unfortunately, methodological doubt is essentially the automatic mode of logic that some mathematics teachers use. As teachers, we may doubt too much. However, we should make conclusions only after considering the results of both believing and doubting when students' ideas, answers, or solutions are deemed incorrect or wrong. When teachers play the believing game they honor and respect students' mathematical thinking [4, 5, 6, 7]. Honoring students' mathematical thinking while at the same time keeping the mathematical goals of the lesson in mind and prominent can be challenging [12]. Thus, the believing game is a tough game to play, but one that has great payoffs.

At the time of this study, Shelly had participated in research on the believing game in the mathematics classroom, but Bethany had not. Shelly's previous research on the believing game in the mathematics classroom focused, in part, on identifying when a professor played the believing game, see [4, 5, 6]. In [7], Bethany and Shelly described the different teacher and student actions in a classroom that prompted the professor to play the believing and doubting games. This article elaborates on one of those specific instances of when a professor played the believing game and describes in great detail the rich whole-class discussion that took place as a result.

One aspect of mathematical discourse that a balanced use of methodological doubt and belief can facilitate is that of *building*. In building, students respond to each other's mathematical ideas and use each other's ideas as a basis for their own mathematical thinking [11]. A student providing additional insight into another's idea or creating his or her own mathematical conjectures based on another's ideas are both evidence that building has occurred in the classroom ([11]).

When teachers purposefully balance the use of methodological doubt and methodological belief, building can occur and rich discourse can transpire. In fact, “[m]athematics and science education reforms encourage teachers to base their instruction in part on the lesson as it unfolds in the classroom, paying particular attention to the ideas that students raise ... and adapt instruction in the moment ... ” [13, page 1]. Teachers must help their students engage in mathematical practices such as providing explanations, building on one another’s contributions, making connections, and using representations; and in the same moments teachers’ facilitating moves should be dependent on what their students say and do [1]. The demands on teachers who engage in these practices in fleeting amounts of time between hearing, listening to, and then orchestrating the discourse are enormous.

In fact, we deem hearing and listening as separate practices. Listening implies hearing and then delving deeper to understand. Kastberg, in [8], described a classroom episode:

I was working through my plan and choosing to answer for them [the students] something that made sense to me. A little voice challenged me to change my direction because an answer I had not even considered made sense to a student ... and all I did was listen, stop what I was doing and allow them to explore their thoughts. (page 158)

We claim that the voice Kastberg heard challenged her to play the believing game. Most teachers have heard this same voice in the classroom asking, “Will you believe? Or will you doubt?” Answering yes to the question, “Will you believe?” can be very difficult. It is a decision that must be made quickly and one which will have consequences for the rest of the classroom discussion.

In this paper Bethany tells the story of what happened in her mathematics classroom when she balanced doubting with believing, when she heard that voice asking, “Will you believe?” and Bethany answered, “Yes.” Shelly and Bethany were studying their research question, “How does a teacher play the believing game in the mathematics classroom?” by collecting classroom observation data, which included field notes and video. We considered the *a priori* categories of believing and doubting, as described by Elbow [2, 3].

The class in which this discussion took place was Mathematics for Elementary and Middle Grades Teachers which had 28 students, about one-fourth of whom were planning to become middle grades teachers and most of whom were sophomores and juniors. The content of the course was geometry and included two- and three-dimensional figures, measurement, congruence, and transformations. The overall goal of the course was for students to develop deep understanding of mathematical concepts important to the teaching of elementary and middle grades mathematics. The first several class meetings of the course were devoted to two-dimensional geometry and establishing terminology and relationships. Before the discussion on three-dimensional figures students had defined and analyzed polygons in general and specifically they had looked closely at different types of quadrilaterals and triangles. They also had some experience with geometric proof; for example, they had proven alternate interior angles are congruent and the sum of the measures of the angles of a triangle is 180° .

Our story focuses on what transpired when students in Bethany's class tried to determine whether or not a certain figure was a polyhedron. In the text for the course, *Mathematics for Elementary Teachers: A Contemporary Approach* [10], the definition of a polyhedron is "the union of polygonal regions, any two of which have at most a side in common, such that a connected finite region in space is enclosed without holes" (page 655). This is the definition that prompted a rich discussion that led to balanced playing of the believing game and the doubting game.

