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Free Market on the Free Way

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FREE MARKET ON THE FREEWAY

by

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APPROVAL OF THE DISSERTATION COMMITTEE

This dissertation has been duly read, reviewed, and critiqued by the Committee listed below, which hereby approves the manuscript of Yuan Cheng as fulfilling the scope and quality requirements for meriting the degree of Doctor of Philosophy in Mathematics.

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Abstract

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Self-driving cars have the potential to decrease congestion and will probably become the future of efficient transportation. This dissertation presents a unique approach to implement sharing lanes on a freeway using the idea of option pricing. A macroscopic physical model (LWR) is implemented by adding noise to the speed which accounts for unexpected events. We then proceed to provide a fair price for lanes in real time.

For my dear Maruko and those who loved her.

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Chapter 1

Introduction

Self driving cars can play a vital role in reducing the congestion of traffic and accidents if implemented in the right manner. Once the cars start communicating with each other we can improve the total travel time and come up with very accurate predictions of the transit times. This is because we have less uncertainty and errors when we computerize the decision making process.

In order to consider the communication we start by having 2 cars, say the incoming car which is on the busier lane and the incumbent car which is on a less busy lane, the incoming car wants to move to the fast lane which can be made possible only if the incumbent car slows down. Human drivers may or may not slow down to let the car into its lane but once we provide a complete computerization of the decision making process as in the case of self driving cars we can be sure of this process.

In order to make this more realistic we suggest a trading mechanism in which a fair price is calculated in real time which is paid by the incoming car to the incumbent car to allow the transit. This immediately raises the question of what the fair price is. This can be calculated by taking into account the difference in the velocities of the two lanes. We implement a stochastic model for velocity which takes into account accidents and drivers' perceived density. We already have mathematical models with the help of traffic equations that are able to forecast the future traffic ahead based on the current state of traffic. The individual preference of drivers plays a major

role. One approach is to model preferences by the traditional financial economic concept of utility function. Instead, we provide a solution inspired by the option pricing approach. The advantage of this approach is that only one macroscopic preference parameter, namely the "price of time", is needed to fit the model. Previously an option pricing by Friesz et al. [9] approach has been used in the congestion pricing literature but in a different context whereby the driver can buy a call option on the price of a toll.

This dissertation is composed of chapter 2 which provides an literature review, chapter 3 which introduces our mechanism, chapter 4 which introduces the models, chapter 5 which provides the numerical analysis, and chapter 6 which provides numerical results, chapter 7 which provides conclusions.

Chapter 2

Literature review

In this dissertation we try to optimize the traffic on a freeway by introducing a trading scheme between cars in real time. To the best of our knowledge, this is the first time this has been done in the literature. In previous the research Wie et al. [25] predicted the evolution of the traffic flow in time on a congested multiple origin-destination taking into account the Wardropian equilibrium. This is an extension of the Beckmann's mathematical programming problem (Beckmann et al. [5]) for a static user equilibrium traffic assignment under the steady state assumptions. Optimal conditions are also derived which are generalizations of the single-destination networks. Klar et al. [15] presented a new model for traffic on a multi-lane freeway in which they track the density and speed across all the lanes but not what is happening in individual lanes. They provided two equilibrium relationships between density and velocity and also provided an easy switching mechanism which allows the car to move from one equilibrium to another. They introduced the idea of pricing a specified origin-destination path at a specified departure time. This is different from our work as their paper models the drivers as C-Nash non-cooperative agents who typically compete for the limited roadway capacity. They provide a numerical analysis which shows that the flow patterns based on congestion call options have lower social costs of congestion than the traditional user optimized flow options.

Yao et al. [26]studied congestion pricing under uncertainty. They show that by introducing con-

gestion derivatives such as an option or forward contract the total cost will be reduced due to commuters departure time changes and risk mitigation. They also state that creating a derivative market is beneficial not only to commuters but also to toll collection agents (such as CALTRANs). Friesz et al. [10] provide an alternative for the scalar conservation law as a flow-based partial differential equation (PDE) and then proceed to solve that semi-analytically by implementing the Lax-Hopf formula which allows a very efficient computational scheme for large scale networks. This has been then implemented into the dynamic user equilibrium (DUE) and solved using a fixed point algorithm and finally the numerical analysis is performed which compares the solutions obtained from the fixed iterations and Kuhn Tucker-conditions and proved them to be valid.

Ban et al. [4] provided a rigorous analysis of the continuous time point-queue models which is a major component in the DUE problem. They focus on the drawbacks of the original point-queue model introduced by Vickrey [24] in which the non-negativity of the queue length is violated. Numerical analysis and convergence of trajectories is established. Han et al. [12] implement the Lighthill–Whitham–Richards model (LWR) in a traffic network, and investigate the convergence of the on-and-off signal model to the continuum model in regimes of diminishing signal cycles. Numerical analysis and error analysis are provided especially when the signal cycles are not small. They also solve a traffic signal optimization problem which shows the novel benefits of implementing the continuum signal model instead of the on and off model. Sometimes we also need to know how much GPS data is needed to accurately predict the total travel time. This has been studied by Patire et al. [21], who propose a hybrid data framework to use real-time, GPS-based, point-speed data from mobile sources to augment previous investments in existing fixed sensors. They conclude that the penetration rates for GPS-based probe data are now suitable for travel time estimation on selected corridors. They also state that fusing a small amount probe data from large number of providers is better than increasing the number of loop detectors for better travel time prediction.

Jabari and Liu [14] offer a new stochastic model of traffic flow which overcomes the issues of negative traffic densities and mean dynamics that are inconsistent with the original deterministic dynamics when the noise part is added to the deterministic equations. The model allows noise/uncertainty specific to the driver gap choice, which is represented by a wide choice of random states dependent

on vehicle time headway. They show that this construction ensures the non-negativity of traffic densities and that the fluid limit of the stochastic model is consistent with cell transmission model (CTM) based deterministic dynamics. Earlier Tie-Qiao et al. [22] developed a stochastic LWR model which is completely based on the influences of the individual drivers on their perceived density and speed deviation. They further show that the drivers individual characteristics have effects on the traffic density only when the initial density is moderate i.e., at this time, oscillating traffic flow will occur and the oscillating phenomena in the traffic system consisting of the conservative and aggressive drivers is more serious than that in the traffic system consisting of the conservative (aggressive) drivers.

Holden and Risebro [13] studied the scalar conservation laws with a noisy nonlinear source. A unique method is then proposed to construct approximate weak solutions, and then proceed to show that this yields a convergent sequence and prove this in the numerical analysis. Feng and Nualart [7] proposed a stochastic entropy solution but by replacing deterministic calculus with Ito's calculus. Uniqueness and existence theories have been developed by adapting compensated compactness arguments to stochastic setting. They also show the dependence of existence of solution on spatial dimension. Mazzia [20] studied the numerical approximation of hyperbolic conservation laws in the one-dimensional scalar case by focusing on Godunov and van Leer's methods. They provided numerical analysis for the linear advection equation and Burgers' equation. Andreianov and Cancès [1] focused on hyperbolic scalar conservation laws with discontinuous bell-shaped flux functions. They proposed a simple formula for the Godunov numerical flux across the interface, for each choice of connection.

Chapter 3

The Lane-switching Mechanism

First we explain why self-driving cars should negotiate with each other in order to change lanes in real time. When people talk about driving, safety is always the most important issue to address. We assume a car can only change to its adjacent lanes if there is enough space and time to accelerate to the same speed of that lane(driver's gap selection). And here is how autonomous cars negotiate to satisfy this assumption.

We define two types of cars that will be used later, the *incoming* cars are cars that are seeking opportunities to change lanes and the *incumbent* cars are cars that are on the lanes which the incoming cars want to move to. Let the position of the incoming car be defined as $(x(t), y(t))$, where $x(t)$ is the longitudinal position at time t, $y(t)$ is the index of lanes at time t. Suppose the incoming car wants to move to lane $y(t) + 1$ on which the incumbent cars are faster. Let the position of the incumbent cars be $(x_n(t), y(t) + 1)$, where $n \in \mathbb{N}$. For all the incumbent cars that can be negotiated, $x_n(t) \leq x(t)$. The incoming car offers a price for the incumbent cars, and it will wait for the first one among the incumbent cars which agrees with that price to slow down and let the incoming car move in. However, the number of incumbent cars that can participate in the negotiation is limited according to the demand of the owner of the incoming car. The following graph show how the mechanism works. There may be some other ways to implement this negotiation process in changing lanes. The key idea is to provide the fair price for different lanes

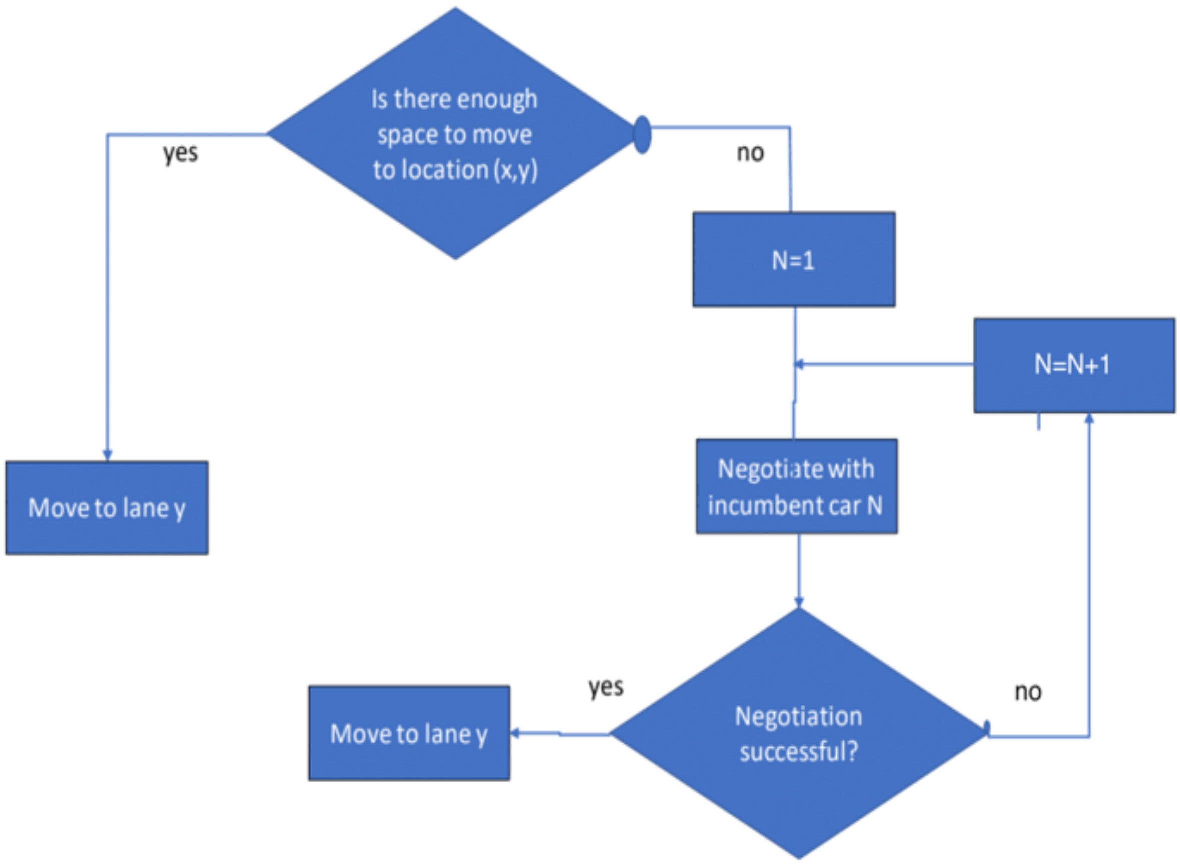

Figure 3.1: The mechanism to change lanes

at different position and time. Once the fair price is set up as a standard for negotiation, we will see how traffic will be affected by this trading mechanism in section 6.

Chapter 4

Continuous Model

Our model can be used in two different ways, *offline*, and *online*:

- offline, in order to calibrate the parameters
- online, in order to determine the lane prices.

One could also imagine a combined model that recursively updates the parameters while determining prices, but we leave this work for future research. We will show in chapter 5 and 6 how we calibrated the parameters of our model using Monte Carlo simulation. We have not implemented the online part yet because we do not have access to an online data feed of density. We highlight in chapter 5.4 how an online system could be implemented.

Our model consists of two parts: traffic model and price model. As for the traffic model, we start by considering the LWR model in multiple lanes. Although we notice that the ARZ(Aw and Rascle [3]) model may perform better when the network of traffic is complex, due to our simple construction of freeway and the complexity of our model, we stick to the LWR model. Also, we found that the deterministic speed can not describe the complex situation on the freeway. Accidents and incoming traffic will affect the traffic speed in a stochastic way. A successful negotiation of changing lanes will also affect the speed of traffic on related lanes. A stochastic term is added to the linear speed and density model in order to explain those random events.

Since we want to explore the behavior of multi-lane traffic, cars should be able to move among adjacent lanes. The corresponding flux is called the transversal flux in the dissertation. The transversal flux consists of two parts:

- "Natural flux": cars that move from the incoming lanes to the incumbent lanes if there is enough space for the incoming cars.
- "Traded flux": cars that move from the incoming lanes to the incumbent lanes if the incumbent cars are paid to leave space.

If we ignore cars that want to exit the freeway, chasing for higher speed is a natural behavior. Natural flux points towards lanes with higher speed than others, which will cause the speed in all lanes to converge eventually. However, the traded flux might behave differently than the natural flux. Cars may pay to move to a slower lane in order (i)to achieve lower speed variation, and (ii) to move to a congested exit lane.

As for the price model, we decide to utilize the idea of option pricing and develop a no-arbitrage model. Although there are many ways to model price in economy, we choose to implement the concept of no arbitrage since no practical data is available to model preference in a sophisticated way, such as utility function. Moreover, many studies have been made to measure the price of time for peak-hour drivers on freeway. Litman [17],Fielding [8] states that drivers are willing to spend $15 for every hour saved on average. **This price of time statistic is** essentially **the only statistic we will need to calibrate the pricing model** in a no-arbitrage model before the system is rolled out. [1]

[1] Obviously the proportion of vehicles equipped with the system is another important parameter. We will also need to calibrate the price noise in a reasonable way. Since we do not have any data before the system is rolled-out, we use heuristic considerations, which are explained in chapter 6.

4.1 Mathematical Preliminaries and Definitions

4.1.1 Mathematical Preliminaries

Brownian Motion A one-dimensional continuous stochastic process $W(t)$ is called a standard Brownian motion if

- *$W(t)$ is almost surely continuous in t,*
- *$W(t)$ has independent increments,*
- *$W(t) - W(s)$ obeys the normal distribution with mean zero and variance $t - s$,*
- *$W(0) = 0$.*

Actually, $W(t)$ is Gaussian process. The *white noise* can be understood as the first-order derivative of $W(t)$ in time, $\xi(t) = \frac{d}{dt}W$. Since Brownian motion is not differentiable, a careful definition is needed.

The Brownian motion and white noise can be constructed by orthogonal expansions. Let $\{m_k(t)\}_{k\geq 1}$ be a complete orthogonal basis in $L^2([0,T])$. The Brownian motion $W(t)$, $t \in [0,T]$ is defined as following

$$W(t) = \sum_{i=1}^{\infty} \theta_i \int_0^t m_i(s)ds,\ t \in [0,T],$$

where ξ_i are mutually independent standard Gaussian random variables. Correspondingly, the white noise is defined as follows:

$$\xi(t) = \sum_{i=1}^{\infty} \theta_i m(s),\ t \in [0,T],$$

Similarly, a one-dimensional Brownian sheet $B(s,x)$ for $0 \leq s \leq 1$ and $0 \leq x \leq T$, which is a Gaussian random field with mean zeros and covariance

$$\mathbb{E}[B(s,x)B(t,y)] = \min(s,t)\min(x,y),$$

can be defined as follows:

$$B(s,x) = \sum_{i=1}^{\infty} W_i(x) \int_0^s m_i(u)du,\ x \in [0,T]$$

In order to move from Eulerian to Lagrangian coordinates we use the Ito-Wentzell formula(Lototsky et al. [18]).

Ito-Wentzell Formula Let $F = F(x,t)$, $x \in [0,T]$, be a random field and let $g = g(t)$, $t \in [0,T]$, be a stochastic process such that

$$F(x,t) = F(x,0) + \int_0^t \mu_F(x,u)du + \int_0^t \int_0^s \sigma_F(x,r,u)B(dr,du),$$

$$g(t) = g(0) + \int_0^t \mu_g(u)du + \int_0^t \int_0^s \sigma_g(r,u)B(dr,du).$$

A pair (F,g) satisfies the Ito-Wentzell conditions if

- *The random variables $F(x,0)$, $x \in \mathbb{R}$ and $g(0)$ are $\mathcal{F}_0$-measurable,*
- *Each of the processes μ_g, σ_g, μ_F, $x \in \mathbb{R}$, and σ_F is $\mathcal{F}_t$-adapted,*
- *F and g are continuous in t,*
- *F is twice continuously differentiable in x and σ_F is continuously differentiable in x,*
- *$\mathbb{E}[\mathcal{I}] < \infty$, where*

$$\begin{aligned}
\mathcal{I} = &\int_0^T |\mu_F(g(u),u)|du + \int_0^T \int_0^1 \sigma_F^2(g(u),s,u)dsdu \\
&+ \int_0^T |\frac{\partial F(g(u),u)}{\partial x}\mu_g(u)|du + \int_0^T \int_0^1 |\frac{\partial F(g(u),u)}{\partial x}\sigma_g(s,u)|^2 dsdu \\
&+ \int_0^T \int_0^1 \frac{\partial^2 F(g(u),u)}{\partial x^2}\sigma_g^2(s,u)dsdu + \int_0^T \int_0^1 |\frac{\partial \sigma_F(g(u),s,u)}{\partial x}\sigma_g(s,u)|^2 dsdu.
\end{aligned}$$

Theorem 4.1.1.1 *Suppose that (F,g) satisfy the Ito-Wentzell conditions. Then*

$$\begin{aligned}
F(g(t),t) - F(g(0),0) = &\int_0^t \mu_F(g(u),u)du + \int_0^t \int_0^s \sigma_F(g(u),r,u)B(dr,du) \\
&+ \int_0^t \frac{\partial F(g(u),u)}{\partial x}\mu_g(u)du + \int_0^t \int_0^s |\frac{\partial F(g(u),u)}{\partial x}\sigma_g(r,u)B(dr,du) \\
&+ \frac{1}{2}\int_0^t \int_0^s \frac{\partial^2 F(g(u),u)}{\partial x^2}\sigma_g^2(r,u)drdu \\
&+ \int_0^t \int_0^s \frac{\partial \sigma_F(g(u),r,u)}{\partial x}\sigma_g(r,u)drdu
\end{aligned}$$

Stochastic Differential Equations We now discuss conditions on the existence and uniqueness of solutions to stochastic ordinary differential equations(SODE). Considering a general form of SODE

$$dX(t) = a(t, X(t))dt + \sum_{r=1}^{m} \sigma_r(t, X(t))dW_r(s)$$

where $W_r(t)_{1\leq r\geq m}$ is a set of independent Brownian motions. A strong solution of such SODE satisfies the following

- $a(t, X(t)) \in \mathbb{L}_{ad}(\Omega, L^1([c, d]))$,
- $\sigma_r(t, X(t)) \in \mathbb{L}_{ad}(\Omega, L^2([c, d]))$,
- *and $X(t)$ satisfies the following integral equation*

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \sum_{r=1}^{m} \int_0^t \sigma_r(s, X(s))dW_r(s)$$

Referring to Mao and Sabanis [19], we have the following theorem:

Theorem 4.1.1.2 *Suppose that $X(0)$ is $\mathcal{F}_0$-measurable and $\mathbb{E}[X^2(0)] < \infty$. Assume that the coefficients a, σ satisfy the following conditions.*

- *(Lipschitz condition) a and σ are Lipschitz continuous, i.e., there is a constant $K > 0$ such that*

$$|a(x) - a(y)| + \sum_{r=1}^{m} |\sigma_r(x) - \sigma_r(y)| \leq K|x - y|,$$

- *(Linear growth) a and σ grow at most linearly i.e., there is a $C > 0$ such that*

$$|a(x)| + |\sigma(x)| \leq C(1 + |x|),$$

then the SODE above has a unique strong solution and the solution has the following properties

- *$X(t)$ is adapted to the filtration generated by X_0 and $W(s)$ $(s \leq t)$,*

- $\mathbb{E}[\int_0^t X^2(s)ds] < \infty$.

4.1.2 Definitions for Traffic Model

We denote by $l(t; s, x, y) = \{(X(\tau; s, x, y), Y(\tau; s, x, y)),\ s \leq \tau \leq t\}$ a path starting from time s and position (x, y) to time t. Define $\mathcal{L}(s, x, y)$ to be the set of these paths. We assume cars in all path starts from the rightmost lane, which means $Y(s; s, x, y) \equiv 0$. And there should be a limit for the amount of lanes on freeway, denoted by $\bar{Y}$. To be specific, we sometimes use $X^l(t)$ and $Y^l(t)$ instead of $X(t)$ and $Y(t)$.

Definition The density ρ at location (x, y) and time t is a stochastic process which value is denoted by $\rho(x, y, t)$, where $\rho \geq 0$ and should be bounded from above by the road capacity ρ_{max}.

Definition The speed u at location (x, y) and time t is a stochastic process which value is denoted by:

$$u(x, y, t) = \begin{bmatrix} u^x(x, y, t) \\ u^y(x, y, t) \end{bmatrix} \tag{4.1}$$

where $u^x \geq 0$ is the speed in x direction, u^y is the speed in y direction, which can be positive or negative. We now move from Eulerian coordinates to Lagrangian coordinates.

