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On Rings of Entire Functions of Finite Order

Melvin Henriksen
Harvey Mudd College

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Madras Christian College
Tambaram

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By MELVIN HENRIKSEN

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In 1940, Helmer showed [1, Theorem 9] that in the ring R of entire functions, every finitely generated ideal is principal. That is, if f, g are entire functions without zeros in common, there exist s, t in R such that

$$sf + tg = 1. \tag{1}$$

He asked if this theorem is true in the ring R^* of all entire functions of finite order. A negative answer to this question existed already in 1936 in a paper of Whittaker [7, p. 256]. In particular if the zeros (a_n), (b_n) of f, g respectively are not sufficiently separated as $n \to \infty$, the equation (1) cannot hold with s, t in R^*. In 1940, making use of results of [7], Mursi showed [6] that if there is an $h > \max$ (ord f, ord g) such that the circles $S(a_n, |a_n|^{-h})$ with center a_n and radius $|a_n|^{-h}$ intersect none of the corresponding circles $S(b_m, |b_m|^{-h})$, then (1) holds with both ord s and ord t no greater than max (ord f, ord g).

In an earlier paper [2], the author showed that if M is any maximal ideal of R, the residue class field R/M is isomorphic with the complex field K. In this paper, under some restrictions, this theorem is extended to the ring R_λ of all entire functions of order no greater than λ, and hence to R^*.

DEFINITION. *Let i_h (f, n) be the number of zeros of f contained in $S(a_n, |a_n|^{-h})$, where a zero of multiplicity m is counted m times.*

THEOREM. *Let M' be a maximal ideal of R_λ containing a function f such that i_h (f, n) is bounded for some h. Then R_λ/M' is isomorphic with K.*

The proof proceeds as in [2, Theorem 6]. It is easily seen that R_λ/M' has degree of transcendency c (where c is the cardinal number of the continuum) over the rational field. By a well-known theorem of Steinitz, it is only necessary to show that R_λ/M' is algebraically closed.

Since a maximal ideal of any integral domain is prime, there is a g in M' such that $i_h(g, n) = 1$, for all n. In particular, all the zeros (b_n) of g are simple. Let $\Phi(z, X) = X^m + f_1(z) X^{m-1} + \dots + f_m(z)$ be a polynomial with coefficients in R_λ/M', of degree $m > 0$. For each n, $\Phi(b_n, X)$ is a polynomial with coefficients in K, which has m complex roots. Choose any such and call it δ_n. It is well known that $|\delta_n| < 1 + \max(|f_1(b_n)|, \dots, |f_m(b_n)|)$. Since the order of the f_i, and the exponent of convergence of (b_n) do not exceed λ, it follows from a theorem of Macintyre and Wilson [5, Theorem 4] (also obtained independently by Leont'ev [4]) that there is a t in R_λ such that $t(b_n) = \delta_n$. So $\Phi(z, t(z))$ is in M', whence the theorem.

Remarks: 1. The author does not know if there is a maximal ideal in R_λ that fails to satisfy the hypothesis of the theorem.

2. There exist prime ideals of R_λ and R^* that fail to satisfy this hypothesis. For, the set B of elements of f of R_λ (or R^*) with $i_h(f, n)$ bounded for some h, is closed under multiplication. Hence, one can construct, with the aid of Zorn's lemma, prime ideals not intersecting B. See also [3].

3. If, in the theorem R_λ is replaced by R^*, the constant h in the definition of $i_h(f, n)$ can be replaced by a positive, increasing function of $|z|$ such that $\limsup_{n\to\infty} \dfrac{\log h(|a_n|)}{\log |a_n|}$ is finite. See [5, Theorem 5].

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Purdue University
Lafayette, Indiana, U. S. A.