Additionally, there was another unexpected aspect of the believing game present during the class discussion on polyhedra. Not only was Bethany intentionally playing both the believing game and the doubting game, the students joined her in these games. They believed and doubted Bethany and each other. Their doubting of her understanding and/or explanations greatly increased the richness of the discussion. Their believing and doubting of each other's arguments gave them the opportunity to engage each other in mathematical discussions.

It was during the reflection of this classroom discussion that Shelly and Bethany invented the new terms, *reserved believing* and *reserved doubting* [7]. Reserved believing appears on a continuum of play where the teacher does not believe her own mathematical understanding to be the only understanding, yet, she does not fully doubt her own mathematical understanding, either.

When a teacher is participating in reserved believing, she tries to find merits in students' mathematical understanding. During the class discussion, Bethany wanted to play the believing game with her students. She wanted to find strength in their arguments, but she needed more from them before she was willing to completely suspend her own beliefs and trust their understanding. Reserved believing was a mechanism to motivate students to convince Bethany of the merit of their thinking. Likewise, reserved doubting occurs when a teacher balances her own mathematical understanding with an attempt to find flaws in students' understanding. In this class discussion, Bethany entered into the believing game by first playing reserved believing, giving students the opportunity to convince her to believe them.

In the remainder of this paper, we describe different components of the classroom discussion that exemplify the believing game, the doubting game and the reserved believing game.

“Like a donut”

The discussion about polyhedra began in a very typical way. Bethany talked about how planes intersect to form dihedral angles and those planes together form the faces of a three-dimensional figure. At that point, she cited the textbook definition of a polyhedron (from the Musser, Burger, and Peterson text [10]) and tried to engage the students in examining the definition by considering it phrase by phrase while looking at an example or a non-example. Bethany held up a rectangular prism and talked about how the prism satisfied each part of the definition. This included addressing if the prism was the union of polygonal regions, if any two of the polygonal regions had at most a side in common, if the inside was connected, and if the figure had no holes.

The class talked about this part of the definition a little bit more to really understand what “holes” meant in this context. According to Musser, Burger, and Peterson [10], the object in Figure 1 is not classified as a polyhedron because it has a “hole.” Bethany described how the “hole” in the definition is a “hole” that goes through the entire figure. This is different from a hole that might appear in one of the faces. In the solid models she used in class, one of the faces had a small hole in it so a teacher can pour water or sand in it to illustrate the concept of volume. In the past, students have thought that that hole was the type of hole that was being described in the definition.

Bethany pointed out to the class that the hole in the definition was the hole that was going through the center of the figure. As she described this, a student said, “Like a donut.” Her reply was, “Literally. Like a donut.” It was at this point that she thought the class had some understanding about the concept of a hole in this context and its relation to determining if a figure was a polyhedron.

Figure 1: “Instead of saying a fly, could you say water?” All images in this article were created by the authors using Geometer’s Sketchpad.

While a hole in one of the faces would also present a problem in the definition of a polyhedron, it is not the same problem as when there is a hole that goes through the entire figure as it does in Figure 1. If one of the faces had a hole in it, the space would not be enclosed, which would prevent the figure from being classified as a polyhedron.

The question, “Is the inside connected?” also deserved special attention in the class discussion. Bethany used the illustration that: in order for the inside to be connected, if a fly were inside the figure, it would be able to access the entire inside of the figure. If it can’t, that means the figure is not connected. After considering a few examples of this, a student had an alternative way to view it. She said, “Instead of saying a fly, could you say water? Would that better explain it?” To which Bethany replied, “What needs to happen to the water?” The student said, “Well, it needs to go like ...

throughout the entire thing.” The idea of connected can be confusing at times, especially depending on the figure under consideration. This was a good alternative way to conceptualize the meaning of the connectedness of a polyhedron. At this point, students seemed to agree that we could use the “water test” or the “fly test” to determine if the inside of a figure was connected or not.

“A box and a plastic Easter egg”

Another component of the definition that received attention in class was that a polyhedron is the union of polygonal regions. In the Musser, Burger, and Peterson text [10], a polygon is defined as a simple closed curve made up of line segments. Because of their understanding of this definition, students were comfortable in determining that the figure below was not a polyhedron.

Figure 2: A non-polyhedron.

It was at this point in Bethany’s very typical class meeting on polyhedra that the discussion turned atypical. A student in the class, Amy, described a figure and asked if it was a polyhedron. She said, “Say you took . . . the box and like you know those plastic Easter eggs you can fill with stuff? You took the top half of one of those and stuck it in. You would still be touching on edges. Would that still be [a polyhedron] if that was inside the box?”