- **Eulerian Coordinates** the observer focus on a specific location on the road through which the traffic fluxs as time passes
- **Lagrangian Coordinates** the observer follows an individual car as it moves through space and time.

Definition Fix (s, x, y). The speed along path $l \in \mathcal{L}(s, x, y)$ is denoted by

$$v^l(t) = \begin{bmatrix} v^{l,x}(t) \\ v^{l,y}(t) \end{bmatrix} \tag{4.2}$$

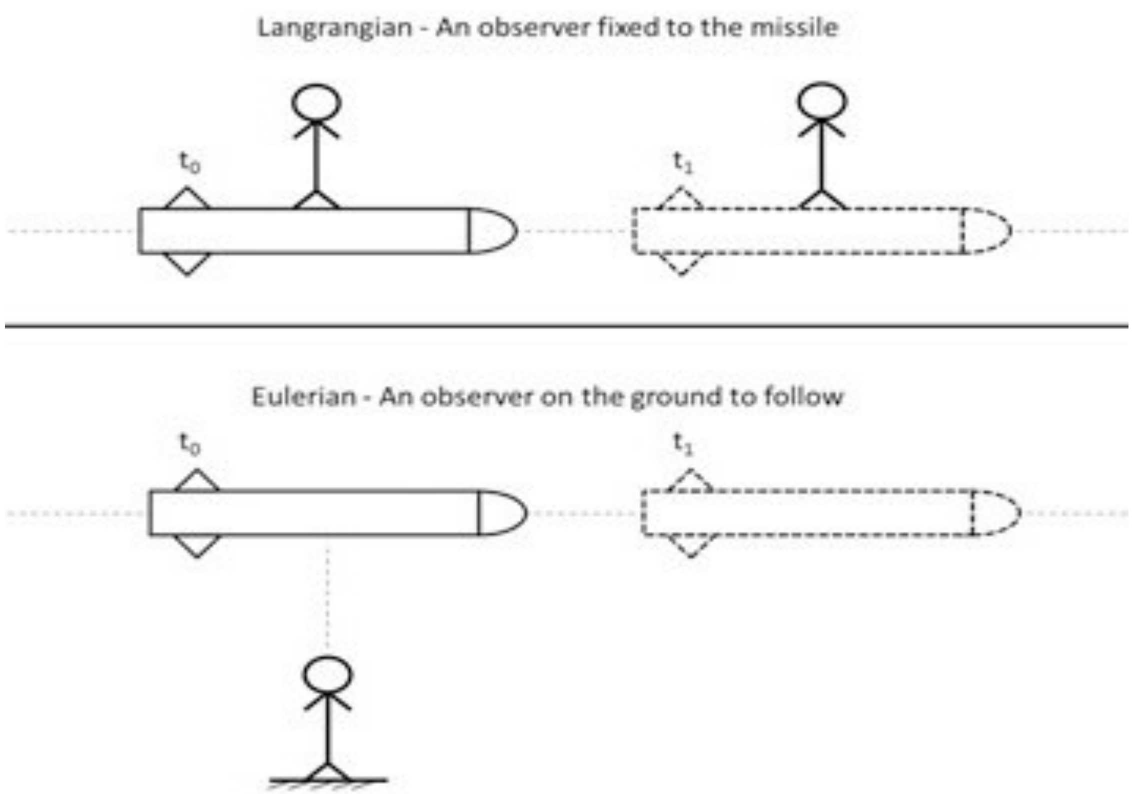

Figure 4.1: A visual way to understand Lagrangian and Eulerian coordinates

Note that rather than putting a label on vehicles, we put a label on the trajectory l that a particular vehicle follows. We have the following relation between Eulerian and Lagrangian speeds:

$$v^l(t) = u(X^l(t), Y^l(t), t) \tag{4.3}$$

Definition The density of cars at location (x, y) at time t is denoted by $\rho(x, y, t)$. The flux of cars is denoted by:

$$q(x, y, t) = \begin{bmatrix} q^x(x, y, t) \\ q^y(x, y, t) \end{bmatrix} \tag{4.4}$$

where q^x is the longitudinal flux(flux in x direction), q^y is the transversal flux(flux in y direction). By definition, the flux is the product of speed and density:

$$q^x(x, y, t) = u^x(x, y, t)\rho(x, y, t) \tag{4.5}$$

$$q^y(x, y, t) = u^y(x, y, t)\rho(x, y, t) \tag{4.6}$$

4.1.3 Definitions for Price Model

All proceeds of buying and selling lanes will be stored and accounted for in a money market account for each car or, equivalently for each path.

Definition The wealth along path $l \in \mathcal{L}(s, x, y)$ is a stochastic process M^l, which can take positive or negative values. We first define lane prices in an Eulerian coordinate system.

Definition The price of lane y at location x and time t is a stochastic field denoted by $p(x, y, t)$, which can take positive or negative values.

Suppose there is not enough room in lane $y + dy$ to let in an incoming car in lane y. A vehicle will be asked to pay $p(x, y + dy, t) - p(x, y, t) \geq 0$ when moving from lane y to lane $y + dy$.
Wealth will depend on the trajectory a particular car follows. To this effect,we define prices in Lagrangian coordinates.

Definition Fix (s, x, y). The lane price along path $l \in \mathcal{L}(s, x, y)$ is a stochastic process π^l, which can take positive or negative values.

As for prices, we have the following relation between Eulerian and Lagrangian prices:

$$\pi^l(t) = p(X^l(t), Y^l(t), t) \tag{4.7}$$

As (4.7) indicates, π^l changes because of three reasons: the x-location of the car, the change in lanes, and the passage of time. Whereas for a change in lanes, the transaction fee will be observable, the others will need to be modeled to reflect future congestion ahead and potential future accidents. By convention,we set the price of lane 0 to zero:

$$p(x, 0, t) = 0 \tag{4.8}$$

Also, for each path l, we start with zero wealth, i.e. : $M^l(s) = 0$. We thus see that

$$M^l(t) = -\pi^l(t) \tag{4.9}$$

Definition: There is an arbitrage between path l_1 and $l_2 \in \mathcal{L}(s, x, y)$if one path(say l_1) is systematically superior, i.e. if there exists a $t \geq s$ such that

$$P(X^{l_1}(t) \geq X^{l_2}(t), M^{l_1}(t) \geq M^{l_2}(t), Y^{l_1}(t) = Y^{l_2}(t)) = 1 \tag{4.10}$$

and

$$P(X^{l_1}(t) > X^{l_2}(t), M^{l_1}(t) > M^{l_2}(t), Y^{l_1}(t) = Y^{l_2}(t)) > 0 \tag{4.11}$$

There will be no arbitrage in the freeway, if for any (s, x, y) there is no arbitrage between any path $l_1 \in (s, x, y)$ and $l_2 \in \mathcal{L}(s, x, y)$.

In a world with arbitrage, there are scenarios where both wealth and longitudinal distance are strictly better. Note that the condition $Y^{l_1}(t) = Y^{l_2}(t)$ is assumed, because cars will exit the freeway eventually.

4.2 Traffic Model

4.2.1 1-D Conservation Laws with Stochastic Source Term

We begin by recalling the LWR model. We know from Lighthill and Whitham [16] that this model is based on the conservation of cars which follows the equation

$$\frac{\partial \rho}{\partial \mathrm{t}} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{4.12}$$

For simplicity, we assume that $u = f(\rho)$, where *(LWR1)* f is a C^2 function, defined on $[0, \rho_{max}]$, where ρ_{max} denotes the maximum possible car density; *(LWR2)* f is a strictly concave function; *(LWR3)* $f(0) = f(\rho_{max}) = 0$. The main assumption for the LWR model is that the average velocity u depends only on the density of the cars. For simplicity, we describe the velocity function $u = u(\rho)$ as

$$u(\rho) = u_{max}(1 - \frac{\rho}{\rho_{max}}) \tag{4.13}$$

where u_{max} denotes the maximum speed of cars.

Normally, there is at least one lane that can allow cars to enter or exit the freeway. The entering or exiting ratio is related to the density of that particular lane and can be represented it in a stochastic form as follows

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = h(\rho) + \sigma g(\rho)\xi \tag{4.14}$$

$$g(\rho) = \sqrt{\rho(\rho^{max} - \rho)} \tag{4.15}$$

where $\xi(t)$ and σ are white noise and volatility parameter respectively, and $h(\rho) = a(b - \rho)$, both a and b are positive.

Actually, if we take out the PDE part in the model, which becomes

$$\frac{\partial \rho}{\partial t} = h(\rho) + \sigma g(\rho)\xi \tag{4.16}$$

the equation has the property of Cox-Ingersoll-Ross model(Cox et al. [6]) due to our definition of h and g, which means that ρ is bounded between 0 and ρ_{max}, and allows us to preserve a non-negative speed.

4.2.2 2-D Conservation Laws with Stochastic Speed

Although lanes in the freeway are discrete and finite, we use a continuous form for the 2-D conservation laws for the convenience of analysis.

$$\frac{\partial \rho(x,y,t,\omega)}{\partial t} + \frac{\partial q^x(x,y,t,\omega)}{\partial x} + \frac{\partial q^y(x,y,t,\omega)}{\partial y} = 0 \tag{4.17}$$

where the initial condition is

$$\rho(x,y,0,\omega) = \rho_0(x,y) \tag{4.18}$$

In order to explain the randomness of speed caused by various reasons, we introduce a stochastic model based on the LWR model

$$u^x(x,y,t,\omega) = u_{max}(1 - \frac{\rho(x,y,t,\omega)}{\rho_{max}})N(x,y,t,\omega) \tag{4.19}$$

where $N(x,y,t,\omega) \geq 0$ is a stochastic field. If $\rho(x,y,t,\omega) \leq \rho_{max}$ then speed will be positive. Let $\alpha = \frac{\rho_{max}}{u_{max}}$ and inverting (4.19), we obtain:

$$\rho(x,y,t,\omega) = \rho_{max} - \alpha\frac{u(x,y,t,\omega)}{N(x,y,t,\omega)} \tag{4.20}$$

Take total derivative with respect to t on both sides,

$$\frac{\partial \rho(x,y,t,\omega)}{\partial t}dt = -\alpha d_t(\frac{u(x,y,t,\omega)}{N(x,y,t,\omega)}) \tag{4.21}$$

Apply (4.17) in (4.21) and expand the equation,

$$N^2(\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y})dt = \alpha[Nd_tu - ud_tN - d_tud_tN + \frac{u}{N}d_tNd_tN] \tag{4.22}$$

or in a better form

$$d_tu = \frac{1}{\alpha N}[N^2(\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y})dt + \alpha ud_tN + \alpha d_tud_tN - \alpha\frac{u}{N}d_tNd_tN] \tag{4.23}$$

Multiplying (4.23) by d_tN, we obtain:

$$d_tud_tN = \frac{u}{N}d_tNd_tN \tag{4.24}$$

Because $dtd_tN = 0$, $d_tud_tNd_tN = 0$ and $(d_tN)^3 = 0$ according to Ito's Lemma. Equation (4.23) is simplified to:

$$d_tu = \frac{1}{\alpha N}[N^2(\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y})dt + \alpha ud_tN] \tag{4.25}$$

Let $g \in C^2(R)$ be bounded both from below and above so that the speed won't be too large or small, a reasonable choice may be

$$g(x) = \begin{cases} \frac{1}{2}(1+e^x) & x < 0 \\ \\ \frac{1}{2}(3-e^{-x}) & x \geq 0 \end{cases} \tag{4.26}$$

which ranges from $[0.5, 1.5]$.

$$N(x,y,t,\omega) = g(U(x,y,t,\omega)) \tag{4.27}$$

where

$$U(x,y,t,\omega) = \sigma(y) \int_0^t \int_0^x f(t,t',x,x')\xi(x',y,t',\omega)dx'dt' \tag{4.28}$$

and $\xi(t,x,y,\omega)$ is a 3-d singular white noise, $f \in C^2$. Then we have the following derivatives:

$$d_tN = g'(U)d_tU + \frac{1}{2}g''(U)d_tUd_tU \tag{4.29}$$

$$\frac{\partial U(x,y,t,\omega)}{\partial x} = \sigma(y) \int_0^t \int_0^x \frac{\partial f(t,t',x,x')}{\partial x}\xi(x',y,t',\omega)dx'dt' \tag{4.30}$$

$$d_tU(x,y,t) = \sigma(x,y)(\int_0^x f(t,t,x,x\prime)\xi(x',y,t)dx'dt + \int_0^t \int_0^x \frac{\partial f(t,t',x,x')}{\partial t}\xi(x',y,t')dx'dt\prime dt) \tag{4.31}$$

Let $\int_0^t \int_0^x \frac{\partial f(t,t',x,x')}{\partial t}\xi(x',y,t')dx'dt\prime dt$ be denoted as $\zeta(t)$. Since $\zeta(t)$ is differentiable in t, thus it has no quadratic variation, or cross-variation with the first term. We have then:

$$\begin{aligned} d_tUd_tU &= \sigma(y)^2 \int_0^x \int_0^x f(t,t,x,x\prime)f(t,t,x,x'')\xi(x',y,t)\xi(x'',y,t)dx'dx''dtdt \\ &:= \Delta \end{aligned} \tag{4.32}$$

Then our model becomes

$$d_tu(x,y,t) = \mu(x,y,t)dt + \int_0^x \sigma_u(x,y,t,x')\xi(x',y,t)dx'dt \tag{4.33}$$

where

$$\mu dt = \frac{1}{\alpha N}\{N^2(\frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y})dt + \alpha u[\frac{1}{2}g''(U)\mathit{\Delta} + \sigma g'(U)\zeta]\} \tag{4.34}$$

$$\sigma_u = \frac{ug'(U)\sigma(y)}{N}f \tag{4.35}$$

4.2.3 Transversal flux

In (4.17) $q^y(x, y, t)$ represents the transversal flux in y direction in continuous sense, which is defined as:

$$q^y(x, y, t) = -k_0 \frac{\partial \rho(x, y, t)}{\partial y} + K_1(x, y, t) \frac{\partial p(x, y, t)}{\partial y} \tag{4.36}$$

where $-k_0 \frac{\partial \rho(x,y,t)}{\partial y}$ represents the "Natural flux" and $k_0 \geq 0$ is the density-elasticity of transversal flux; $K_1(x, y, t) \frac{\partial p(x,y,t)}{\partial y}$ represents the "Traded flux" and K_1 is the price-elasticity of transversal flux. We model it by:

$$K_1(x, y, t) = k_1 \frac{\rho(x, y, t)}{\rho_{max}} \tag{4.37}$$

For a freeway with exits, see chapter 6.2. If we consider only the natural flux, then (4.17) would be:

$$\frac{\partial \rho}{\partial t} = k_0 \frac{\partial^2 \rho}{\partial y^2} \tag{4.38}$$

which is a standard heat equation when $k_0 > 0$. It's sensible that cars will move from a slower lane to a faster lane, which causes diffusion across lanes.

If we consider only the traded flux, and treat $K_1(x, y, t)$ as a constant, then (4.17) would be:

$$\frac{\partial \rho}{\partial t} = -K_1(x, y, t) \frac{\partial^2 p}{\partial y^2} \tag{4.39}$$

If $k_1 \geq 0$ cars are move from the lane with a lower price to the lane with a higher price since it's natural to define that a lane with a higher speed is worthier than the lane with a lower speed.

4.3 Price Model

Like (4.33), we define a forward in time SPDE for price

$$d_t p(x, y, t) = \mu_p(x, y, t)dt + \int_0^x \sigma_p(x, y, t, x')\xi(x', y, t)dx'dt \tag{4.40}$$

Notice that both traffic model and price model are driven by the same white noise ξ. This is not only realistic, but also critical for the implementation of the online price determination system, as we shall see in chapter 5.4. A price model needs to satisfy at least four requirements:

- realistic correlation properties between time and location
- absence of arbitrage
- stationarity
- plausible relation with speed and position.

Like for density, the presence of a volatility kernel σ_p allows to model all types of correlation observed between the price at different times and locations. Before the system is rolled-out however it will be impossible to determine the volatility σ_p exactly, and we will in the empirical chapter consider only a diagonal volatility structure.

Absence of arbitrage is discussed in the next chapter. We note that the requirement of stationarity may look to contradict the no-arbitrage requirement, which forces prices in a Lagrangian coordinate system to be martingales. However we remind the reader that these prices need to be martingales only in the risk-neutral measure, and not in the physical measure (i.e., the measure under which simulations are carried). Here we have full flexibility to determine a model in the physical measure.The requirement of stationarity is important. The model is highly nonlinear and there must be a built-in mechanism from preventing prices to explode. We implement stationarity by making the price process an Ornstein-Uhlenbeck (OU) field(Uhlenbeck and Ornstein [23]). Fixing x, y, this field becomes an OU process. The speed of mean-reversion of this OU process is denoted by $\beta \geq 0$. Finally, we expect lane prices to be roughly proportional to speed u in an infinite road without on-ramps and off-ramps. Thus, up to a constant, the current speed $u(x, y, t)$ can be taken as the long term value (see at time t) of the price in a OU process. In a way, the price follows speed like a moving target. This leads us to consider a drift of the form (for freeways without on-ramps and off-ramps):

$$\mu_p(x, y, t) = \beta(Cu(x, y, t) - p(x, y, t)). \tag{4.41}$$

The *price-adjustment factor* C will be used to scale prices to their correct level. Indeed, if the fair price of a lane is 15, should it be 15 cents or \$15? We will show in chapter 5 that, by linearity, any

price level can be attained without impacting density if we set the initial condition to be:

$$p(x, y, 0) = Cu(x, y, 0). \tag{4.42}$$

We call this fact the numeraire invariance property of the trading mechanism. We emphasize that, due to the presence of the noise term, prices will not be fully correlated with speed in this model, as we should not expect them to be in real life.

On a freeway with on-ramps, we expect that vehicles should pay to move to the right-most lane, even if it is slower than the other lanes. We will see how to model this is chapter 6.2.

4.4 Model Properties

4.4.1 Traffic Model

In the traffic model, we construct the noise in a way that it satisfies the following conditions:

$$\lim_{t\to\infty} \frac{Var(N(x, y, t))}{t} = 0 \tag{4.43}$$

$$\frac{\partial Cov(N(x_1, y, t), N(x_2, y, t))}{\partial |x_1 - x_2|} < 0 \tag{4.44}$$

$$\frac{\partial Cov(N(x, y, t_1), N(x, y, t_2))}{\partial |t_1 - t_2|} < 0 \tag{4.45}$$

Based on historical traffic data, the volatility of speed in the freeway is mostly related to its lane position and time, and the covariance between two longitudinal positions should be negatively related to their distance, which can be interpreted as that the effect of an accident at a certain position will be smaller as the traffic moves further and further away from the accident. Similarly, time plays the same role as longitudinal position. Also, the volatility of speed should be stationary in time, in other words, it should not grow faster than time. Based on these conditions, let

$$f(t, t', x, x') = e^{-a(t-t')^2 - b(x-x')^2} \tag{4.46}$$

where $a > 0$ and $b > 0$. Define

$$\varphi(a, t) = \int_0^t \frac{1}{\sqrt{\frac{\pi}{2a}}} e^{-2a(t-t')^2} dt'. \tag{4.47}$$

We see that $\varphi(a,t)$ is an monotonically increasing, non-negative function with respect to t which has the following property with a positively bounded,

$$\lim_{t\to\infty} \varphi(a,t) = \frac{1}{2}. \tag{4.48}$$

Since $g(U)$ is a monotonically increasing function, we only need to compute the variance and covariance of $U(x,y,t)$

$$\begin{aligned}
Var(U(x,y,t)) &= \sigma^2(y)\int_0^t\int_0^x f^2(t,t',x,x')dx'dt'(ito\ isometry)\\
&= \sigma^2(y)\int_0^t\int_0^x e^{-2a(t-t')^2}e^{-2b(x-x')^2}dx'dt'\\
&= \sqrt{\frac{\pi}{2b}}\sqrt{\frac{\pi}{2a}}\sigma^2(y)\int_0^t \frac{1}{\sqrt{2\pi\frac{1}{4a}}}e^{-2a(t-t')^2}dt'\int_0^x \frac{1}{\sqrt{2\pi\frac{1}{4b}}}e^{-2b(x-x')^2}dx'\\
&= \frac{\pi}{2\sqrt{ab}}\sigma^2(y)\varphi(a,t)\varphi(b,x)
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
Cov(U(x_1,y,t),U(x_2,y,t)) &= \sigma^2(y)\int_0^t\int_0^{min(x_1,x_2)} f(t,t',x_1,x_1')f(t,t',x_2,x_1')dx_1'dt'\\
&= \frac{\pi}{2\sqrt{ab}}\sigma^2(y)\int_0^t \frac{1}{\sqrt{2\pi\frac{1}{4a}}}e^{-2a(t-t')^2}e^{-\frac{b(x_1-x_2)^2}{2}}dt'\\
&\quad \int_0^{min(x_1,x_2)} \frac{1}{\sqrt{2\pi\frac{1}{4b}}}e^{-2b(x_1'-\frac{x_1+x_2}{2})^2}dx_1'\\
&= \frac{\pi}{2\sqrt{ab}}\sigma^2(y)e^{-\frac{b(x_1-x_2)^2}{2}}\varphi(a,t)\varphi(b,min(x_1,x_2)).
\end{aligned} \tag{4.50}$$

Similarly

$$Cov(U(x,y,t_1),U(x,y,t_2)) = \frac{\pi}{2\sqrt{ab}}\sigma^2(y)e^{-\frac{b(t_1-t_2)^2}{2}}\varphi(a,min(t_1,t_2))\varphi(b,x). \tag{4.51}$$

Thus

$$\lim_{t\to\infty}\frac{Var(U(x,y,t))}{t}=\lim_{t\to\infty}\frac{\pi}{2\sqrt{ab}}\sigma^2(y)\frac{\varphi_a(t)}{t}\varphi(b,x)=0. \tag{4.52}$$

$$\frac{\partial Cov(U(x_1,y,t),U(x_2,y,t))}{\partial|x_1-x_2|}=-b|x_1-x_2|\sigma^2(y)e^{-\frac{b(x_1-x_2)^2}{2}}\varphi(a,t)\varphi(b,min(x_1,x_2))<0 \tag{4.53}$$

$$\frac{\partial Cov(U(x,y,t_1),U(x,y,t_2))}{\partial|t_1-t_2|}=-b|t_1-t_2|\sigma^2(y)e^{-\frac{b(t_1-t_2)^2}{2}}\varphi(a,min(t_1,t_2))\varphi(b,x)<0 \tag{4.54}$$

4.4.2 Price Model

We introduce a no-arbitrage theorem between time and money on a freeway.