After some clarification about the figure that Amy was describing, Bethany and the students determined it to look something like the figure in Figure 3, where the open end of the egg was still open, creating a concave figure. It is worth noting that Bethany had a prism and, and believe it or not, another student had the top of a plastic egg in her book bag that they used to model the figure.

Figure 3: Figure described by a student that prompted the question, “Is this a polyhedron?”

The discussion continued, with a student, not Amy, commenting, “There’s [sic] no angles being formed within the egg.” After being asked what aspect of the definition of a polyhedron that violated, the student replied, “the polygonal regions.” So, the student understood that a polyhedron needed to be made up of polygonal regions, which implied figures made up of line segments, which in turn would create angles. The egg shape did not have any angles, and thus was not made up of polygonal regions.

“What if the egg was a cube?”

At this point, I removed the plastic egg from the prism and replaced it with a cube. Almost immediately, a student said, “You still can’t get to all the regions, though.” Another student said, “But the fly could not...” It was Bethany’s sense at this point in the discussion that students were not seeing this as a concave figure, but as a small cube inside a prism, where the small cube was sitting on one of the faces of the prism, as shown in Figure 4.

Figure 4: Cube within a prism — non-concave.

Bethany clarified that the square region at the bottom of the small cube was not really there and that the figure was actually concave as seen in Figure 5.

Figure 5: Cube within a prism — concave.

To do this she put her fingers on the square at the bottom of the cube and said, “If that square is cut out and I . . . put my fingers up there . . . If that square’s cut out, then I’d feel like that part where my fingers would be . . . I feel like that’s the exterior of the figure.” Students seemed to agree with this and Bethany assumed that there was consensus about what the figure looked like. At this point, she decided they were ready to discuss whether the figure was, or was not, a polyhedron. It was at this point that the discussion again took a turn to the unexpected.

“If that’s not a polygon . . . then that cannot be a polygonal region.”

A student in the class, Paul, said, “When we were talking about what a polygon is, you had something similar that had the box inside the box. It can’t be a polygon.” Bethany alleged she knew what Paul was talking about. A few weeks prior to this, the focus on the lesson was on the definition of a polygon and she had put several examples and non-examples on the board to discuss. One of the non-examples was the figure shown in Figure 6.

Figure 6: A non-polygon.

This was discussed as a non-example because a polygon is a curve; this figure is not a curve, because you would have to pick up your pencil to trace it. Paul had determined that Figure 6, “the box inside a box” was the face of Figure 5 that had the smaller square cut out (the bottom of Figure 5). Paul had also decided that, based on a previous class discussion, this face was not a polygonal region, thus violating one of the requirements of the definition of a polyhedron. When Bethany asked Paul to clarify his comment, he said,

“... if that [Figure 6] is not a polygon, which we have said that it's not, then that [bottom of Figure 5] cannot be a polygonal region, thus violating [the definition of polyhedron].” This was a thoughtful observation made by Paul. He had made a connection between polyhedra and polygonal regions.

It did not take long for another student, Jan, to chime in, “Oh, wait. Is that the whole thing, or is that just the polygonal regions touching its edges? So, it would really just break it up into several polygonal regions. That isn't [a polygonal region] — that's the connection [of] several [polygonal regions].” This comment prompted lots of discussion among students. The class, with input from Bethany and Shelly, determined that Jan was referring to the possibility that the face may look like the figure in Figure 7a. However, in that figure, one of the polygonal regions (rectangle in top left of figure) has one side that is not fully shared with one of the other (rectangular) polygonal regions. This prevents the figure from satisfying the definition of a polyhedron which requires polygonal regions to have one side in common. At this point in the conversation, Shelly played the believing game. She believed that Jan's idea of partitioning the face in question (Figure 6) into polygonal regions was a way to view the face as the union of polygonal regions. She recognized that Jan's particular partitioning did not work because of the side-sharing requirement but attempted to envision a way in which the polygonal regions shared one side. Shelly then described the figure in Figure 7b as an alternative way to partition the face into polygonal regions that had only one side in common and Bethany drew it on the board.

(a) Does not allow figure to satisfy the definition of a polyhedron.

(b) Allows figure to satisfy the definition of a polyhedron.

Figure 7: Possible partitionings of the face in question.

Because of Shelly's participation in the believing game, she and Jan were able to determine a way to visualize the face in Figure 6 that could address Paul's issue. Jan and Shelly found a loophole in the definition that would enable us to classify the figure in Figure 5 as a polyhedron.

One could argue that because Jan's suggested partitioning in Figure 7a was not viable, Shelly also played the doubting game. However, Shelly used her belief in Jan's understanding, along with a balance of her own mathematical understanding and an attempt to find the flaws in Jan's figure, to determine a partitioning (Figure 7b) that would give them what they wanted. Shelly's figure corroborated Jan's observation that the face was the union of polygonal regions with at most a side in common, and the rest of the polyhedron was clearly the union of polygonal regions; therefore, the figure satisfied that portion of the polyhedron definition. This was something that Bethany had not seen or thought of when trying to reply to Paul's comments. The class seemed to be at a point where the students might be ready to agree that this was a polygonal region. Bethany asked Paul what he thought. He still was not convinced. He said, "If you say it is, it is."

"I'm stubborn. I still say no."

At this point, other students started asking questions about the figure, refining their understanding of the discussion so far, clarifying that the figure in question was enclosed and that the figure had no holes. The class was still trying to come to a conclusion regarding the question of whether this figure was a polyhedron or not.

After some further discussion, Bethany returned to Paul. "Where do you stand?" she asked. Paul replied, "Hmm... I'm stubborn. I still say no." After further prompting, he continued, "I still say... I still don't say it's a polygonal region." He expressed that the face in Figure 6 was what bothered him, saying, "if that isn't a polygon, and we've said that it isn't, then that is not a polygonal region." At this point, Bethany was ready to move on and accept that some of the class saw the figure as a polyhedron and that some of them, namely Paul, did not. But Paul would not let her ignore the voice asking, "Will you believe?" Paul did not want to move on.

As Bethany tried to doubt her own mathematical understanding of the situation, she wanted to hear more about his views. He said, "Like this [Figure 7a] and this [Figure 7b] would be the union of polygonal regions,

but if this [Figure 6] isn't a polygon, how can this [Figure 6] be a polygonal region?" He had a good point. In order to try to further explain the argument that had been presented, Bethany said, "I'm saying it's [Figure 7b] the union of polygonal regions because it is the union of these four trapezoids . . . So I can take polygonal regions, put them together like a puzzle and make something that's not a polygon." To Bethany's surprise, Paul replied, "Okay, I get it now." And another student said, "That made sense." Bethany asked Paul specifically, "That made sense?" to which he answered, "Yeah."

While she was not completely sure that Paul understood the argument in a way that convinced him the figure could be a polyhedron, Bethany was happy with the discussion that had taken place. The discussion was student-initiated and Bethany had allowed herself to believe in students' thinking and their ideas, which in turn had prompted a rich mathematical discussion. Not only had the students learned more about polyhedra, but so had Bethany. Whether or not the figure was actually a polyhedron was secondary to students believing and doubting each other's thinking and doubting her. As long as the students understood the mathematical arguments, Bethany was okay with them formulating their own opinions regarding what the arguments told them.

Bethany's Reflection

When Shelly approached me about observing my class for evidence of the believing game and the doubting game, I did not hesitate to say, "Yes." Upon reflection, I don't think I really knew what I was getting myself into. While I had some basic understanding of what the believing game was, based on brief conversations with Shelly, I had not purposefully played the believing game in my classroom. During the semester that Shelly observed my class, I thought about how to play the game more than ever before.

When asked to describe the believing game, I say that you play by allowing students to express their own mathematical understanding and validating that understanding as having merit. This is much easier said than done. Through this experience, I have learned that in order to play the believing game I had to be willing to be vulnerable in the classroom. First, I had to be willing to be mathematically vulnerable. I had to be willing to put my own mathematical understanding aside and consider someone else's under-

standing as perfectly valid, even if it was different from mine. This required me to be “put on the spot” in the classroom, trying to make mathematical sense of an understanding that I had not considered before. Second, I had to be vulnerable with respect to class planning. I had to be willing to allow a classroom conversation to take longer than I had planned. When students are allowed to fully explore and express their understanding, discussions can take much longer than anticipated. I had to be okay with that. It had been a fear of mine that when these mathematical discussions take place, some students might “tune out” because the conversation lasted longer than was typical. In reality, what happened was that many of the students rose to the occasion, participating in the discussion and even contributing to the class in ways that they had not before.