Theorem 4.4.2.1 *(Schellhorn) Assume that there exists an equivalent martingale measure such that, for each $s<t$, l_1 and $l_2 \in \mathcal{L}(s,x,y)$:*

$$\tilde{E}_s[X^{l_1}(t)-X^{l_2}(t)]=0 \tag{4.55}$$

$$\tilde{E}_s[M^{l_1}(t)-M^{l_2}(t)]=0 \tag{4.56}$$

then there is no arbitrage.

Proof Suppose that there is arbitrage, then both (4.10) and (4.11) should hold. According to (4.10), we have

$$P(X^{l_1}(t)\geq X^{l_2}(t),Y^{l_1}(t)=Y^{l_2}(t))=1 \tag{4.57}$$

$$P(M^{l_1}(t)\geq M^{l_2}(t),Y^{l_1}(t)=Y^{l_2}(t))=1 \tag{4.58}$$

By (4.55) and (4.56) we know that

$$P(X^{l_1}(t)>X^{l_2}(t),Y^{l_1}(t)=Y^{l_2}(t))=0 \tag{4.59}$$

$$P(M^{l_1}(t)>M^{l_2}(t),Y^{l_1}(t)=Y^{l_2}(t))=0 \tag{4.60}$$

thus

$$P(X^{l_1}(t)>X^{l_2}(t),M^{l_1}(t)>M^{l_2}(t),Y^{l_1}(t)=Y^{l_2}(t))=0 \tag{4.61}$$

which contradicts (4.11), then arbitrage condition doesn't hold under the assumption of (4.55) and (4.56), in other words, there is no arbitrage. Q.E.D.

By applying the above theorem, we are able to solve the system of forward in time stochastic partial differential equations for speed and price, since the martingale conditions are satisfied under the same risk neutral measure.

First, we need to move from Eulerian Coordinates to Lagrangian Coordinates. Then we have, for each $l \in \mathcal{L}(s, x, y)$

$$X^l(t) = \int_s^t v_x^l(\tau) d\tau \tag{4.62}$$

$$M^l(t) = -\pi^l(t; s, x, y) \tag{4.63}$$

where v and π are speed and price in Lagrangian coordinates.

Fact 4.4.2.1 If $v_x^l$and π^l are both $\tilde{P}$ martingales under any path l, (4.55) and (4.56) will be satisfied.

Then we relate the Eulerian Coordinates to the Lagrangian Coordinates,

$$v_x^l(t) = u(X^l(t), Y^l(t), t) \tag{4.64}$$

$$\pi^l(t) = p(X^l(t), Y^l(t), t) \tag{4.65}$$

We now drop the path superscript l on X, Y to emphasize the fact that the following relations must hold for each path. The processes $X(t)$ and $Y(t)$ satisfy the following differential equation system

$$dX(t) = u^x(X(t), Y(t), t)dt \tag{4.66}$$

$$dY(t) = u^y(X(t), Y(t), t)dt \tag{4.67}$$

where $u^x(X(t), Y(t), t)$ and $u^y(X(t), Y(t), t)$ are both $\mathcal{F}_t$-measurable process. Apply Ito-Wentzell formula to (4.64) and (4.65),

$$\begin{aligned} du(X(t), Y(t), t) = \mu_u(X(t), Y(t), t)dt + \int_0^x \sigma_u(X(t), Y(t), t, x', t)\xi(x', Y(t), t, \omega)dx'dt \\ + \frac{\partial u}{\partial x} u^x(X(t), Y(t), t)dt + \frac{\partial u}{\partial y} u^y(X(t), Y(t), t)dt \end{aligned} \tag{4.68}$$

$$dp(X(t),Y(t),t) = \mu_p(X(t),Y(t),t)dt + \int\limits_0^x \sigma_p(X(t),Y(t),t,x',t)\xi(x',Y(t),t,\omega)dx'dt$$
$$+ \frac{\partial p}{\partial x}u^x(X(t),Y(t),t)dt + \frac{\partial p}{\partial y}u^y(X(t),Y(t),t)dt \tag{4.69}$$

Let the *market price of risk* λ be a stochastic field adapted to the filtration. Define $\tilde{\xi} = \xi + \lambda$. Suppose that the market price of risk equations below are satisfied:

$$\int\limits_0^x \sigma_u(X(t),Y(t),t,x')\lambda(x',Y(t),t,\omega)dx' = \mu_u(X(t),Y(t),t)$$
$$+ \frac{\partial u^x}{\partial x}u^x(X(t),Y(t),t) + \frac{\partial u^x}{\partial y}u^y(X(t),Y(t),t) \tag{4.70}$$

Then we can rewrite equation (4.68) as:

$$du(X(t),Y(t),t) = \int\limits_0^x \sigma_u(X(t),Y(t),t,x',t)\tilde{\xi}(x',Y(t),t,\omega)dx'dt \tag{4.71}$$

According to Girsanov's theorem(Girsanov [11]), provided some integrability conditions are satisfied, there exists a risk-neutral measure $\tilde{P}$ under which $\tilde{\xi}$ is white noise, and thus v is a martingale. For the price π to be a $\tilde{P}$-martingale, we must transform equation (4.69) into:

$$dp(X(t),Y(t),t) = \int\limits_0^x \sigma_p(X(t),Y(t),t,x',t)\tilde{\xi}(x',Y(t),t,\omega)dx'dt \tag{4.72}$$

Since the market price of risk λ is fixed by the market price of risk equation (4.70), the equation above imposes a relation between the price drift μ_p and the price volatility σ_p. Recursively, we can apply λ found in (4.70) to solve for p in (4.72) at each time step by assuming a functional form either for μ_p or for σ_p, and then solving for the other.

Chapter 5

Numerical Analysis

We start by introducing a table of notations we will use in the following sections.

Name	**Continuous variable**	**Discrete variable**
Density	$\rho(j\Delta x, l\Delta y, i\Delta t)$	$\rho_l^{i,j}$
Longitudinal speed	$u^x(j\Delta x, l\Delta y, i\Delta t)$	$u_l^{i,j}$
Longitudinal flux	$q^x(j\Delta x, l\Delta y, i\Delta t)$	$q_{x,l}^{i,j}$
Transversal flux	$q^y(j\Delta x, l\Delta y, i\Delta t)$	$q_{y,l}^{i,j}$
Longitudinal speed drift	$\mu^u(j\Delta x, l\Delta y, i\Delta t)$	$\mu_{u,l}^{i,j}$
Longitudinal speed factor	$f(j\Delta x, j'\Delta x, i\Delta t, i'\Delta t)$	$f_{i,i'}^{j,j\prime}$
Derivative of speed factor	$\frac{\partial f(j\Delta x, j'\Delta x, i\Delta t, i'\Delta t)}{\partial t}$	$\frac{\partial f_{i,i'}^{j,j\prime}}{\partial t}$
Longitudinal speed volatility	$\sigma_u(j\Delta x, l\Delta y, j'\Delta x, i'\Delta t)$	$\sigma_{u,l}^{i,j,j'}$
Lane price	$p(j\Delta x, l\Delta y, i\Delta t)$	$p_l^{i,j}$
Lane price drift	$\mu^p(j\Delta x, l\Delta y, i\Delta t)$	$\mu_{p,l}^{i,j}$
Lane price volatility	$\sigma^p(j\Delta x, l\Delta y, i\Delta t)$	$\sigma_{p,l}^{i,j}$
Noise Term	$U(j\Delta x, l\Delta y, i\Delta t)$	$U_l^{i,j}$

Table 5.1: Notations of all field variables

Name	Variables & parameters
C	price adjustment factor
B	total money paid
τ	total time saved
k_1	price-elasticity of transversal flux
k_0	density-elasticity of transversal flux
β	speed of mean reversion of the price

Table 5.2: Notations of other variables & parameters

5.1 Offline Parameter Estimation

5.1.1 Predetermined Parameters

Recall that in (4.36), we define the transversal flux as the sum of natural flux and traded flux. k_0 is the density-elasticity of the transversal flux which shows that the difference between the densities of two adjacent lanes will affect the transversal flux linearly. We believe that this is a parameter that is able to be estimated using the real-world data. Thus we follow the records of freeway camera and record the data.(see Appendix A)

5.1.2 Undetermined Parameters

Before the system is deployed we assume we have access to two macroscopic statistics:

- the total amount of cash drivers currently spend on toll roads to driver faster, which we call B
- the total amount of time that drivers currently gain by using toll roads, which we call τ

Before deploying the system, we must calibrate its parameters so that, on average, the buyers pay the same amount to gain the same time advantage. By averaging, we mean averaging by time,

distance and scenario. To this effect, we simulate our system and vary three unsettled parameters: k_1 which is price-elasticity of transversal flux, β which represents the speed of mean reversion in the price model in a Eulerian referential, C which is the price adjustment factor.

We will assume initially that the coefficient of variation of speed is the same as the coefficient of variation of price, which is

$$CV(u^x) = CV(p) \tag{5.1}$$

We start from a reasonable initial condition for these three parameters, $[k_1, \beta, C]$ and updated them if the following conditions are not satisfied:

$$|CV(u^x) - CV(p)| < \varepsilon_1 \tag{5.2}$$

$$|B^n - B| < \varepsilon_2 \tag{5.3}$$

$$|\tau^n - \tau| < \varepsilon_3 \tag{5.4}$$

for any given ε_1, ε_2, ε_3.

According to Litman [17], Fielding [8], the price of travel time savings is \$12.5 per hour on average in local travel, whereas the average price of time is \$18 per hour in intercity travel. Air and high-speed rail is not considered in both cases. Also, in peak hours, drivers are willing to spend \$2.5 for the time saved in 10 miles. We will demonstrate in section 5.1.4 the numeraire-invariance property of our trading mechanism. Thanks to this result, the problem of determining (B, τ) will be decoupled into determining first τ and then selecting the price adjustment factor C that makes us achieve a particular value of B. Here is an important fact that we are going to use to calculate the total time that's saved by changing lanes.

For convenience, we want to compute the time gained using our mechanism by adding up travel times computed only along the same lane. In other terms, we want to use a Eulerian referential in the y direction rather than a Lagrangian referential. We need the following definitions. Let $T(\bar{X}, l(.; s, x, y))$ be the time taken to go from x to $\bar{X}$ along path $l(.; s, x, y)$. Let $\mu^*(s, x, y)$ be the density of cars that trade and $\mu(s, x, y)$ the density of cars that would have liked to trade but can't because the mechanism does not exist. Let $l^*(s, x, y)$ be the path obtained with our trading

and $l(s,x,y)$ be the path obtained without the trading mechanism. The total time taken using our trade-switching mechanism over a cube $[0,\bar{X}] \times [0,\bar{Y}] \times [0,\bar{T}]$ is given by:

$$\textbf{TimeMechanism} = \int\limits_{x=0}^{\hat{X}} \int\limits_{y=0}^{\hat{Y}} \int\limits_{s=0}^{\hat{T}} T(\bar{X}, l(.;s,x,y))\mu^*(s,x,y)dsdydx \tag{5.5}$$

The total time taken when we turn o the trade-switching mechanism (referred to as TimeNormal in the sequel) is given by

$$\textbf{TimeNormal} = \int\limits_{x=0}^{\hat{X}} \int\limits_{y=0}^{\hat{Y}} \int\limits_{s=0}^{\hat{T}} T(\bar{X}, l(.;s,x,y))\mu(s,x,y)dsdydx \tag{5.6}$$

The total time saved is given by:

$$\textbf{TotalTimeSaved} = \textbf{TimeNormal} - \textbf{TimeMechanism} \tag{5.7}$$

In this sequel, we assume that $\mu = \mu^*$ for simplicity. We observed (see Appendix C) that our trading mechanism has a negligible impact on the total throughput. That is, in all the cases we observed, lane speed is barely impacted by the traded flux values. We call this regime the regime of low global impact of the trading mechanism.

Fact 5.1.2.1 :Let $q^{y,t}$ be the traded flux and $T^{fl}(\bar{X}, s, x, y)$ be the time taken to go from s, x, y to $\bar{X}$ along a fixed lane, namely y. In a regime of low global impact of the trading mechanism, **TotalTimeSaved** can be approximated by:

$$\tau = \int\limits_{x=0}^{\hat{X}} \int\limits_{y=0}^{\hat{Y}} \int\limits_{s=0}^{\hat{T}} \frac{\partial T^{fl}(\bar{X}, s, x, y)}{\partial y} q^{y,t}(s,x,y)dsdxdy \tag{5.8}$$

Here is a general example to show the proof of this fact.

In the graph above, there are two trajectories that share the same origin x and destination $\bar{X}$: the red one is the trajectory without lane change, and the green one is the traded trajectory l^* . Following l^* , there are three time point where lanes are traded(changed) s, $s+s_1$, $s+s_1+s_2$ and three longitudinal position x , $x+x_1$, $x+x_1+x_2$ respectively. We define the time saved through the three changes of lanes considering only simple trajectories as τ_1 at time s, τ_2 at time $s+s_1$ and

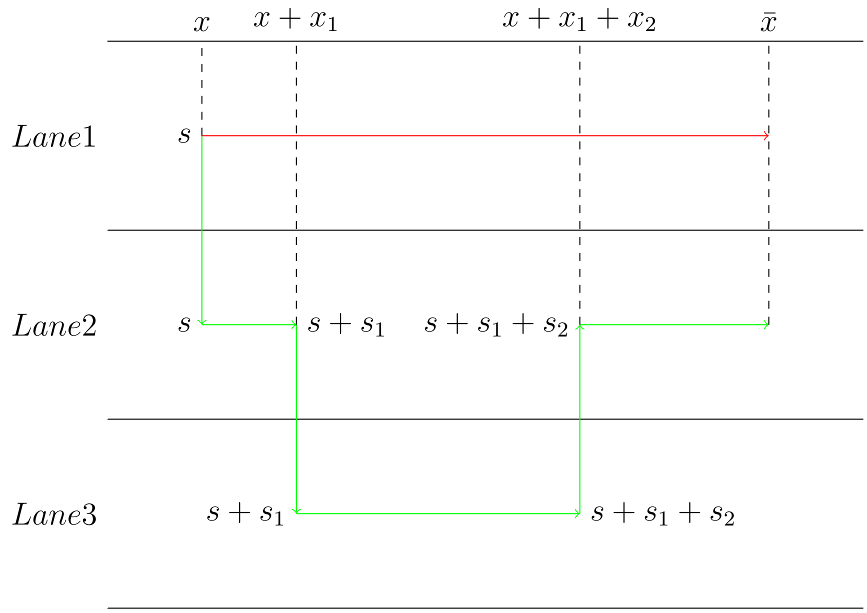

Figure 5.1: Example of Fact 1

τ_3 at time $s+s_1+s_2$. Thus

$$\tau_1 = T^{fl}(\bar{X}, s, x, 1)) - T^{fl}(\bar{X}, s, x, 2)) \tag{5.9}$$

$$\tau_2 = T^{fl}(\bar{X}, s+s_1, x+x_1, 2)) - T^{fl}(\bar{X}, s+s_1, x+x_1, 3)) \tag{5.10}$$

$$\begin{aligned}\tau_3 = &T^{fl}(\bar{X}, s+s_1+s_2, x+x_1+x_2, 3)) \\ &- T^{fl}(\bar{X}, s+s_1+s_2, x+x_1+x_2, 2))\end{aligned} \tag{5.11}$$

Notice that

$$-T^{fl}(\bar{X}, s, x, 2)) + T^{fl}(\bar{X}, s+s_1, x+x_1, 2)) = s_1 \tag{5.12}$$

and

$$-T^{fl}(\bar{X}, s+s_1, x+x_1, 3)) + T^{fl}(\bar{X}, s+s_1+s_2, x+x_1+x_2, 3)) = s_2 \tag{5.13}$$

thus the total time saved is

$$\begin{aligned}\tau_1+\tau_2+\tau_3 = &T^{fl}(\bar{X}, s, x, 1)) \\ &- (s_1+s_2+T^{fl}(\bar{X}, s+s_1+s_2, x+x_1+x_2, 2)))\end{aligned} \tag{5.14}$$

it is obvious that

$$T(\bar{X}, l^*(\cdot; s, x, 1)) = s_1 + s_2 + T^{fl}(\bar{X}, s + s_1 + s_2, x + x_1 + x_2, 2)) \tag{5.15}$$

then we have the following result

$$\tau_1 + \tau_2 + \tau_3 = T^{fl}(\bar{X}, s, x, 1)) - T(\bar{X}, l^*(\cdot; s, x, 1)) \tag{5.16}$$

which proves Fact 5.1.2.1 without loss of generality.

5.1.3 Algorithms

Solving Systems of SPDES

It is hard to solve the theoretical solution for the SPDEs in (4.17), thus we introduce the following methods to find the numerical solution for u and p.

A splitting method is introduced in Holden and Risebro [13], the equation is split into two phases: SDE phase and PDE phase. Then the two phases would be:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (q^y)}{\partial y} = 0 \tag{5.17}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (q^x)}{\partial x} = 0 \tag{5.18}$$

where the solution from (5.17) is taken as the initial condition for (5.18). We define a $t - x$ grid, where time step and space step are Δt and Δx respectively. Also, $\rho(i\Delta t, j\Delta x, l) := \rho_l^{i,j}$, $p(i\Delta t, j\Delta x, l) := p_l^{i,j}$is the density and price at lane k, longitudinal position $i\Delta x$, and time $j\Delta t$ respectively, where $0 < i < I$, $0 < j < J$ and $0 < l < L$ and the rest follows the same rule of notation. Before we start our algorithm to to solve the systems of SPDEs, we need to generate the stochastic terms, $U(x, y, t)$ and so on.

The discrete form of $U(x, y, t)$ and its derivatives are

$$U(j\Delta x, l, i\Delta t) \approx U_l^{i,j} = \sigma(l) \sum_{p=0}^{i} \sum_{q=0}^{j} f(i\Delta t, p\Delta t, j\Delta x, q\Delta x) Z(i, q) \tag{5.19}$$

where $Z(i,q) \sim N(0, \sqrt{\Delta x \Delta t})$ are independent for all i and q.

Also, let

$$U1(x,y,t) = \sigma(l) \int_0^x f(t,t,x,x\prime)\xi(x',y,t,\omega)dx'dt \tag{5.20}$$

and

$$U2(x,y,t) = \sigma(l) \int_0^t \int_0^x \frac{\partial f(t,t',x,x')}{\partial t} \xi(x',y,t',\omega)dx'dt\prime dt \tag{5.21}$$

then $d_t U(x,y,t) = U1(x,y,t) + U2(x,y,t)$ and

$$U1(j\Delta x, l, i\Delta t) \approx U1_l^{i,j} = \sigma(l) \sum_{q=0}^{j} f(i\Delta t, i\Delta t, j\Delta x, q\Delta x) Z(i,q) \tag{5.22}$$

$$U2(j\Delta x, l, i\Delta t) \approx U2_l^{i,j} = -2\sigma(l) \sum_{p=0}^{i} \sum_{q=0}^{j} (i-p)\Delta t f(i\Delta t, p\Delta t, j\Delta x, q\Delta x) Z(i,q) \tag{5.23}$$

$$U^2(j\Delta x, l, i\Delta t) d_t U(j\Delta x, l, i\Delta t) = (U1_l^{i,j})^2 \tag{5.24}$$

Then we can start our algorithms to solve the systems of SPDEs as the follows.

Phase 1

We use forward Euler scheme(Atkinson [2]) to discretize (5.17), and the discretization for the transversal flux in μ_u is defined as following,

$$\frac{\partial q^y(j\Delta x, l, i\Delta t)}{\partial y} \approx q_{y,l}^{i,j} = -k_0 \frac{\rho_{l+1}^{i,j} - 2\rho_l^{i,j} + \rho_{l-1}^{i,j}}{\Delta y} + \frac{K_1(p_l^{i,j}, p_{l+1}^{i,j}) + K_1(p_{l-1}^{i,j}, p_l^{i,j})}{\Delta y} \tag{5.25}$$

where

$$K_1(p_l^{i,j}, p_{l+1}^{i,j}) = k_1 \left(\frac{\rho_l^{i,j}}{\rho_m} \frac{\rho_{l+1}^{i,j}}{\rho_m}\right)^{\frac{1}{2}} (p_l^{i,j} - p_{l+1}^{i,j}) \tag{5.26}$$

and k_1 is an unknown constant.