An interesting consequence of playing the believing game with my students was their playing of the doubting game with me. When I was more willing to believe them, they were more willing to doubt me. Paul doubted me basically the entire class period. Because he was willing to express his doubt, I believe other students were more willing to doubt as well. Paul doubted me when I presented arguments in support that Figure 5 was a polyhedron. His doubting and persistence motivated me to find merits in his arguments. Because of this interaction, other students in the class thought more deeply about the definition of *polyhedron* and wanted to defend their thoughts on whether or not Figure 5 was a polyhedron.

Perhaps the students learned more than what a polyhedron was that day. I hope they learned that you can discuss, agree, disagree, and discuss some more and you may not ever come to a clear consensus on a mathematical concept. I believe that students often think that mathematics is “black and white,” when, in fact, it is far from it. Perhaps, after this discussion, they learned that mathematics is shades of grey.

And on my side, overall, I have gained a new understanding of how a rich mathematical discussion can take place when I allow myself to play the believing game with a student, and allow students to play the doubting game with me. My students were deeply engaged and participated in the ensuing class discussion in meaningful ways.

But there are other benefits of playing the believing game in the mathematics classroom. For example I also gained a better understanding of my students’ mathematical thinking. But the benefit that I had not really anticipated was

gaining a deeper mathematical understanding myself. I knew the definition of a polyhedron before this class period. But I had not explored the idea of partitioning one face of the figure into polygonal regions in order to satisfy the part of the definition that says a polyhedron must be “the union of polygonal regions, any two of which have at most a side in common.” This was genius! And a student thought of it, not me. Not only did I learn more about the definition of the polyhedron, I also learned a problem-solving strategy to use in other problems. I now find myself thinking of this as a technique to solve other, unrelated, mathematical problems. I ask myself, “what loophole might I find that would give me an answer to this question?” I truly learned mathematics from my students that day.

Class meetings can do more than teach the students a lesson. Sometimes, when mathematics teachers allow themselves to be vulnerable, they learn lessons too. Choosing to believe in students’ thinking and allowing students to doubt theirs can enhance both students’ and teachers’ own mathematical understanding. I learned that from experience.

Shelly’s Reflection

It is definitely easier to play the believing game while sitting in the back of the classroom and observing and listening to the conversation. I contend that one way to practice playing it is to do just that [6]. Ask a trusted colleague if you can practice playing the believing game while observing one or more of their classes and explain the nature of and reason for your visit.

I also tend to play the believing game when I get the opportunity to observe preservice teachers in mathematics classrooms. I look for times during the classroom conversations when the preservice teachers missed opportunities to believe. Based on anecdotal evidence I am convinced that most preservice teachers doubt answers they deem incorrect or wrong without at least considering their students’ assumptions or trying to tease out what might be right about the thinking behind a “wrong” answer. It is not unusual for me, upon reflection, sometimes while driving back to campus from the observation or while reading my field notes, to have an “aha” moment about something mathematical that a student said or did.

Field notes are helpful, but the power of video is immense [14]. Because we obtained Institutional Review Board permission to videotape the classroom conversations in Bethany’s class, I was able to transcribe the discourse.

The transcripts were helpful in recapturing what occurred but we were, additionally, able to view the videotape multiple times and talk about the nuances of Bethany's and the students' actions in light of playing the believing and doubting games. In fact, we video-recorded these conversations, too. There were multiple layers of research conversations.

It was during the interview conversations that Bethany and I grappled with what occurred during the times when she moved back and forth between doubting to believing. We created the terms *reserved believing* and *reserved doubting* to describe these times [7]. Reserved believing happened with one or more of the following occurred:

- Students questioned something that Bethany said or did; they doubted her.
- Bethany asked for clarification of students' thinking or justifications and then found some merit in their answers or comments.

During times when she was using reserved believing, Bethany alternated between believing and doubting. She sought justifications for students' answers, thinking, or comments. And, meaningful mathematical conversations occurred. Indeed, at the end of the class described in this paper, one of Bethany's colleagues, whose office was across the hall from the classroom, stuck her head in the door and said something to the effect that "that was the best discussion she had ever heard." Is there a better testimonial than that? Perhaps. Personally, my own understanding of what constitutes a definition of a polyhedron was enhanced as we played with mathematics and found a possible loophole in the definition stated in the textbook. Bethany and I are still amazed each time we watch the video and discuss the conversation that transpired about, "What is a polyhedron?"

Conclusion

This sound and rich mathematical discussion was a powerful experience for all involved. Paul's questions were a driving force in the discussion, evidence once again that student thinking is honored when teachers play the believing game [4, 5, 6, 7]. Paul's voice was heard and his thinking was honored. His mathematical understanding was considered, valued, and enhanced by the discussion. Not only was his thinking honored, the thinking of his classmates played a vital role in the discussion as well. Building, as defined by Sherin,

Louis, and Mendez [11], occurred in this classroom episode, as evidenced by Jan's use of Paul's ideas to move her own mathematical understanding forward. Paul's classmates had opportunities to provide insight into his ideas and to make their own conjectures based on some of his thoughts, which enhanced their classroom experience. Building was facilitated by the playing of the believing game. According to Boerst *et al.* [1], teachers must react to what students say and do in ways that facilitate students' engagement in practices such as providing explanations, making connections, and using representations. By balancing believing, reserved believing, reserved doubting, and doubting, Bethany and Shelly were in a position to foster the aforementioned practices. By listening to the voice asking "Will you believe?" as Kastberg did in [8] and answering the challenge to play the believing game, Bethany and Shelly gained a deeper understanding, not only of the students' mathematical understanding, but of their own mathematical understanding as well. What started as a simple question, "Is this a polyhedron?", ended as a valuable learning experience for all.

References

- [1] Timothy Boerst, Laurie Sleep, Deborah Loewenberg Ball, and Hyman Bass, "Preparing teachers to lead mathematics discussions," *Teachers College Record*, Volume **113** (2011), pages 2844-2877.
- [2] Peter Elbow, *Embracing Contraries*, Oxford University Press, New York, 1986.
- [3] Peter Elbow, "The believing game and how to make conflicting opinions more fruitful," in *Nurturing the Peacemakers in our Students: A Guide to Teaching Peace, Empathy, and Understanding*, edited by C. Weber (Heinemann, Portsmouth, NH, 2006), pages 16-25.
- [4] Shelly Sheats Harkness, "Social constructivism and the Believing Game': A mathematics teacher's practice and its implications," *Educational Studies in Mathematics*, Volume **70** (2009), pages 243-258.
- [5] Shelly Sheats Harkness, Beatriz D'Ambrosio, and Anastasia Morrone, "Preservice elementary teachers' voices describe how their teacher motivated them to do mathematics," *Educational Studies in Mathematics*, Volume **65** (2007), pages 235-254.

- [6] Shelly Sheats Harkness, Catherine Lane, Sue Mau, and Amber Brass, "The Believing Game' in mathematics: Stories in a discipline of doubt," *The Journal of the Assembly for Expanded Perspectives on Learning*, Volume **15** (Winter 2009-2010), pages 37-49.
- [7] Shelly Sheats Harkness and Bethany Noblitt, "Playing the believing game: Enhancing productive discourse and mathematical understanding," *Journal of Mathematical Behavior*, Volume **45** (2017), pages 63-77.
- [8] Signe Kastberg and Wendy Otoupal-Hylton, "Learning from voices in classrooms," *Mathematics Teaching in the Middle School*, Volume **12** (2006), pages 156-160.
- [9] Signe Kastberg, Wendy Otoupal-Hylton, and Sherri Farmer, "Self-differentiating in inclusion classrooms: Opportunities to learn," in *Mathematics for Every Student: Responding to Diversity, Grades 6-8*, edited by M. W. Ellis (National Council of Teachers of Mathematics, Reston VA, 2008), pages 85-93.
- [10] Gary Musser, William Burger and Blake Peterson, *Mathematics for Elementary Teachers: A Contemporary Approach*, 9th ed., Wiley, Hoboken, NJ, 2011.
- [11] Miriam Gamoran Sherin, David Louis, and Edith Prentice Mendez, "Students' building on one another's mathematical ideas," *Mathematics Teaching in the Middle School*, Volume **6** (2000), pages 186-190.
- [12] Margaret Smith, Elizabeth Hughes, Randi Engle, and Mary Kay Stein, "Orchestrating discussions," *Mathematics Teaching in the Middle School*, Volume **14** (2009), pages 549-556.
- [13] Elizabeth van Es and Miriam Gamoran Sherin, "Learning to notice: Scaffolding new teachers' interpretations of classroom interactions," *Journal of Technology and Teacher Education*, Volume **10** (2002), pages 571-596.
- [14] Elizabeth van Es, Shari Stockero, Miriam Gamoran Sherin, Laura Van Zoest, and Elizabeth Dyer, "Making the most of teacher self-captured video," *Mathematics Teacher Educator*, Volume **4** (2015), pages 6-19.