Phase 2

We implement Godunov's method to find a unique weak solution for (5.18), which takes the results from Phase 1 as the initial condition. Assume that $\rho_k^{i,j}$ is know for certian i, k and all j, how can

we find ρ at time $(i+1)\Delta t$. First, we integrate (5.18) over the domain $D = [(j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x] \times [i\Delta t, (i+1)\Delta t)$, we have,

$$\iint_D [\rho_t(x,y,t) + (q^x(x,y,t))_x] dxdt = 0 \tag{5.27}$$

Applying Green's Theorem, we have,

$$\oint_{\partial D} [\rho(x,y,t)dx - (q^x(x,y,t))dt] = 0 \tag{5.28}$$

Thus

$$\rho_l^{i+1,j} = \rho_l^{i,j} - \frac{\Delta t}{\Delta x}(\bar{F}_l^{i,j+\frac{1}{2}} - \bar{F}_l^{i,j-\frac{1}{2}}) \tag{5.29}$$

where

$$\bar{F}_l^{i,j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{i\Delta t}^{(i+1)\Delta t} F(\rho((j+\frac{1}{2})\Delta x, l, t), N((j+\frac{1}{2})\Delta x, l, t)) dt$$

and $F(x,y) = x(1-\frac{x}{\rho_{max}})y$ is the flux function.

Notice that $F(\rho_l^{i,j}, N_l^{i,j}) = \rho_l^{i,j}(1-\frac{\rho_l^{i,j}}{\rho_{max}})N_l^{i,j}$, which has spatially varying coefficient $N_l^{i,j}$. Andreianov and Cancès [1] provides an explicit representation of interface flux with any choice of interface coupling as following:

$$\bar{F}_l^{i,j+\frac{1}{2}} = min(\bar{F}_{(A,B)}, F(min(\rho_l^{i,j}, b_L), N_l^{i,j}), F(max(b_R, \rho_l^{i,j+1}), N_l^{i,j+1})) \tag{5.30}$$

Moreover, when $F^{opt}(\rho_l^{i,j}, \rho_l^{i,j+1}) > \bar{F}_{(A,B)}$, i.e., the constraint is active, one has

$$\gamma_L(\rho) = A, \qquad \gamma_R(\rho) = B$$

where $b_L = b_R = \frac{\rho_{max}}{2}$ are the local maximum points with respect to x. And

$$F^{opt}(\rho_l^{i,j}, \rho_l^{i,j+1}) = min(F(min(\rho_l^{i,j}, b_L), N_l^{i,j}), F(max(b_R, \rho_l^{i,j+1}), N_l^{i,j+1})) \tag{5.31}$$

Naturally, the constraint on the flux $\bar{F}_{(A,B)}$ is given by $max_{\rho\in[0,\rho_{max}]}\rho u_{max}(1-\frac{\rho}{\rho_{max}})$ which is $\frac{1}{4}\rho_{max}u_{max}$, thus the simplified equation for the interface flux is

$$F_l^{i,j+\frac{1}{2}} = min(\bar{F}_{(A,B)}, F^{opt}(\rho_l^{i,j}, \rho_l^{i,j+1})) \tag{5.32}$$

Moreover, the choice of Δx and Δt should satisfy the CourantFriedrichsLewy(CFL) condition

$$\max_j |F'(\rho_l^{i,j}, N_l^{i,j})| \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \tag{5.33}$$

so that the adjacent waves from the Riemann problem do not intersect before reaching the next time step.

Solving SPDEs

By solving the systems of equations (4.68) – (4.72) with the results from phase 1 and phase 2, we will have the solutions for prices.

To simplify the equation, we use the following definitions,

$$A = \frac{1}{2}g''(U_l^{i,j})(U1_l^{i,j})^2 + g'(U_l^{i,j})U2_l^{i,j} \tag{5.34}$$

According to equation (4.34) and (4.68),

$$\mu_{u,l}^{i,j} = \frac{u_m}{\rho_m g(U_l^{i,j})}(\frac{\rho_m}{u_m}u_l^{i,j}A + g^2(U_l^{i,j})[q_{y,l}^{i,j} + (u_l^{i,j}\rho_l^{i,j-1} - u_l^{i,j}\rho_l^{i,j-1})]) \tag{5.35}$$

$$\tilde{\mu}_{u,l}^{i,j} = \mu_{u,l}^{i,j} + \frac{\Delta t}{2\Delta x}(u_l^{i,j+1} - u_l^{i,j-1})u_l^{i,j} \tag{5.36}$$

Notice that in equation (5.36) we implement the central difference to approximate $\frac{\partial u}{\partial x}$, this needs to be addressed because when we use first-order approximation, either forward or backward finite difference, the results of price will always blow up. An obvious advantage of central difference over forward or backward difference is that it provides second-order accuracy, which is shown as follows:

$$\begin{aligned}
\frac{u_l^{i,j+1} - u_l^{i,j-1}}{2\Delta x} &\simeq \frac{1}{2\Delta x}(u_l^{i,j} + \frac{\partial u_l^{i,j}}{\partial x}\Delta x + \frac{\partial^2 u_l^{i,j}}{\partial x^2}(\Delta x)^2 + \frac{\partial^3 u_l^{i,j}}{\partial x^3}(\Delta x)^3 \\
&- u_l^{i,j} + \frac{\partial u_l^{i,j}}{\partial x}\Delta x - \frac{\partial^2 u_l^{i,j}}{\partial x^2}(\Delta x)^2 + \frac{\partial^3 u_l^{i,j}}{\partial x^3}(\Delta x)^3 + \mathcal{O}((\Delta x)^4)) \\
&= \frac{\partial u_l^{i,j}}{\partial x} + \frac{\partial^3 u_l^{i,j}}{\partial x^3}(\Delta x)^2 + \mathcal{O}((\Delta x)^3) \\
&= \frac{\partial u_l^{i,j}}{\partial x} + \mathcal{O}((\Delta x)^2)
\end{aligned} \tag{5.37}$$

The method for calculating the market price of risk in a Eulerian referential given in (4.70) is as following. The vector of drift is given by $\vec{\mu}_{u,l}^{i} = [\tilde{\mu}_{u,l}^{i,0}, ..., \tilde{\mu}_{u,l}^{i,J}]$. The volatility matrix is given by

$$B_{u,i,l} = \begin{bmatrix} b_{u,l}^{i,0,0} & 0 & & .... & 0 \\ b_{u,l}^{i,1,0} & b_{u,l}^{i,1,1} & 0 & .... & 0 \\ b_{u,l}^{i,2,0} & b_{u,l}^{i,2,1} & b_{u,l}^{i,2,2} & .... & 0 \\ .... & & & & \\ b_{u,l}^{i,J,1} & b_{u,l}^{i,J,2} & b_{u,l}^{i,J,3} & .... & b_{u,l}^{i,J,J} \end{bmatrix} \tag{5.38}$$

where

$$b_{u,l}^{i,j,j'} = g'(U_l^{i,j}) u_l^{i,j} f_{i,i}^{j,j'} / g(U_l^{i,j}) \tag{5.39}$$

Since $B_{u,i,l}$ is a lower triangular matrix which is invertible, the market price of risk is given by

$$\vec{\lambda}_{u,l}^{i} = \frac{B_{u,i,l}^{-1} \vec{\mu}_{u,l}^{i}}{\sigma(l)} \tag{5.40}$$

In order to solve (4.68) combined with (4.70), we need to assume a structure of either μ_p or σ_p. Here we choose to assume a mean-reverting model where the drift is given by

$$\mu_{p,l}^{i,j} = \beta(C u_l^{i,j} - p_l^{i,j}) \tag{5.41}$$

The drift of price in a Lagrangian referential is given by

$$\tilde{\mu}_{p,l}^{i,j} = \mu_{p,l}^{i,j} + \frac{\Delta t}{2\Delta x}(p_l^{i,j+1} - p_l^{i,j-1}) u_l^{i,j} \tag{5.42}$$

Then the volatility of price is given by

$$\sigma_{p,l}^{i,j} = \frac{\tilde{\mu}_{p,l}^{i,j}}{\lambda_{u,l}^{i,j}} \tag{5.43}$$

The price in a Eulerian referential is given by

$$p_l^{i+1,j} = p_l^{i,j} + \mu_{p,l}^{i,j} \Delta t + \sum_{j'}^{j} \sigma_{p,l}^{i,j,j'} Z_l^{i,j'} \tag{5.44}$$

All the techniques above are implemented in algorithm 1. In algorithm 2 we define a function, $TravelTime(\bar{X}, u, i, j, l)$ which is the discrete equivalent of $T^{fl}(\bar{X}, u, s, x, y)$ to calculate the time that's taken starting from time $i\Delta t$ and position $j\Delta x$ to position $\bar{X}$ along lane l. Also, the total money gained, B and the total time saved, τ are calculated in algorithm 2.

Remark 5.2.1.1 In algorithm 1 step 11 and 14, we ignore the following terms which can be found in equation (4.68) and (4.69): $\frac{\partial u}{\partial y} u^y(X(t), Y(t), t)dt$ and $\frac{\partial p}{\partial y} u^y(X(t), Y(t), t)dt$. In Appendix B we show that these terms are negligible.

5.1.4 Impact of Parameters on Total Money Paid, B and Total Time Saved τ

After we introduced algorithm 1 and 2, it is possible to analyze the impact of parameters such as C, k_1 and β on the results of total money gained, B and total time saved, τ. According to our

Algorithm 1 Splitting Methods to Solve SPDEs

Initialize $\rho(0,j,l)$ and all other parameters for all j and l

$\quad u(0,j,l) = u_m(1 - \frac{\rho(0,j,l)}{\rho_m})$, $p(0,j,l) = C * u(0,j,l)$

Generate $[U_l^{u,i,j}, U_{1,l}^{u,i,j}, U_{2,l}^{u,i,j}]$ for all i,j,l and k as (5.22),(5.23) and (5.24)

for k from 0 to $K-1$ (scenario) **do**

$\quad$ **for** i from 0 to $I-1$ (time) **do**

$\quad\quad$ **for** j from 0 to $J-1$ (x direction) **do**

$\quad\quad\quad$ **for** l from 1 to L (lanes) **do**

$\quad\quad\quad\quad$ Phase 1

$\quad\quad\quad\quad\quad$ Define $Tradf(i\Delta t, j\Delta x, l)$ as the traded flux

$\quad\quad\quad\quad\quad$ $Tradf_l^{i,j} = k_1(\frac{\rho_l^{i,j}}{\rho_m}\frac{\rho_{l+1}^{i,j}}{\rho_m})^{\frac{1}{2}}(p_l^{i,j} - p_{l+1}^{i,j}) + k_1(\frac{\rho_{l-1}^{i,j}}{\rho_m}\frac{\rho_l^{i,j}}{\rho_m})^{\frac{1}{2}}(p_{l-1}^{i,j} - p_l^{i,j})$

$\quad\quad\quad\quad\quad$ $q_{y,l}^{i,j} = k_0(\rho_{l+1}^{i,j} - 2\rho_l^{i,j} + \rho_{l-1}^{i,j}) + Tradf_l^{i,j}$

$\quad\quad\quad\quad\quad$ $\rho_l^{i+\frac{1}{2},j} = \rho_l^{i,j} + \Delta t q_{y,l}^{i,j}$ Note: decrease $|q_{y,l}^{i,j}|$ so that $\rho_l^{i+\frac{1}{2},j} \geq 0$ for all j

$\quad\quad\quad\quad$ Phase 2

$\quad\quad\quad\quad\quad$ $F_l^L = \min(\bar{F}_{(A,B)}, F(\min(\rho_l^{i+\frac{1}{2},j}, \frac{\rho_m g(U_l^{u,j})}{2}), g(U_l^{u,j})),$

$\quad\quad\quad\quad\quad$ $F(\min(\rho_l^{i+\frac{1}{2},j+1}, \frac{\rho_m g(U_l^{u,j+1})}{2}), g(U_l^{u,j+1})))$

$\quad\quad\quad\quad\quad$ $F_l^R = \min(\bar{F}_{(A,B)}, F(\min(\rho_l^{i+\frac{1}{2},j+1}, \frac{\rho_m g(U_l^{u,j+1})}{2}), g(U_l^{u,j+1})),$

$\quad\quad\quad\quad\quad$ $F(\min(\rho_l^{i+\frac{1}{2},j+2}, \frac{\rho_m g(U_l^{u,j+2})}{2}), g(U_l^{u,j+2})))$

$\quad\quad\quad\quad\quad$ $\rho_l^{i+1,j} = \rho_l^{i+\frac{1}{2},j} + \frac{\Delta t}{\Delta x}(F_l^L - F_l^R)$

$\quad\quad\quad\quad\quad$ $u_l^{i+1,j} = u_m(1 - \frac{\rho_l^{i+1,j}}{\rho_m})g(U_l^{u,j})$

$\quad\quad\quad$ **end for**

$\quad\quad$ **end for**

Solving SPDE to find price

for j from 0 to $J-1$ (x direction) **do**

for l from 1 to L (lanes) **do**

Define $A = \frac{1}{2}g''(U_l^{u,j})(d_t U_l^{u,j})^2 + g'(U_l^{u,j}) d_t U_{2,l}^{u,j}$ and

$\mu_{u,l}^{i,j} = \frac{u_m}{\rho_m g(U_l^{u,j})}(\frac{\rho_m}{u_m} u_l^{i,j} A$

$+g^2(U_l^{i,j})[q_{y,l}^{i,j} + (u_l^{i,j}\rho_l^{i,j} - u_l^{i,j-1}\rho_l^{i,j-1})])$

update $\tilde{\mu}_{u,l}^{i,j}$ in algorithm 3, $b_l^{i,j,j'} = g'(U_l^{i,j}) u_l^{i,j} \frac{f_{i,ij,j'}}{g(U_l^{i,j})}$

end for

end for

$\vec{\mu}_{u,l}^{i} = [\tilde{\mu}_{u,l}^{i,0}, .., \tilde{\mu}_{u,l}^{i,J}]$ for all l

$$B_{u,i,l} = \begin{bmatrix} b_{u,l}^{i,0,0} & 0 & & .... & 0 \\ b_{u,l}^{i,1,0} & b_{u,l}^{i,1,1} & 0 & .... & 0 \\ b_{u,l}^{i,2,0} & b_{u,l}^{i,2,1} & b_{u,l}^{i,2,2} & .... & 0 \\ .... & & & & \\ b_{u,l}^{i,J,0} & b_{u,l}^{i,J,1} & b_{u,l}^{i,J,2} & .... & b_{u,l}^{i,J,J} \end{bmatrix} \text{ for all } l$$

$\vec{\lambda}_{u,l}^{i} = \frac{B_{u,i,l}^{-1} \vec{\mu}_{u,l}^{i}}{\sigma(y)}$ for all l

for j from 0 to $J-1$ (x direction) **do**

for l from 1 to L (lanes) **do**

$\lambda_{u,l}^{i,j} = sgn(\lambda_{u,l}^{i,j}) \max(|\lambda_{u,l}^{i,j}, \epsilon|)$

$\mu_{p,l}^{i,j} = \beta(C u_l^{i,j} - p_l^{i,j})$

update $\tilde{\mu}_{p,l}^{i,j}$ (see algorithm 3)

$\sigma_{p,l}^{i,j} = \frac{\tilde{\mu}_{p,l}^{i,j}}{\lambda_{u,l}^{i,j}}$

$p_l^{i+1,j} = p_l^{i,j} + \mu_{p,l}^{i,j}\Delta t + \sum_{j'}^{j} \sigma_{p,l}^{i,j'} Z_l^{i,j'}$

end for

end for

end for

end for

subtract price of lane 0 from price of all lanes.

Algorithm 2 Calculating B and τ

Define function $TravelTime(\bar{X}, u_l^{i,j}, i, j, l)$:

$tt = 0,\ t = i\Delta t,\ x = j\Delta x$

$while \quad \bar{X} > x$:

$if \quad \bar{X} - x \geq \frac{u_l^{i,j} + u_l^{i,j+1}}{2}$

$tt = tt + \Delta t$

$x = x + \frac{u_l^{i,j} + u_l^{i,j+1}}{2} \Delta t$

$i = i + 1,\ j = floor(\frac{x}{\Delta x})$

$else$:

$tt = tt + \frac{x - \bar{X}}{(u_l^{i,j} + u_l^{i,j+1})/2}$

$Return \quad tt$

$B = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{l=1}^{L-1} Tradf_l^{i,j} (p_l^{i,j} - p_{l+1}^{i,j}) \Delta x$

$\tau = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{l=1}^{L-1} Tradf_l^{i,j} (TravelTime(\bar{X}, u, i, j, l)$

$-TravelTime(\bar{X}, u, i, j, l-1)) \Delta x$

Algorithm 3 Update $\tilde{\mu}_u$ and $\tilde{\mu}_p$

If $j = 0$:

$\tilde{\mu}_{u,l}^{i,j} = \mu_{u,l}^{i,j} + \frac{\Delta t}{\Delta x}(u_l^{i,j+1} - u_l^{i,j}) u_l^{i,j}$

$\tilde{\mu}_{p,l}^{i,j} = \mu_{p,l}^{i,j} + \frac{\Delta t}{\Delta x}(p_l^{i,j+1} - p_l^{i,j}) u_l^{i,j}$

Else if $j = J$:

$\tilde{\mu}_{u,l}^{i,j} = \mu_{u,l}^{i,j} + \frac{\Delta t}{\Delta x}(u_l^{i,j} - u_l^{i,j-1}) u_l^{i,j}$

$\tilde{\mu}_{p,l}^{i,j} = \mu_{p,l}^{i,j} + \frac{\Delta t}{\Delta x}(p_l^{i,j} - p_l^{i,j-1}) u_l^{i,j}$

Else:

$\tilde{\mu}_{u,l}^{i,j} = \mu_{u,l}^{i,j} + \frac{\Delta t}{2\Delta x}(u_l^{i,j+1} - u_l^{i,j-1}) u_l^{i,j}$

$\tilde{\mu}_{p,l}^{i,j} = \mu_{p,l}^{i,j} + \frac{\Delta t}{2\Delta x}(p_l^{i,j+1} - p_l^{i,j-1}) u_l^{i,j}$

definition of drift of price in a Eulerian referential, which makes the price satisfy the Vasicek model, thus an explicit discrete form of price is able to be obtained as following:

$$p_l^{i,j} = p_l^{0,j} e^{-\beta i \Delta t} + C \sum_{i'=0}^{i} u_l^{i',j}(1 - e^{-\beta i' \Delta t}) + e^{-\beta i \Delta t} \sum_{i'=0}^{i} \sum_{j'}^{j} \sigma_{p,l}^{i',j,j'} Z_l^{i',j'} \tag{5.45}$$

For the convenience of proof, we consider a 2-lane freeway and the general case should be proved using similar ideas. The total money gained, B is given by:

$$\begin{aligned} B &= \sum_{i=1}^{m} \sum_{j=1}^{n} Tradf_l^{i,j}(p_l^{i,j} - p_{l+1}^{i,j}) \Delta x \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} k_1 (p_l^{i,j} - p_{l+1}^{i,j})^2 \Delta x \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} k_1 [C(u_l^{i,j} - u_{l+1}^{i,j})(1 - e^{-\beta i \Delta t}) + \Phi]^2 \Delta x \end{aligned} \tag{5.46}$$

$$\tag{5.47}$$

where $Tradf_l^{i,j}$ is defined in algorithm 1 step 10 and Φ is a function independent of k_1 and C. Thus, B has a linear relation with k_1 and a quadratic relation with C.

In the meanwhile, as we defined in algorithm 2, the difference of $TravelTime$ between lanes could be approximated as the following:

$$\begin{aligned} &TravelTime(\bar{X}, u, i, j, l) - TravelTime(\bar{X}, u, i, j, l-1) \\ &= \frac{\bar{X}}{\bar{u}_{l,\bar{X}}^{i,j}} - \frac{\bar{X}}{\bar{u}_{l-1,\bar{X}}^{i,j}} \\ &= \bar{X} \frac{\bar{u}_{l-1,\bar{X}}^{i,j} - \bar{u}_{l,\bar{X}}^{i,j}}{\bar{u}_{l-1,\bar{X}}^{i,j} \bar{u}_{l,\bar{X}}^{i,j}} \end{aligned} \tag{5.48}$$

where $\bar{u}_{l,\bar{X}}^{i,j}$ is the average speed starting from $(i\Delta t, j\Delta x, l)$ ending at

$(i\Delta t + TravelTime(\bar{X}, u, i, j, l), \bar{X}, l)$ along lane l. As we define in (4.19), speed is a linear function of density, it is reasonable to approximate the average speed as a linear function of average density. Thus the difference between the average speed can be represented by the following.

$$\bar{u}_{l-1,\bar{X}}^{i,j} - \bar{u}_{l,\bar{X}}^{i,j} = \Omega - k_2 tradf_l^{i,j} \tag{5.49}$$

where Ω and k_2 are both independent of C and k_1, and k_2 is positive. Thus the total time gained, τ is given by:

$$\begin{aligned}\tau &= \sum_{i=1}^{m}\sum_{j=1}^{n} Tradf_l^{i,j}(TravelTime(\bar{X}, u_{l+1}^{i,j}, i, j, l+1) \\ &- TravelTime(\bar{X}, u_l^{i,j}, i, j, l))\Delta x \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} Tradf_l^{i,j}(\Omega - k_2 tradf_l^{i,j})\frac{\bar{X}}{\bar{u}_{l-1,\bar{X}}^{i,j}\bar{u}_{l,\bar{X}}^{i,j}}\Delta x\end{aligned} \tag{5.50}$$

Since $Tradf_l^{i,j}$ is a linear function of both C and k_1, τ is a concave quadratic function of both parameters, where according to the definition of $tradf$, C and k_1 have almost the same impact on τ when Φ is relatively small.

5.2 Online Fair Price Estimation

Rather than simulating random variables $Z_l^{i,j}$ at each time step, our method consists in backing out estimators $\hat{Z}_l^{i,j}$ from synchronous traffic density data, which make the equations of our model consistent. These variates $\hat{Z}_l^{i,j}$ are then fed into the pricing model to produce accurate estimates $\hat{p}_l^{i,j}$ of the price of lane l at time $i\Delta t$ and longitudinal distance $j\Delta x$.

We suppose we have access to real-life speed data $\hat{u}_l^{i,j}$ and $\hat{\rho}_l^{i,j}$ at every (i, j, l). Recall that from (4.19), we have the following system of equations at each time step $i\Delta t$ and for $0 < j < J$, $0 < l < L$:

$$\hat{u}_l^{i,j} = u_{max}(1 - \frac{\hat{\rho}_l^{i,j}}{\rho_{max}})g(\sigma(l)\sum_{p=0}^{i}\sum_{q=0}^{j} f_{j,q}^{i,p}\hat{Z}_l^{i,q}) \tag{5.51}$$

Then we apply the results of $\hat{\vec{Z}}_l^i = [\hat{Z}_l^{i,j}]$ for $0 < j < J$, $0 < l < L$ from (5.51) to the following equation:

$$\hat{p}_l^{i+1,j} = p_l^{i,j} + \mu_{p,l}^{i,j}\Delta t + \sum_{j'}^{j} \sigma_{p,l}^{i,j'}\hat{Z}_l^{i,j'} \tag{5.52}$$

Algorithm 4 shows how this method works.

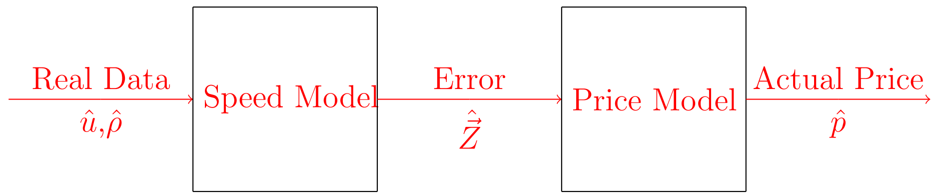

Figure 5.2: Online fair price estimation

Algorithm 4 Online Fair Price Estimation

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for i from 2 to I **do**

  for l from 1 to L **do**

    $\sigma(l)\sum_{j=0}^{J}\sum_{q=0}^{j} f_{j,q}^{i,i}\hat{Z}_l^{i,q} = g^{-1}(\frac{\frac{\hat{u}_l^{i,j}}{u_{max}}}{1-\frac{\hat{\rho}_l^{i,j}}{\rho_{max}}}) - g^{-1}(\frac{\frac{\hat{u}_l^{i-1,j}}{u_{max}}}{1-\frac{\hat{\rho}_l^{i-1,j}}{\rho_{max}}})$

$$F_{i,l} = \begin{bmatrix} f_{0,0}^{i,i} & 0 & & .... & 0 \\ f_{1,0}^{i,i} & f_{1,1}^{i,i} & 0 & .... & 0 \\ f_{2,0}^{i,i} & f_{2,1}^{i,i} & f_{2,2}^{i,i} & .... & 0 \\ .... & & & & \\ f_{J,0}^{i,i} & f_{J,1}^{i,i} & f_{J,2}^{i,i} & .... & f_{J,J}^{i,i} \end{bmatrix}$$

    $\hat{\vec{Z}}_l^i = \frac{F_{i,l}^{-1}}{\sigma(l)}(g^{-1}(\frac{\frac{\hat{u}_l^{i,j}}{u_{max}}}{1-\frac{\hat{\rho}_l^{i,j}}{\rho_{max}}}) - g^{-1}(\frac{\frac{\hat{u}_l^{i-1,j}}{u_{max}}}{1-\frac{\hat{\rho}_l^{i-1,j}}{\rho_{max}}}))$

    $\hat{p}_l^{i+1,j} = p_l^{i,j} + \mu_{p,l}^{i,j}\Delta t + \sum_{j'}^{j} \sigma_{p,l}^{i,j'}\hat{Z}_l^{i,j'}$

  end for

end for

Chapter 6

Numerical Results

In section 6.1, we try to verify the relation between parameters price adjustment factor, C, price-elasticity of transversal flux, k_1,speed of mean reversion of price, β and total money paid, B, total time saved, τ. In section 6.2, we analyze the impact of adding a source term to the rightmost lane on the freeway on the traffic density. In section 6.3, a different volatility structure is introduced and we will find how the traffic density will be affected when accidents occur. In section 6.4, we focus on the case wherein speed and price differentiation are preserved.

6.1 Impact of Parameters and Initial Density Conditions

In the simulation, the parameters of the numerical method are $0 \leq x \leq 40$ miles, $0 \leq t \leq t_{max} = 0.3$ hours, $\Delta x = 0.5$ miles, $\Delta t = 0.002$ hours, $\Delta y = 1$ and the number of scenarios $K = 20$. The conditions of the traffic model are $u_{max} = 75$ miles/hour and $\rho_{max} = 180$ cars per longitudinal mile per lane. For other parameters in the model, $a = 0.5$, $b = 1$, $k_0 = 0.01$. We set up 4 lanes with

different volatility which is :

$$\sigma_u = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.15 \\ 0.2 \end{bmatrix} \tag{6.1}$$

As we stated in 5.1.2, the total money paid, B and the total time saved, τ within a certain amount of time and distance is what we are interested in. Obviously, by taking different parameters such as C, k_1 and β, we would expect different values of B and τ. It is also important to find out how these parameters behave under different initial density conditions. For this particular reason, we initialize 6 different initial density conditions in the 4-lane traffic, where each lane takes a constant initial density in each case.(Table 6.1) For all the cases, we set $\beta = 5$ and test the impact of parameters C and k_1 on B and τ separately with the other one fixed as a constant.

	Lane 1	**Lane 2**	**Lane 3**	**Lane 4**
Case 1	80	100	120	140
Case 2	140	120	100	80
Case 3	95	125	125	95
Case 4	125	95	95	125
Case 5	65	95	95	65
Case 6	82.9	123.6	122.7	110.8

Table 6.1: Initial Density

6.1.1 Impact of Price Adjustment Factor, C, Price-Elasticity of Transversal Flux, k_1

Notice that we add *Timenormal* and *TotalDistance* in the table, where *Timenormal* means the time that it would have taken to travel for the cars that traded to change lanes if they didn't trade.

The measure $TotalDistance$ is the total length that the all cars in the system travel within 0.3 hours, which represents how far the system moves forward, and the larger the $TotalDistance$ is, the more benefit the whole system will gain.

Table c.1, table c.2, and figure 6.1 are for case 1. Table c.3, table c.4, and figure 6.2 are for case 2. Table c.5, table c.6, and figure 6.3 are for case 3. Table c.7, table c.8, and figure 6.4 are for case 4. Table c.9, table c.10, and figure 6.5 are for case 5. Table c.12, table c.13, and figure 6.6 are for case 6.

B shows a quadratic relation with C and a linear relation with k_1, which coincide with the conclusion in 5.1.4. In the meanwhile, τ shares a concave quadratic relation with both C and k_1. Moreover, C and k_1 have the almost the same impact on τ and if the value of $C * k_1$ is the same, τ will be the same. The conclusions in 5.3 are verified.

In table 14, we find the optimal $C * k_1$ to maximize τ for each case. In other words, there is always a unique point that maximizes the travel-time saved under a certain initial density condition within a fixed amount of time.

Another impact of C and k_1 on B and τ that we are interested in is whether the system will benefit differently with different parameters. By comparing results of $TotalDistance$, under the same initial density condition, changing parameters C and k_1 have a negligible impact on the behaviour of the whole system.

For a vertical analysis in table 1, we define the following three terms for the use of later analysis,the total density which is given by:

$$TD = \sum_{j=1}^{m} \sum_{l=1}^{L} (\rho_l^{0,j}) \tag{6.2}$$

the total first-order variation between adjacent lanes, which is given by :

$$TFV = \sum_{j=1}^{m} \sum_{l=1}^{L-1} |\rho_l^{0,j} - \rho_{l+1}^{0,j}| \tag{6.3}$$

and the total quadratic variation of the initial density between adjacent lanes, which is given by:

$$TQV = \sum_{j=1}^{m} \sum_{l=1}^{L-1} (\rho_l^{0,j} - \rho_{l+1}^{0,j})^2 \tag{6.4}$$

- Comparing either $Case1$ with $Case2$ or $Case3$ with $Case4$, they have the same TD, TFV, TQV and B, which means that equal TD, TFV and TQV is a sufficient condition for B to be the same.
- Comparing $Case1$ with $Case3$, they only share the same TD and TFV, which means TQV is a necessary condition for B to be the same.
- Comparing $Case3$ with $Case5$, they only share the same TFV and TQV, which means TD is a necessary condition for B to be the same.
- Comparing $Case3$ with $Case6$, they only share the same TD and TQV, which means TFV is a necessary condition for B to be the same.

Combining all the results, we have the following conclusion:

Conclusion 6.1.1.1 With all other parameters staying the same, different initial conditions results in the same B if and only if they share the same TD, TFV, and TQV.

	$C * k_1$	τ	**Timenormal**	**Percentage of Time Gained**	**TotalTime**	r
Case1	0.08	1.6355	14.0899	11.60%	207.48	6.73%
Case2	0.08	1.5666	14.1036	11.11%	207.49	6.74%
Case3	0.02	0.3671	3.2771	11.20%	200.17	1.62%
Case4	0.02	0.3534	3.2723	10.79%	200.16	1.63%
Case5	0.045	0.2564	2.7482	9.33%	102.48	2.66%
Case6	0.025	0.3659	3.2793	11.15%	202.18	1.62%

Table 6.2: Optimal τ for $\beta = 5$, r is the percentage of drivers using the system

6.1.2 Impact of Speed of Mean Reversion, β

According to chapter 6.1.1, we know that different initial density will lead to different optimal Ck_1 and τ. In this section, we use the same initial condition as in $Case1$ and focus on the relation

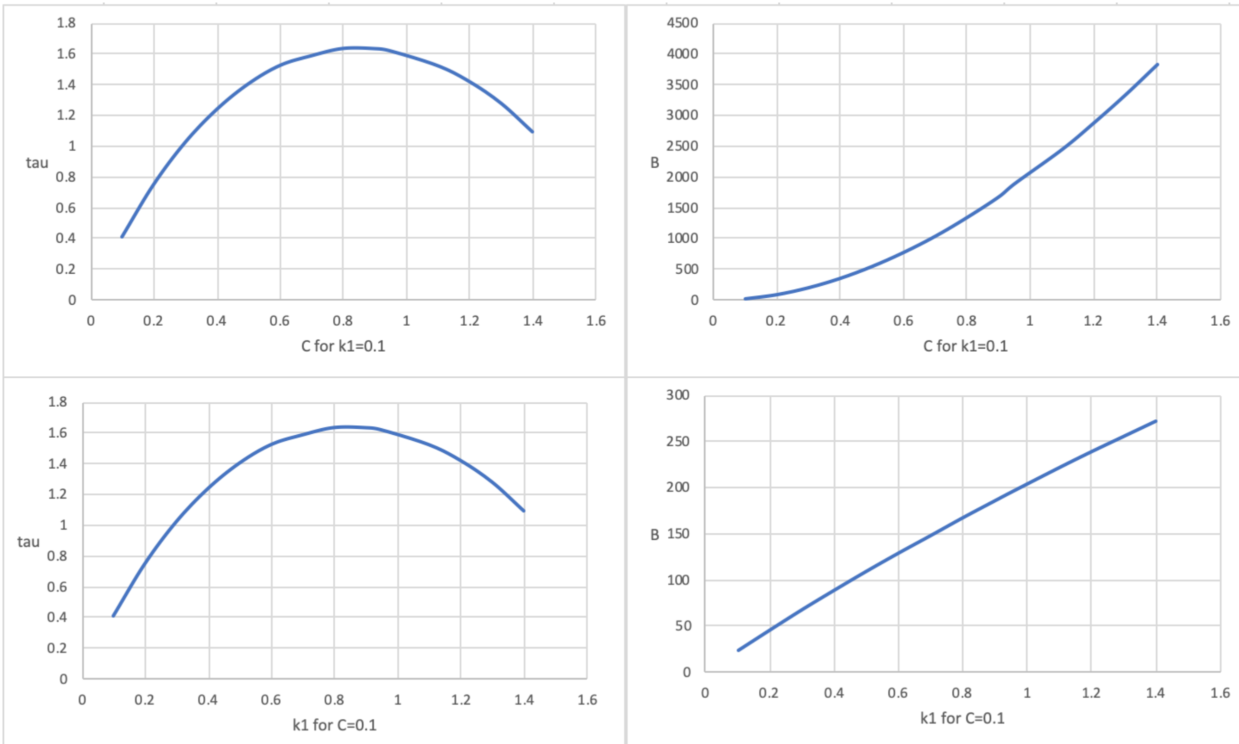

Figure 6.1: Case 1

between β and τ or $\tau/Timenormal$. In table 6.3 and figure 6.7, there exists a monotonic positive relation between β and both τ and $\tau/Timenormal$.

We also test how the average variance of price across transversal and longitudinal direction evolves with time under different β conditions. It is surprising that higher β leads to higher variance of price. Since we use a drift from the Ornstein-Uhlenbeck process for the price model, we expected an inverse relation. We will leave more analysis of β for the future research.

6.2 Impact of the Source Term

6.2.1 Impact of On-ramp and Off-ramp

In reality, cars are entering or exiting the freeway all the time. For the convenience of analysis, we ignored the source term in the previous sections. In this section, we are able to add a source term

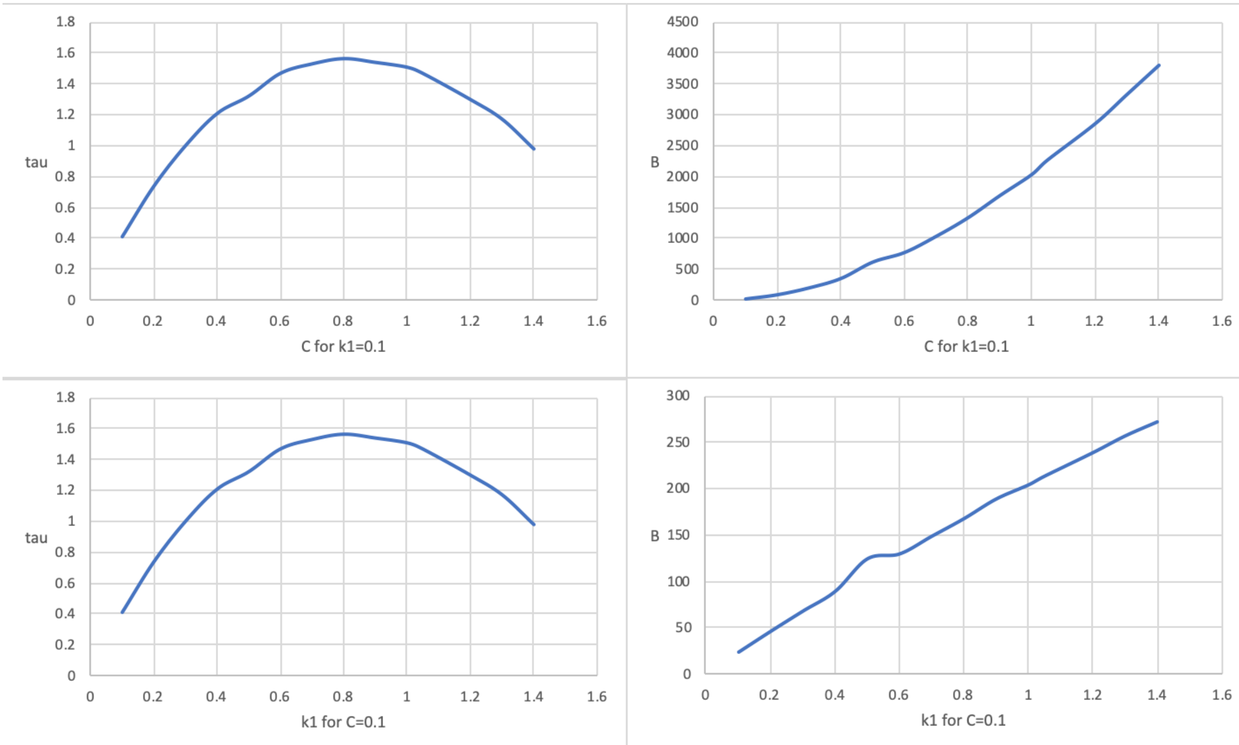

Figure 6.2: Case 2

β	τ	$\tau/Timenormal$
5	1.43	15.4%
10	1.45	15.9%
15	1.48	16.3%
20	1.49	16.6%
25	1.51	16.8%
30	1.52	17.0%
35	1.54	17.1%
40	1.55	17.2%
45	1.57	17.3%
50	1.59	17.5%
55	1.60	17.5%
60	1.61	17.6%

Table 6.3: relation between β, τ and $\tau/Timenormal$ for $C * k_1 = 0.05$

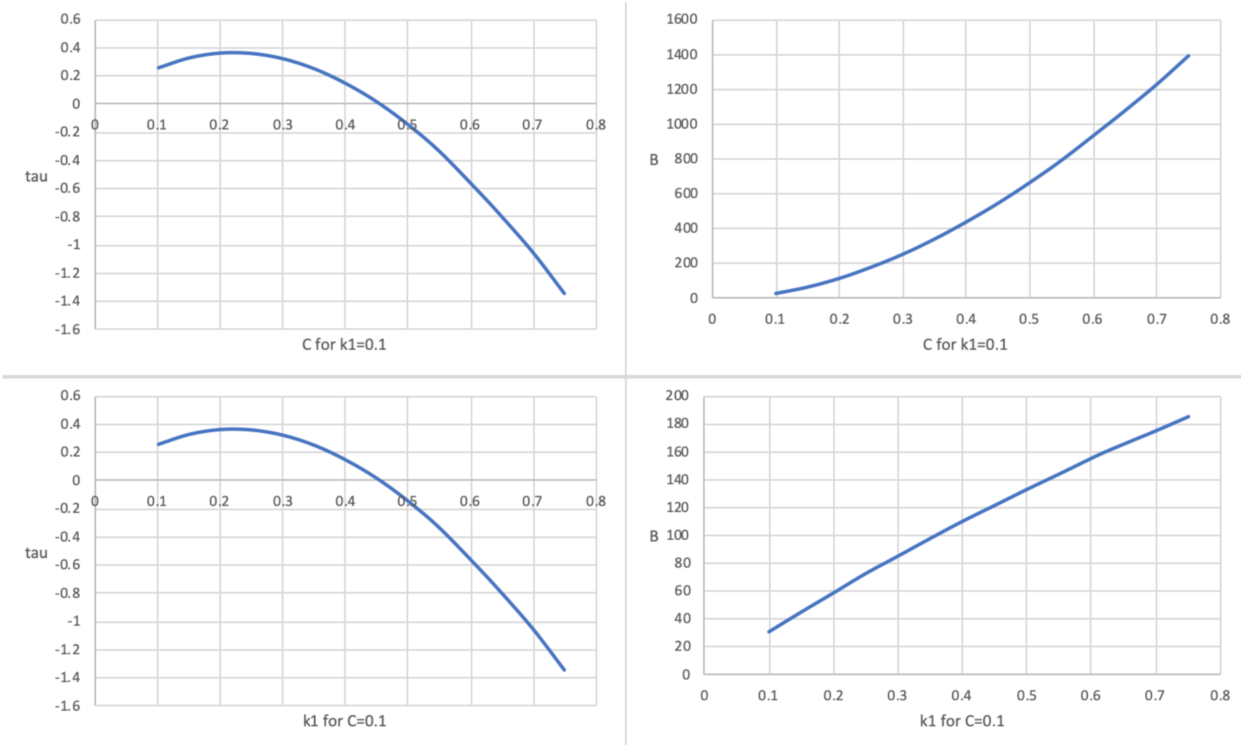

Figure 6.3: Case 3

on the rightmost lane(lane 1) to see how it will affect the traffic density. We use the same initial density as in $Case1$, and choose $C = 0.1$, $k_1 = 0.1$ and $\beta = 5$. We define an entering traffic flux at longitudinal position of 20 miles and an exiting traffic flux at longitudinal position of 30 miles with the same derivative of the flux coming from lane 0 to lane 1 which is given by:

$$\sqrt{(1 - \frac{\rho_1^{i,j}}{\rho_{max}})\rho_1^{i,j}/\Delta t} \tag{6.5}$$

By replacing step 6 in algorithm 1 with algorithm 3, we obtain the algorithm to simulate the traffic flux with a certain source term.

Equation (6.5) ensures that the density will not go down below 0 or up above ρ_{max} because of the source term. Figure 6.9 shows how the density evolves with a source term compared with figure 6.10 which have no source term.

Since we set lane 1 as the rightmost lane where cars enter or exit the freeway from lane 0, there is a peak at 5 miles and a drop at 20 miles in lane 1 in figure 6.9 compared with lane 1 in figure 6.10. And lane 2 is influenced in the same way. Lane 3 and lane 4 show less symptoms of being affected. In the

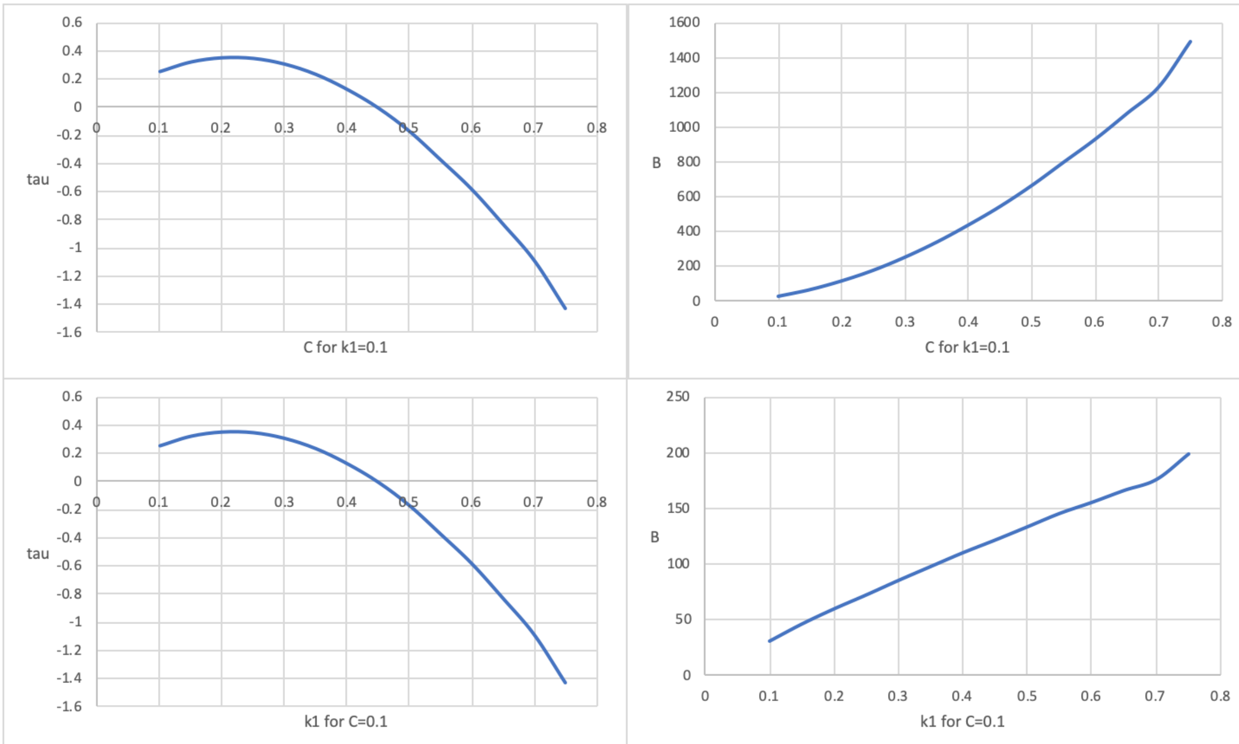

Figure 6.4: Case 4

Algorithm 5 adding a source term

Phase 1

Define $Tradf(i\Delta t, j\Delta x, l)$ as the traded flux

$Tradf_l^{i,j} = k_1(\frac{\rho_l^{i,j}}{\rho_m}\frac{\rho_{l+1}^{i,j}}{\rho_m})^{\frac{1}{2}}(p_l^{i,j} - p_{l+1}^{i,j}) + k_1(\frac{\rho_{l-1}^{i,j}}{\rho_m}\frac{\rho_l^{i,j}}{\rho_m})^{\frac{1}{2}}(p_{l-1}^{i,j} - p_l^{i,j})$

$q_{y,l}^{i,j} = k_0(\rho_{l+1}^{i,j} - 2\rho_l^{i,j} + \rho_{l-1}^{i,j}) + Tradf_l^{i,j}$

if $l = 1$

if $j = \frac{n}{2}$, $\rho_l^{i+\frac{1}{2},j} = \rho_l^{i,j} + q_{y,l}^{i,j} + \sqrt{(1 - \frac{\rho_l^{i,j}}{\rho_m})\rho_l^{i,j}}$

if $j = \frac{3n}{4}$, $\rho_l^{i+\frac{1}{2},j} = \rho_l^{i,j} + q_{y,l}^{i,j} - \sqrt{(1 - \frac{\rho_l^{i,j}}{\rho_m})\rho_l^{i,j}}$

else, $\rho_l^{i+\frac{1}{2},j} = \rho_l^{i,j} + q_{y,l}^{i,j}$

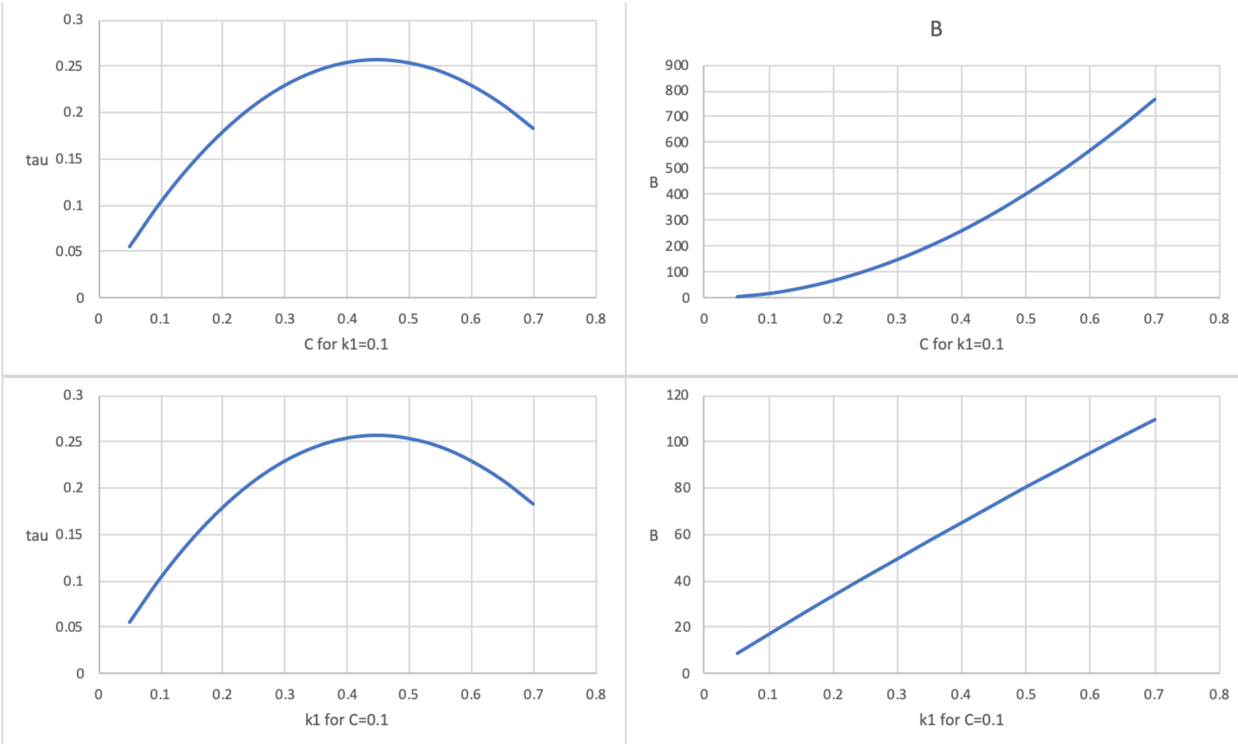

Figure 6.5: Case 5

meanwhile, the effect of a source term gets smaller when the longitudinal position gets further away.

6.2.2 Price Impact

More importantly, we need to consider a very realistic problem: what if many cars want to exit, which makes the rightmost lane(lane 1) more valuable, even if it may suffer extremely low speed(cars pile up in order to exit). We assume that a high density in the right lanes before an exit is an indication that there exists a high proportion of cars want to exit. Thus we need to value the right lanes. In algorithm 5, we show how we reconstruct C to meet this need. And this algorithm should be embedded between step 12 and 13 in algorithm 1. In order to test the new algorithm, we construct a special case wherein there is no entrance and only one exit at $x = 30$ miles in lane 1 and the initial density for lane 1, lane 2, lane 3 and lane 4 are 170, 170, 120,80 respectively. Comparing figure 6.11 with 6.12, we can tell that from the beginning the speed in lane 1 is lower than that in

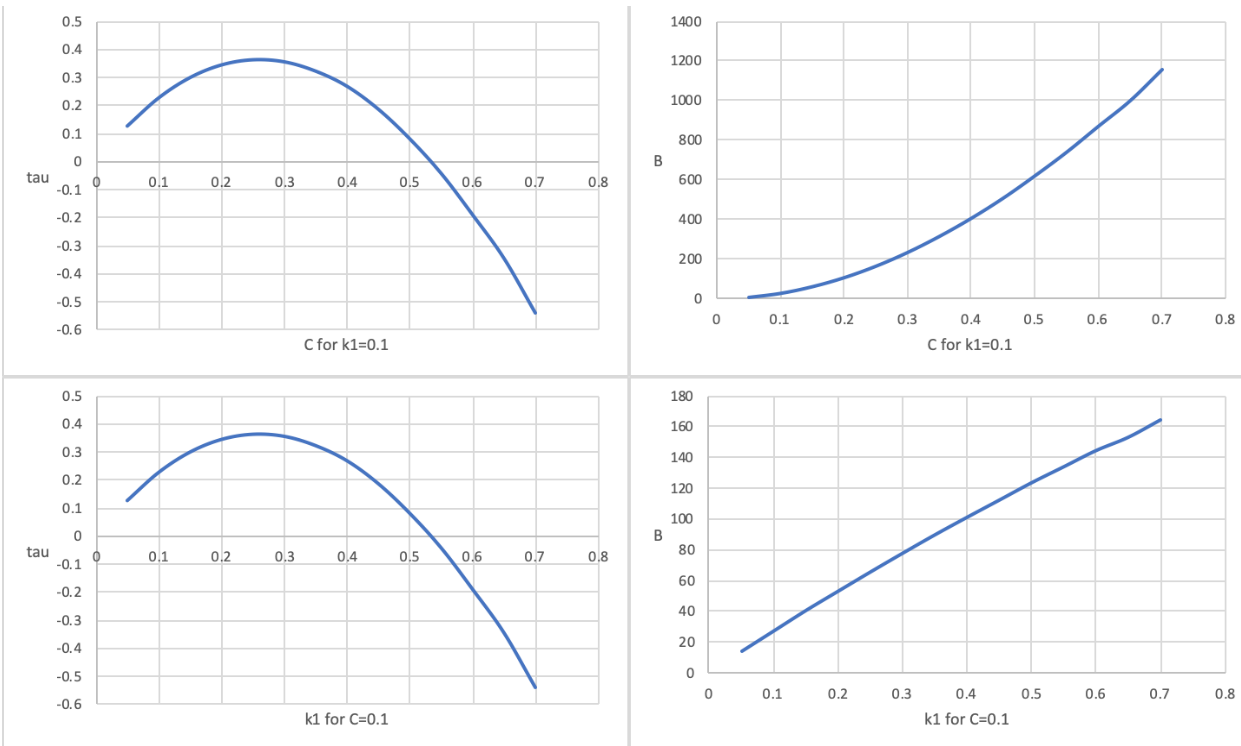

Figure 6.6: Case 6

lane 2, however, the price of lane 1 is higher than that in lane 2, which disobeys the normal patent that higher price means higher speed. In this special case, the demand caused by cars that are in lane 2 and want to exit from lane 1 makes lane 1 worthier than lane 1 even though the speed in lane 1 is lower than the speed in lane 2.

6.3 Impact of the Volatility Structure

In 6.1 and 6.2, the volatility structure of the freeway is given by (6.1), which means that volatility only differs from lanes. However, in reality, volatility may vary with the position. For example, there are some location on the freeway that have a higher possibility of accidents. We would like to see how such a volatility structure would affect the results. We set that at longitudinal position of 20 miles in lane 2, the volatility becomes 5 and the initial density for lane 1, lane 2, lane 3, lane 4 are 80, 100, 100, 80 respectively with all other conditions the same as in 6.2 without a source term

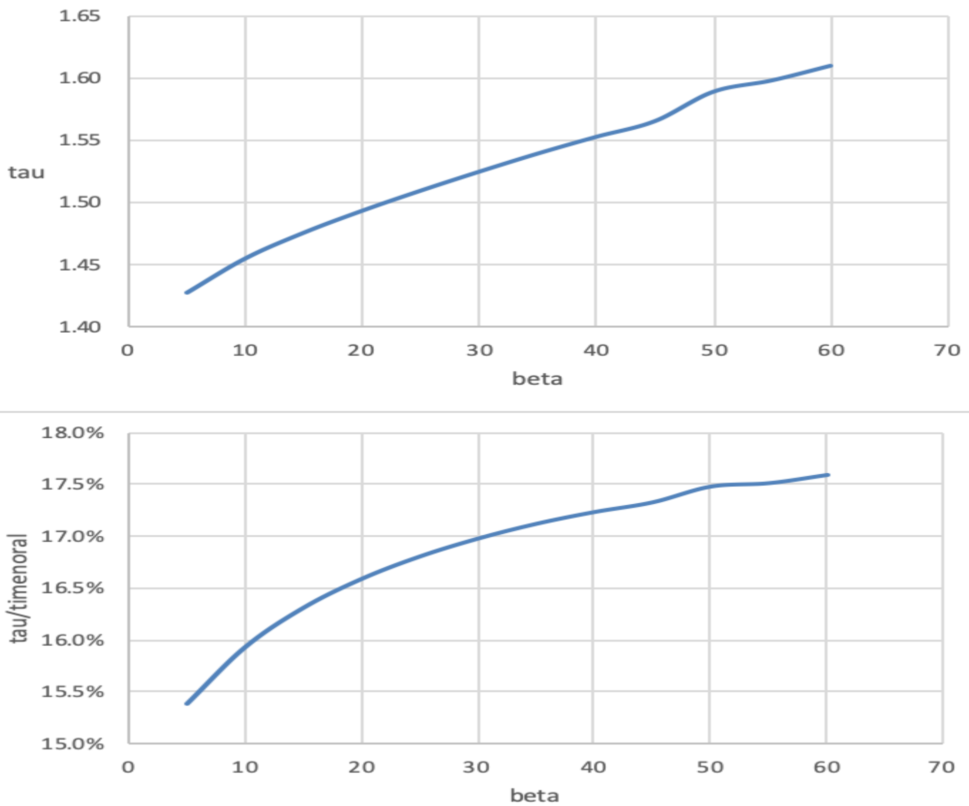

Figure 6.7: Relation between β, τ and $\tau/Timenormal$

Algorithm 6 Exiting problem

Initialize $C = C_0$

for j from 1 to n **do**

  for l from 1 to$L-1$ **do**

    if $(\frac{1}{2} - \frac{1}{20})n < j < \frac{1}{2}n$:

    if $\rho_l^{i,j} < 0.8\rho_{max}$:

      $C = C_0$

    if $0.8\rho_{max} \leq \rho_l^{i,j} < 0.9\rho_{max}$:

      $C_{l+1}^{i,j} = C_0(1 - \frac{\rho_l^{i,j} - 0.8\rho_{max}}{0.05\rho_{max}})$

    else: $C = -C_0$

  end for

end for

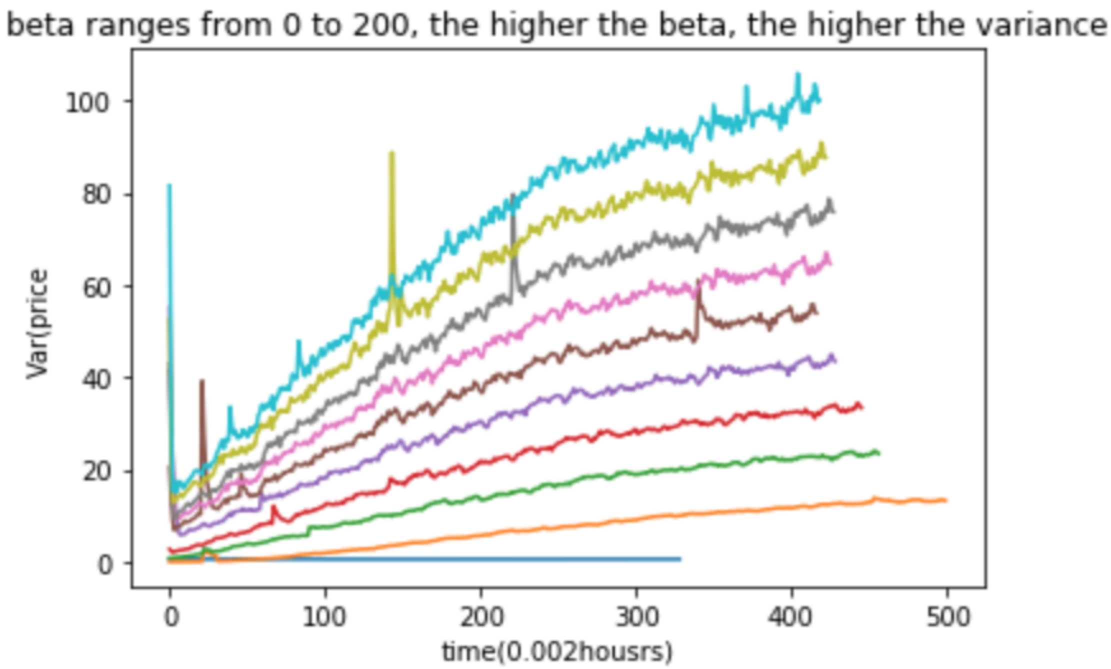

Figure 6.8: Average variance of the price with different β

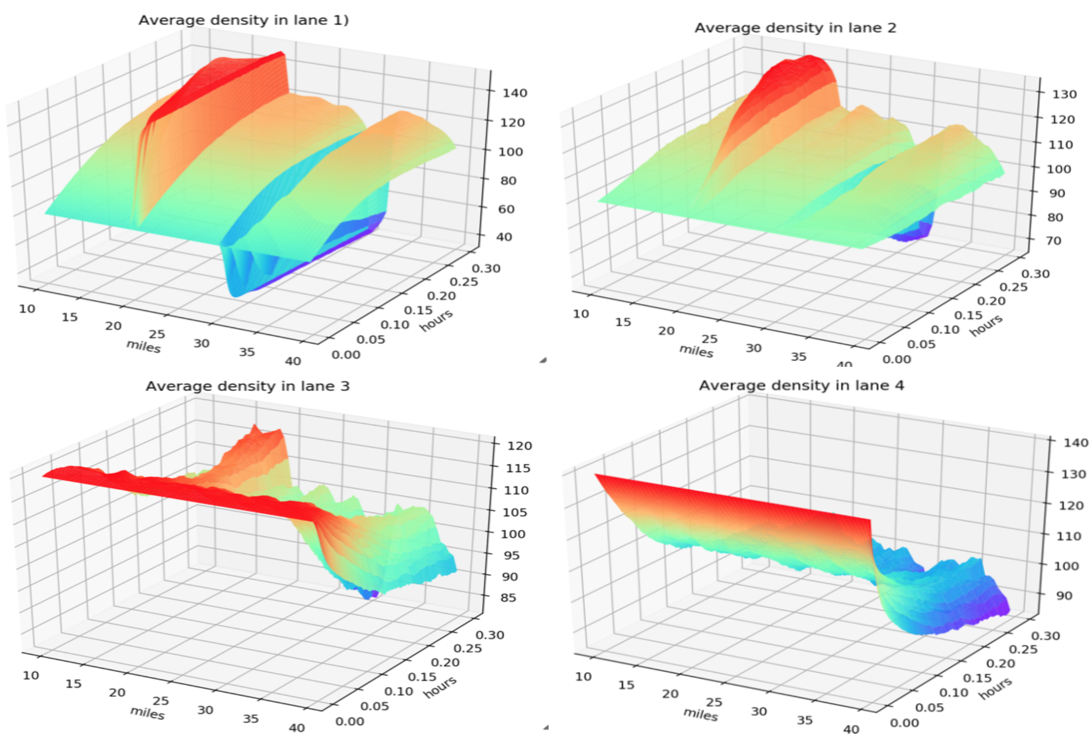

Figure 6.9: Average density in 4 lanes with a source term

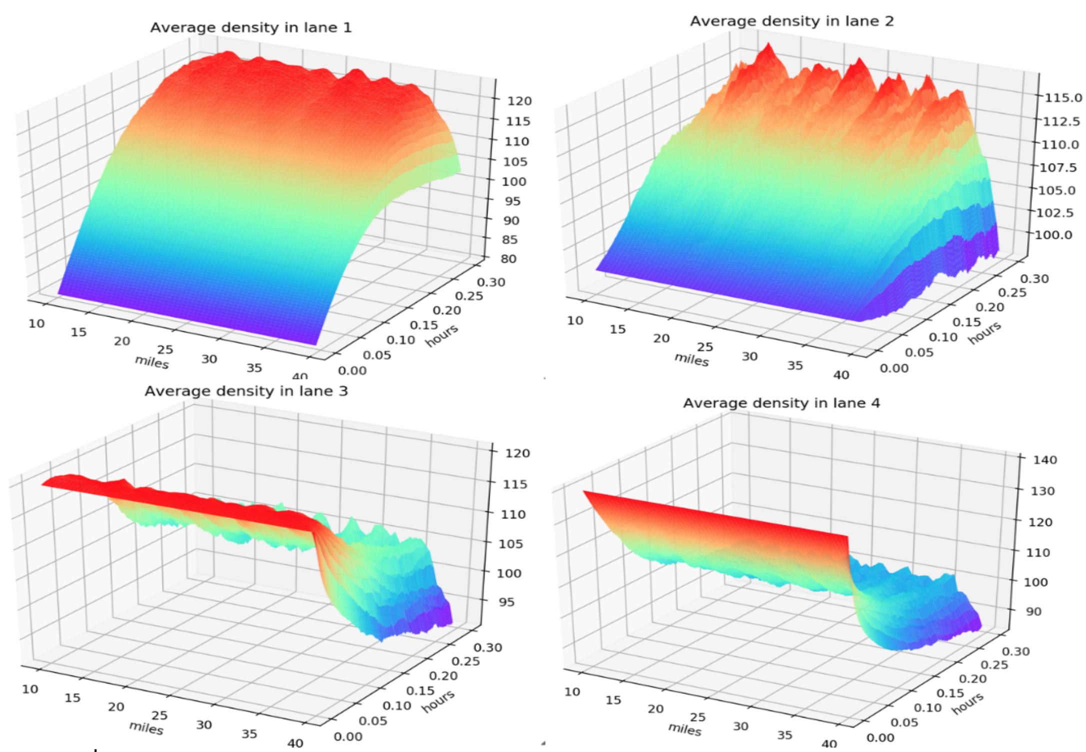

Figure 6.10: Average density in 4 lanes without a source term

wherein the new volatility structure is given as follows:

$$\sigma(\vec{y})f(x,x',t,t') = \begin{bmatrix} 0.05 \\ 0.1 \\ 0.15 \\ 0.2 \end{bmatrix} e^{-a(x-x')^2-b((t-t')^2)}1[x<20 \; or > 20]$$
$$+ \begin{bmatrix} 0.05 \\ 5 \\ 0.15 \\ 0.2 \end{bmatrix} e^{-a(x-x')^2-b((t-t')^2)}1[x=20] \tag{6.6}$$

Figure 6.13 shows how the density evolves. Cars pile up at the longitudinal position of 20 miles in all the lanes, which may indicate that there is an accident happening in lane 2 and all other lanes

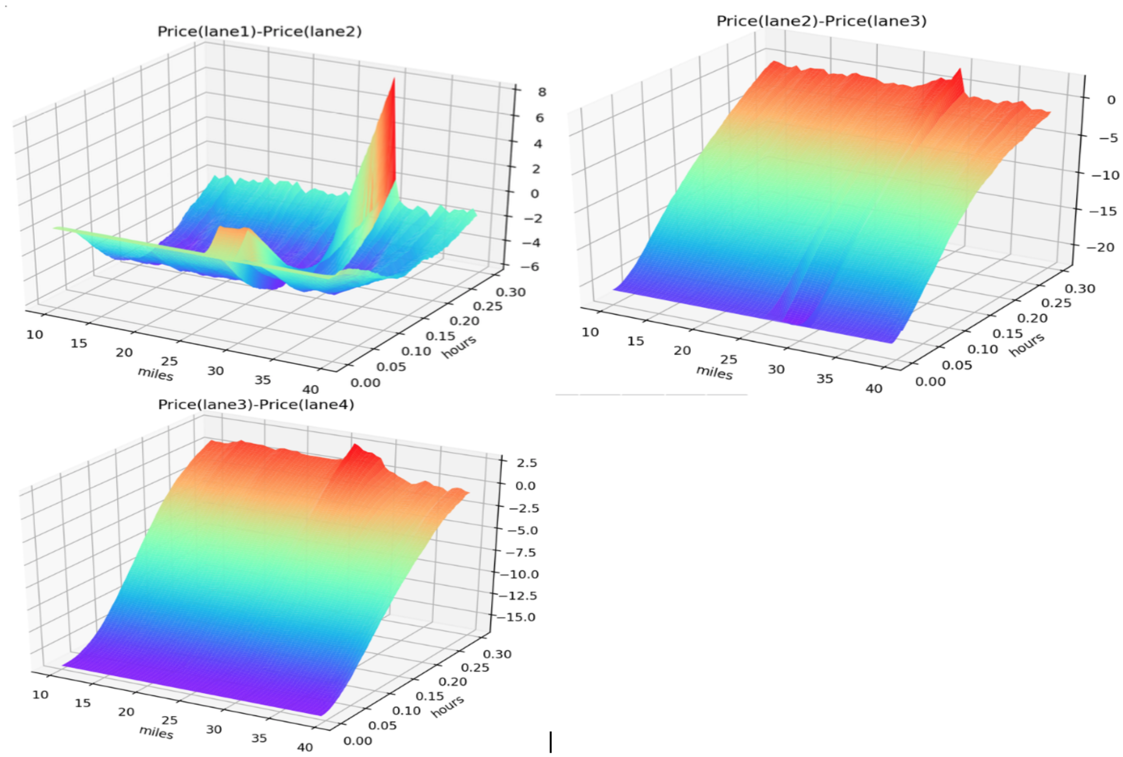

Figure 6.11: Price difference with one exit at $x = 30$

at the same longitudinal position are influenced compared with figure 6.14 which is the normal structure case. By comparing figure 6.15 and 6.16, there shows a relatively high possibility that accidents may happen at the longitudinal position of 20 miles in lane 2. This effect is weakened in its adjacent lanes. As for the standard deviation of price, there are some outliers which we may need to omit. Except for that, the price risk is relatively high at the particular location and this effect is also weakened in its adjacent lanes.

In conclusion, if there is a special location that accidents have a relatively higher possibility to occur, the speed is expected to be lower and the price face a higher risk at this location.

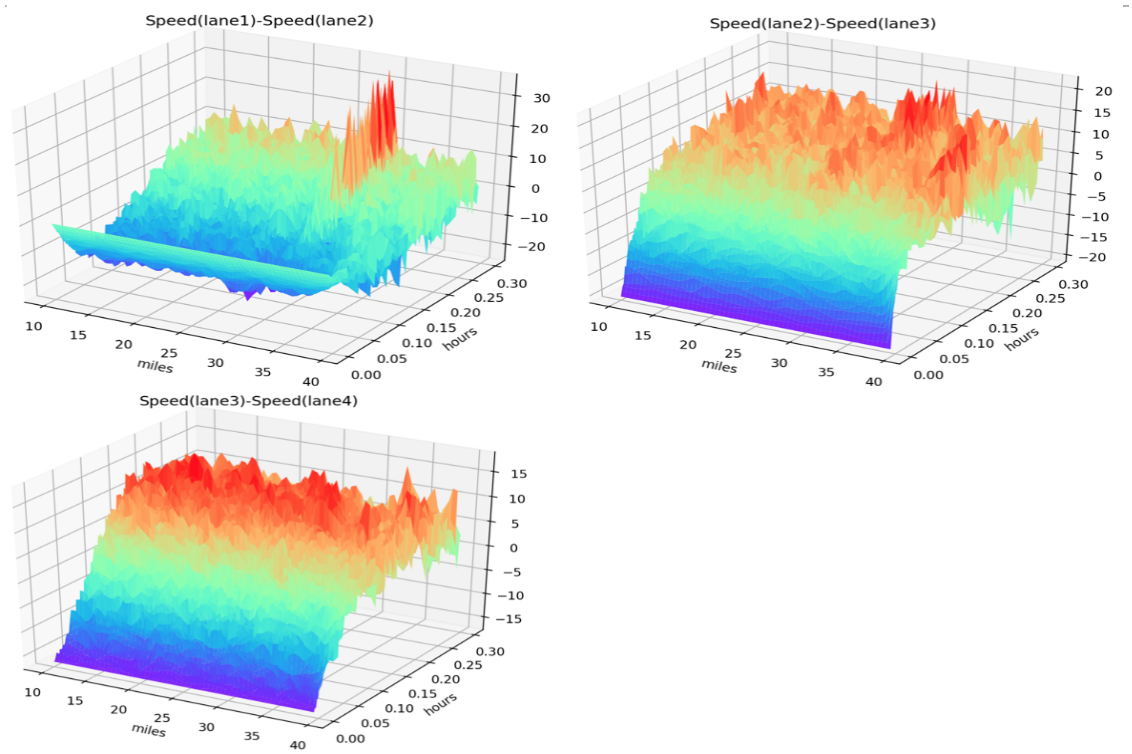

Figure 6.12: Average speed difference with one exit at $x = 30$

6.4 Stable Speed and Price Differentiation

As we can see from the cases in the previous sections, speed(or density) in lanes tends to converge and equalize, which leads to the same property of price. We would also like to see a case wherein the difference in speed and price between lanes are preserved along with time. Before we conduct our simulation, an asymptotic analysis that determines the parameters that satisfy this situation is necessary.

Assume that density is consist of a main term, $\rho_0(x,t)$ and a perturbation term $\epsilon\rho_1(y,t)$ as the follows:

$$\rho(x,y,t) = \rho_0(x,t) + \epsilon\rho_1(y,t) \tag{6.7}$$

By construction, the main term and the perturbation term should satisfy the conservation laws in longitudinal and transversal direction when there is no source term.

$$\frac{\partial \rho_0(x,t)}{\partial t} + \frac{\partial q^x(x,t)}{\partial x} = 0 \tag{6.8}$$

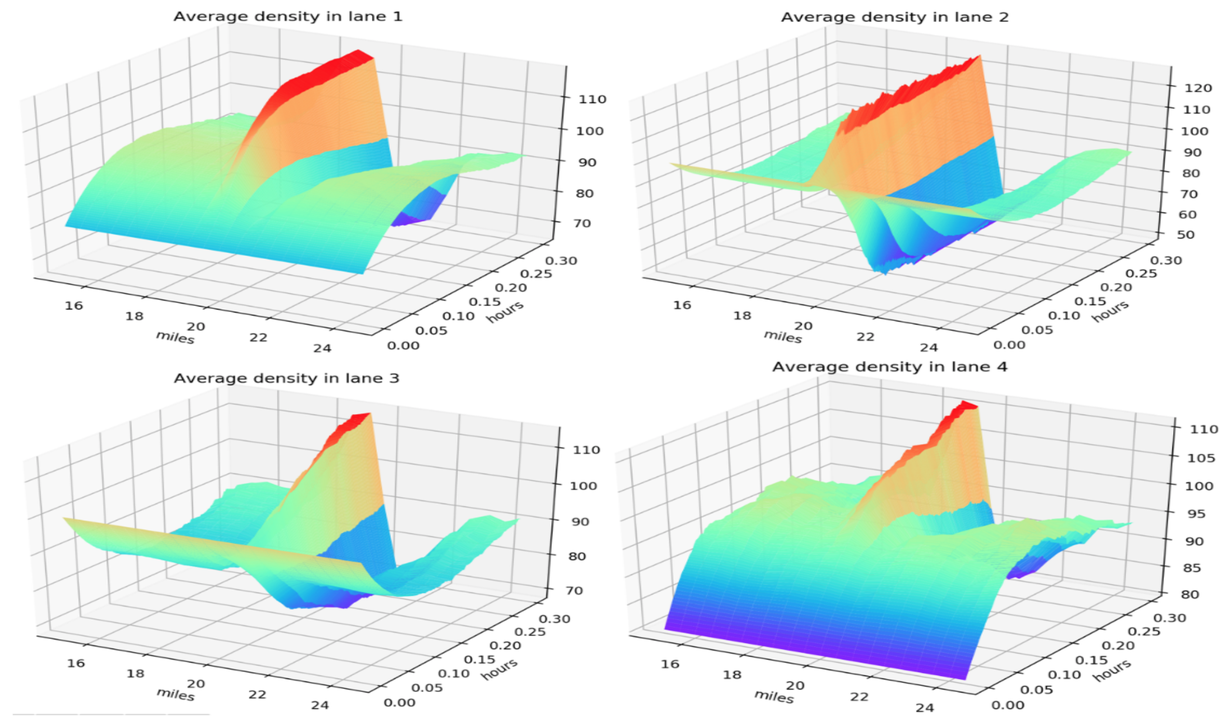

Figure 6.13: Average density in 4 lanes with accidents in lane 2 at $x = 20$

$$\frac{\partial \epsilon \rho_1(y,t)}{\partial t} + \frac{\partial q^y(y,t)}{\partial y} = 0 \tag{6.9}$$

When the perturbation term $\rho_1(y,t)$ comes to its steady state where $\frac{\partial \rho_1(y,t)}{\partial t} = 0$, the differentiation in speed and price between lanes may be preserved. Now, we discuss the steady state of (6.9). Applying (4.36) and (6.7) in (6.9), we have the follows:

$$\frac{\partial \epsilon \rho_1(y,t)}{\partial t} - k_0 \epsilon \frac{\partial^2 \rho_1(y,t)}{\partial y^2} - \alpha \epsilon^2 (\frac{\partial \rho_1(y,t)}{\partial y})^2 - \alpha(\rho_0(x,t) + \epsilon \rho_1(y,t)) \epsilon \frac{\partial^2 \rho_1(y,t)}{\partial y^2} = 0 \tag{6.10}$$

where $\alpha = k_1 \frac{C u_{max}}{\rho_{max}^2}$, let $\epsilon \to 0$, then

$$\frac{\partial \rho_1}{\partial t} - (k_0 + \alpha \rho_0) \frac{\partial^2 \rho_1}{\partial y^2} = 0 \tag{6.11}$$

the steady state condition is given by:

$$k_0 + \alpha \rho_0 = 0 \tag{6.12}$$

then

$$C k_1 = -\frac{k_0 \rho_{max}^2}{\rho_0 u_{max}} \tag{6.13}$$

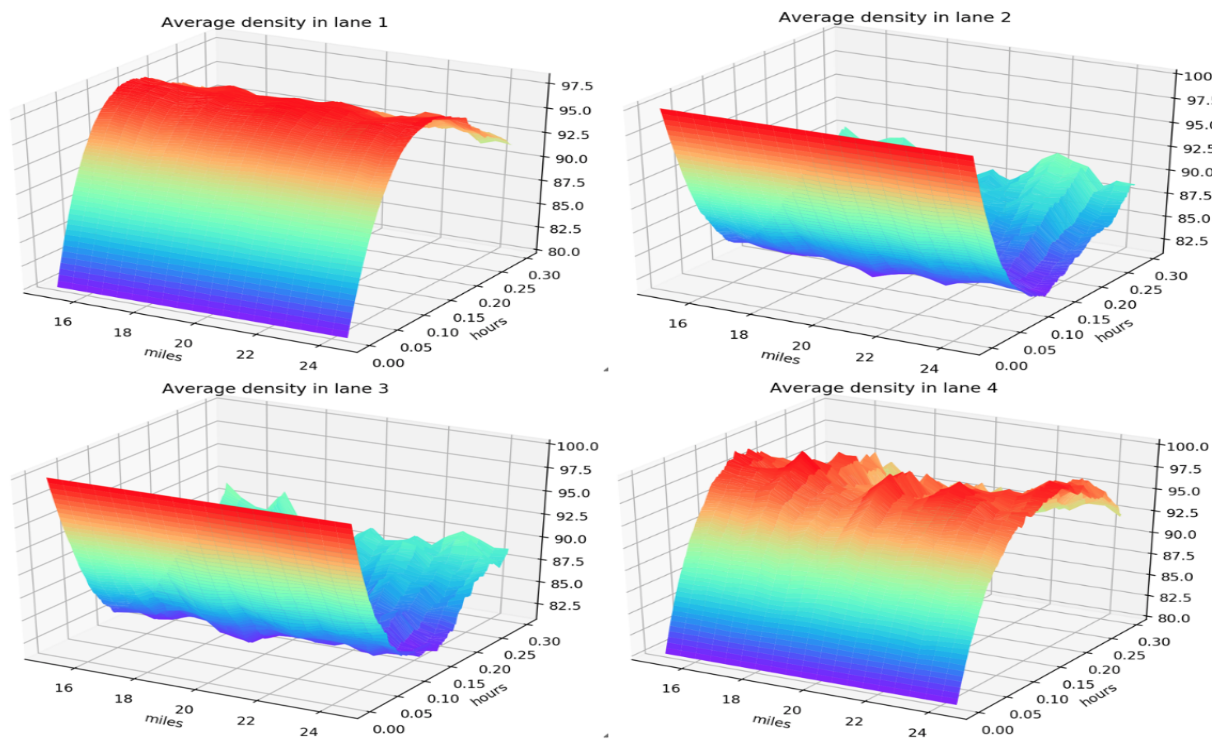

Figure 6.14: Average density in 4 lanes without accidents

According to this result, we should update the value of k_1 every time step so that Ck_1 matches the condition in (6.13). By adding algorithm 5 after step 16 in algorithm 1, we have a new algorithm to keep the speed and price differentiation between lanes.

One very important thing to notice that we add a lower bound to k_1, $k_{1,min}$. This is because when ρ_0 tends to 0, k_1 will get quite large and the rule of conservation will be broken. Due to the trade-off between the price differentiation and conservation of cars, we need to adjust $k_{1,min}$ moderately.

We test two new cases called $Case7$ and $Case8$. They both have the same initial density conditions, $\rho_1^{i,j} = 40$, $\rho_2^{i,j} = 80$, $\rho_3^{i,j} = 120$ and $\rho_4^{i,j} = 160$ for $i \in [0, I]$ and $j \in [0, J]$. In $Case7$, we apply algorithm 2 wherein $k_0 = 0.01$, $k_1 = 0.1$ $C = 0.1$ and all other parameters are the same as in $Case1$. In $Case8$, we apply algorithm 2 combined with algorithm 5 wherein $k_0 = 0.01$, $C = 0.1$ and all other parameters are the same as in $Case1$. We compare the results of the average speed and price at every time step.

In figure 6.21, we can see that price and speed both converge between lanes in $Case7$ and this

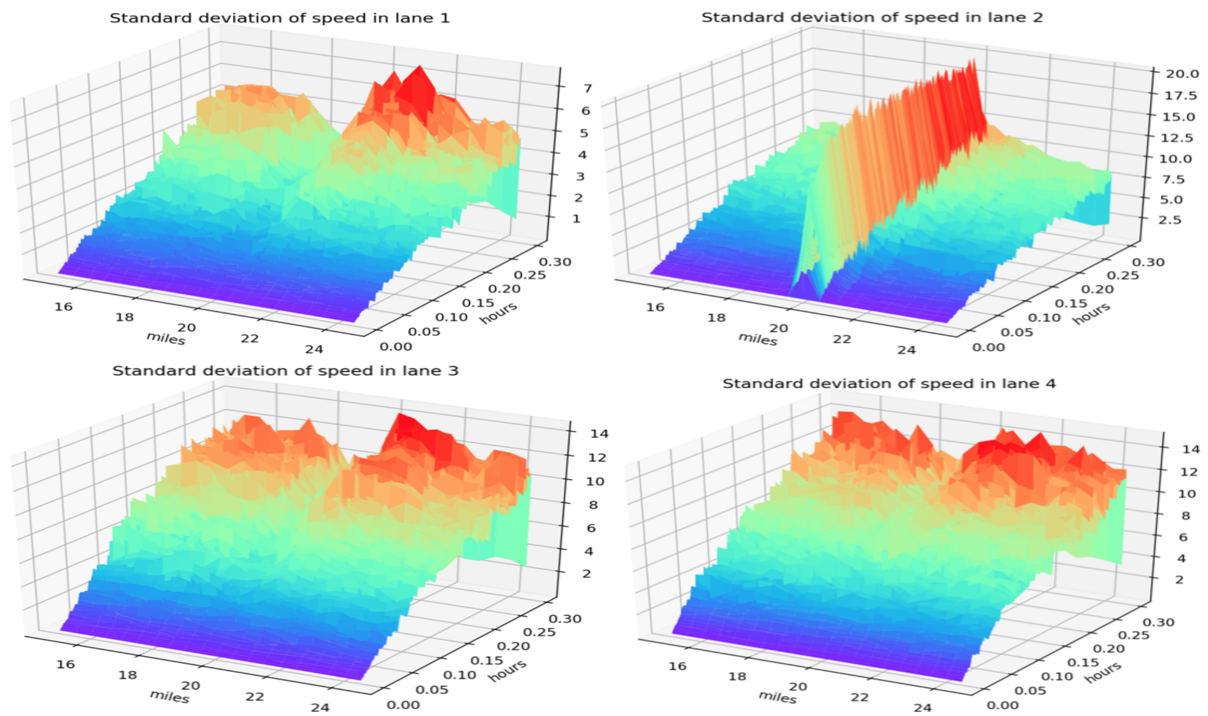

Figure 6.15: Standard deviation of speed in 4 lanes with accidents in lane 2 at $x = 20$

Algorithm 7 price/speed differentiation

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for l from 1 to $L-1$ **do**

    for j from 1 to n **do**

        if $\rho(i,j,l,k) \neq 0$ and $\rho(i,j,l+1,k) \neq 0$:

                $k1 = min(-\frac{k_0\rho_{max}^2}{C\rho_0 u_{max}}, k_{1,min})$

        else: k_1 takes its initial value.

    end for

end for

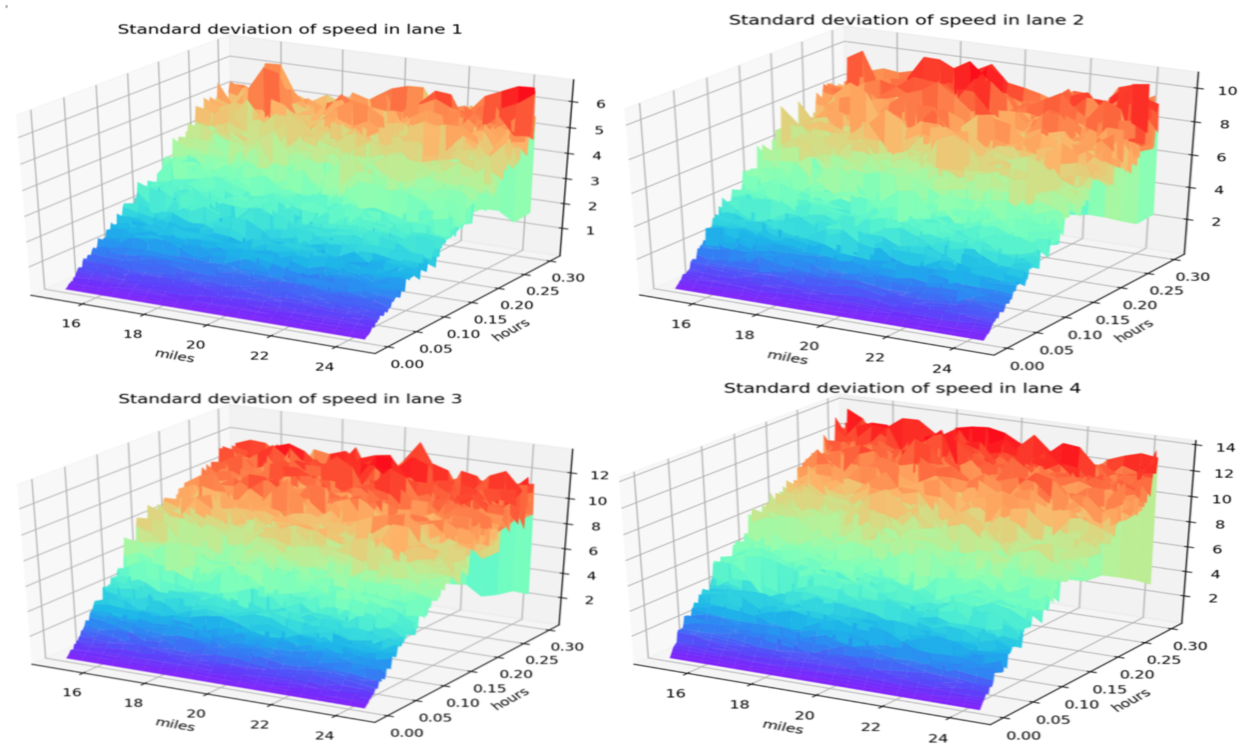

Figure 6.16: Standard deviation of speed in 4 lanes without accidents

situation is controlled well in $Case8$ due to our design of parameters. However, there is still a small tendency that they may converge in the long run in $Case8$, which is the result of trade-off between differentiation and conservation of cars as we mentioned. Table shows how cars are conserved as time evolves in $Case7$ and $Case8$. The number of cars are conserved well for both cases. However, it performs a little bit better in $Case7$ than in $Case8$ since eventually the total number of cars in $Case8$ increase from 8408.14 to 8411.769 and the total number of cars in $Case7$ never exceed its initial value.

6.4.1 Correlation of Speed and Price as a Function of a and b

Recall that in equation (4.46), we use two parameters a and b to define the function that determines the noise. We test the relation between these parameters and the average correlation of speed and price across space and time using the same initial conditions and parameters as in $Case1$. When

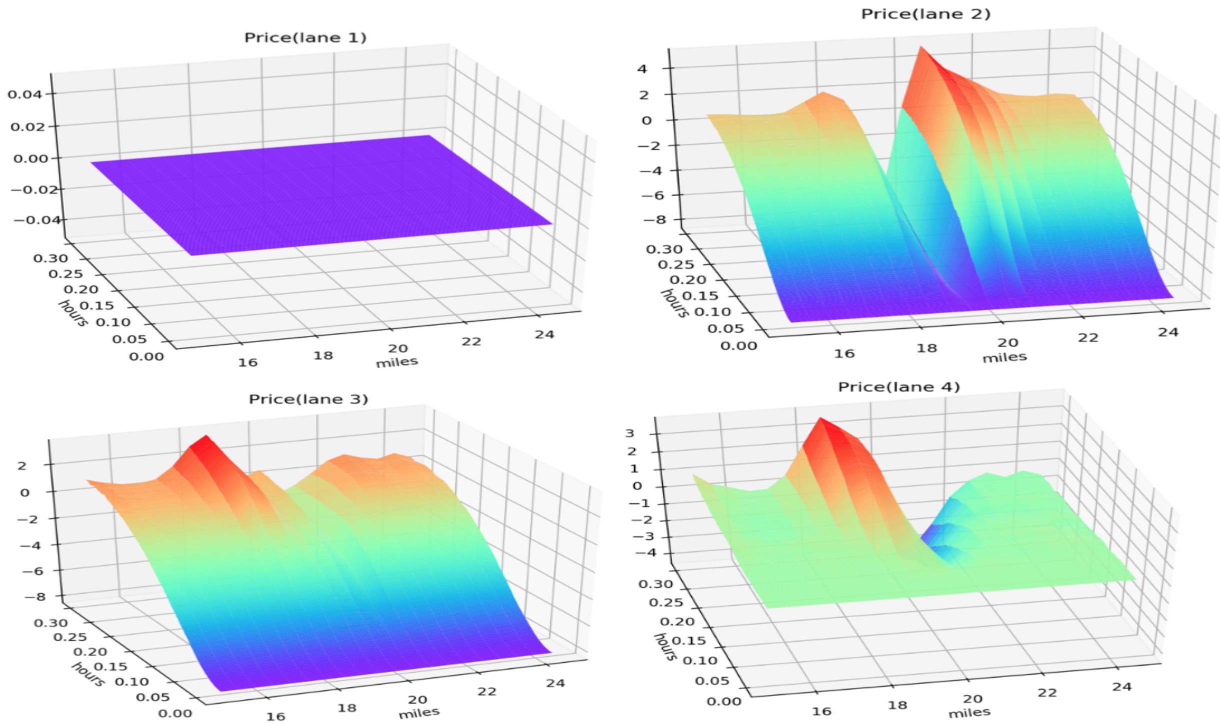

Figure 6.17: Average price in 4 lanes with accidents in lane 2 at $x = 20$

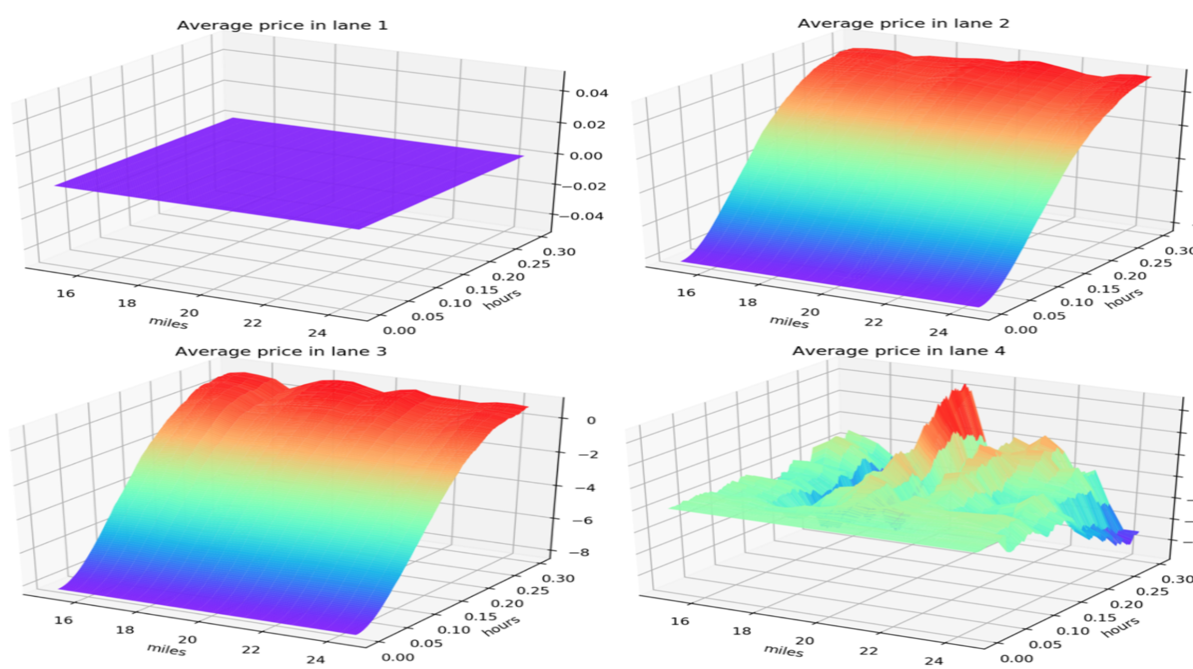

Figure 6.18: Average price in 4 lanes without accidents

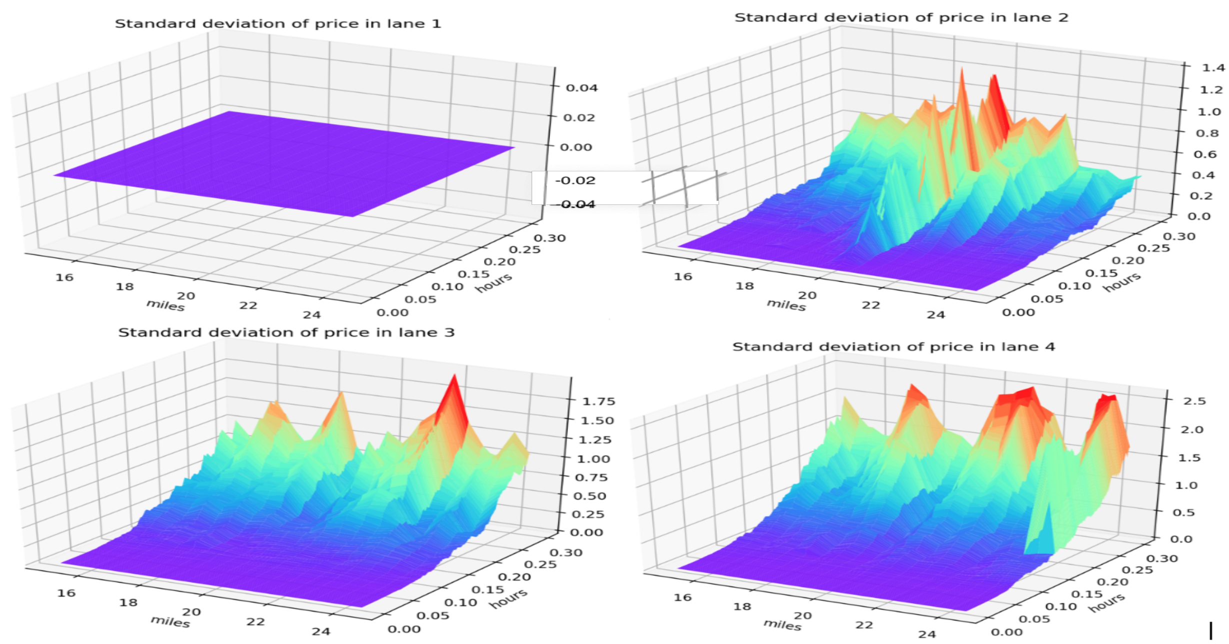

Figure 6.19: Standard deviation of price in 4 lanes with accidents in lane 2 at $x = 20$

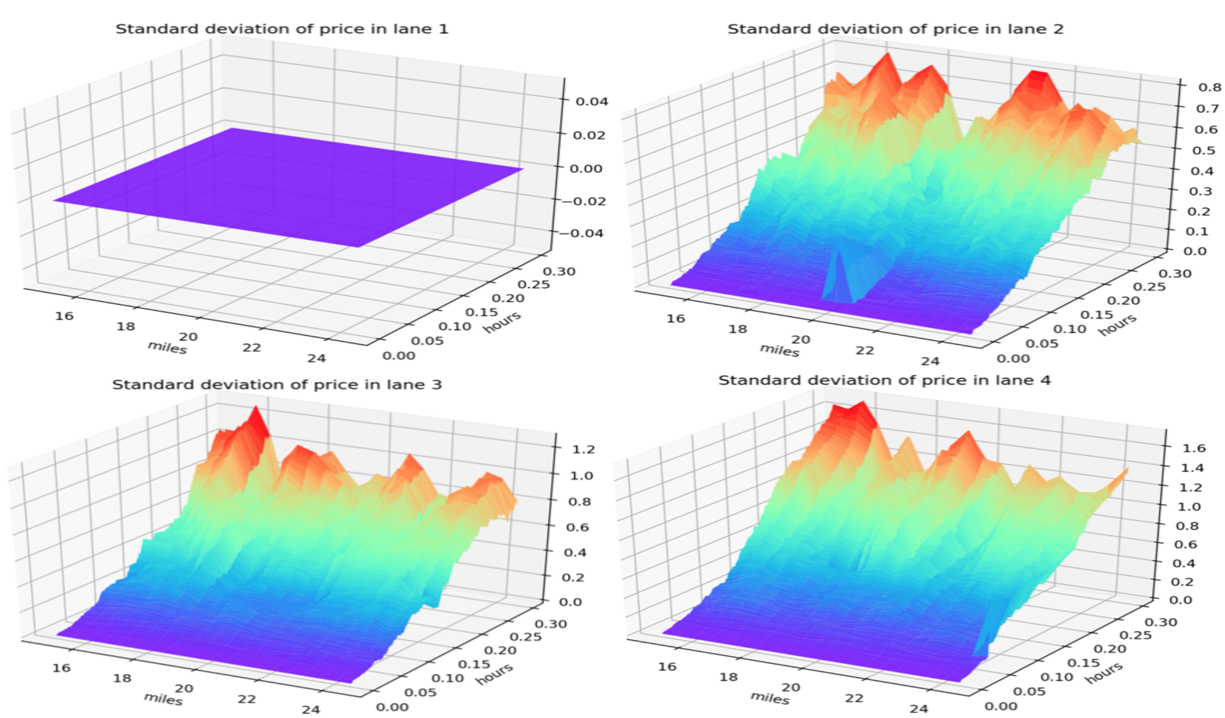

Figure 6.20: Standard deviation of price in 4 lanes without accidents

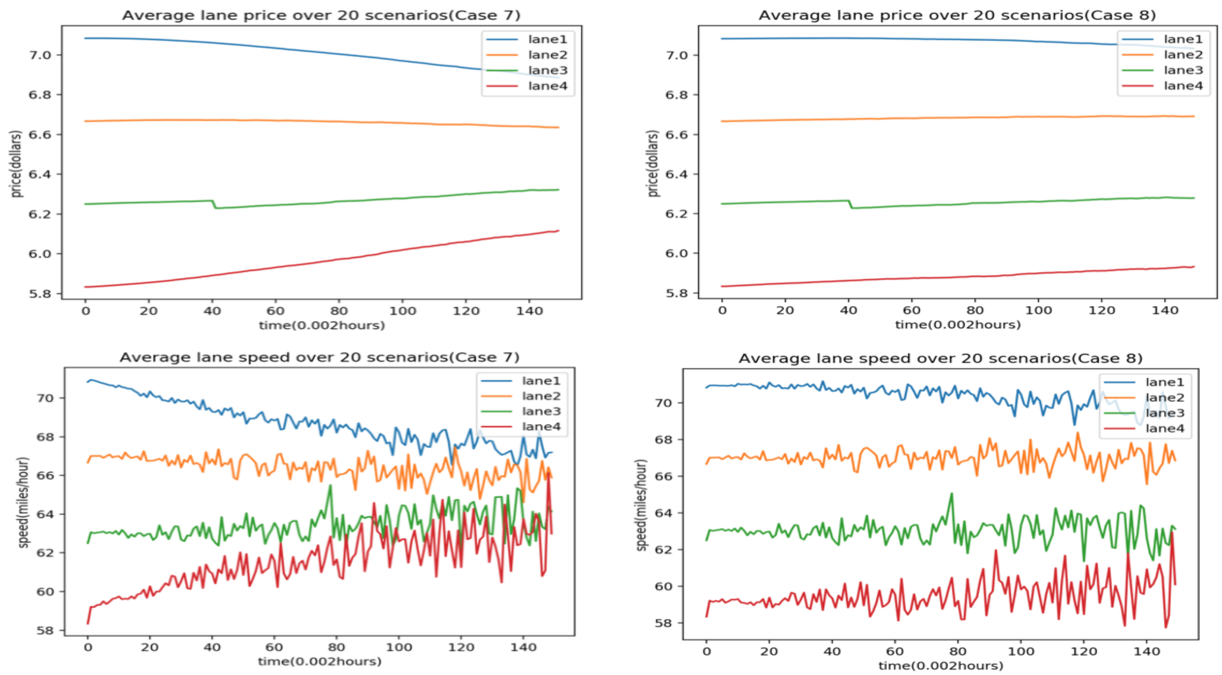

Figure 6.21: Price and Speed Differentiation

we test a, $b = 1$ and when we test b, $a = 1$.

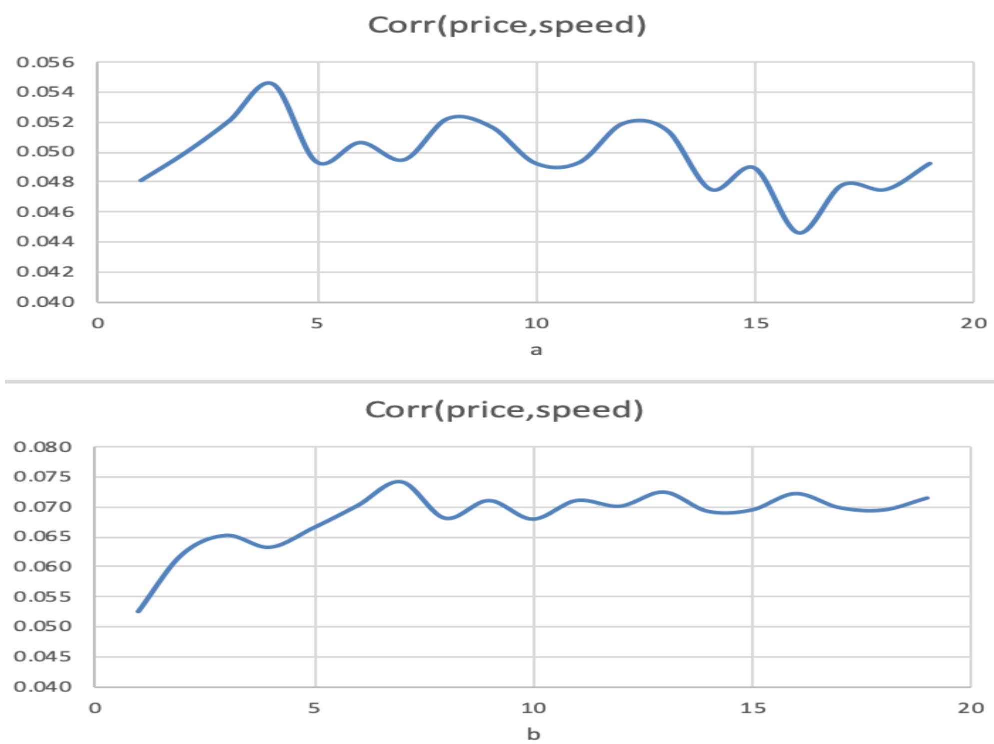

Figure 6.22: relation between $Corr(price, speed)$ and a/b

Chapter 7

Conclusion

We built a stochastic traffic and price model that satisfies the following conditions. Traffic density is positive and bounded above. Speed is positive. The algorithm satisfies the conservation law. Lane prices can be positive or negative, and they do no not explode. The no-arbitrage condition is satisfied. The maximum speed gain for cars that use the trading system is modest (of the order of 17 percent in our simulations). However, the total throughput of the freeway is not affected by the trading system. This dissertation could be extended with more analysis on how parameters affect the traffic and price, especially the analysis of the speed of mean-reversion β, for which we have few theoretical or numerical results.

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Appendix A

Estimation of Density-Elasticity of Transversal Flux k_0

We apply a linear regression to estimate the value of k_0 wherein the data in table (c.16) to (c.19) comes from the live camera of CALTRANS. The original equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q^x}{\partial x} = ab - a\rho - k_0 \frac{\partial \rho}{\partial y}$$

Letting

$$\frac{\partial \rho}{\partial t} + \frac{\partial q^x}{\partial x} = y$$

$$ab = \gamma_0$$

$$a = \gamma_1$$

$$k_0 = \gamma_2$$

By American standards we measure everything in mph. We measure the speed at $t = 5, 10, ...60mins$ respectively.

We then measure the flux from the recorded data at $t = 5, 10, ...60mins$.

We then calculate the difference in flux of 2 different cameras at the same time and divide this by 3 [This is because the approximate distance between the cameras is 3 miles] to evaluate $\frac{\partial q}{\partial x}$.

We then evaluate density at each time interval using the formula $\rho = \frac{q}{v}$. Using this we can evaluate $\frac{\Delta\rho}{\Delta t}$. [Here $\Delta t = 5mins = \frac{1}{12}hrs$].

Then we obtain the optimal parameters by solving the follwoing problem:

$$y_l^{i,j} = \frac{\rho_l^{i,j} - \rho_l^{i-1,j}}{\Delta t} + \frac{q_{x,l}^{i,j} - q_{x,l}^{i,j-1}}{\Delta x} \tag{A.1}$$

$$\min_{\gamma_0,\gamma_1,\gamma_2} \sum_i \sum_j \sum_l (y_l^{i,j} - \gamma_0 - \gamma_1 \rho_l^{i,j} - \gamma_2 \frac{\rho_l^{i,j} - \rho_{l-1}^{i,j}}{\Delta y})^2 \tag{A.2}$$

The values of $\gamma_0, \gamma_1, \gamma_2$ are 12.8, 29.69,43.83 respectively.

Hence

$$k_0 = 43.83$$

Appendix B

Inclusion of Transversal Derivatives

In algorithm 1 step 11 and 14, we ignore the following terms which can be found in equation (4.68) and (4.69): $\frac{\partial u}{\partial y}u^y(X(t), Y(t), t)dt$ and $\frac{\partial p}{\partial y}u^y(X(t), Y(t), t)dt$ which we consider have very small influence on the results. We test if this modification is acceptable by comparing the results from adding the terms back with the original one. Except for few outliers, figure 7.1 and 7.2 show that the difference in both price and speed as well as their standard deviation are close to 0 most of the time.

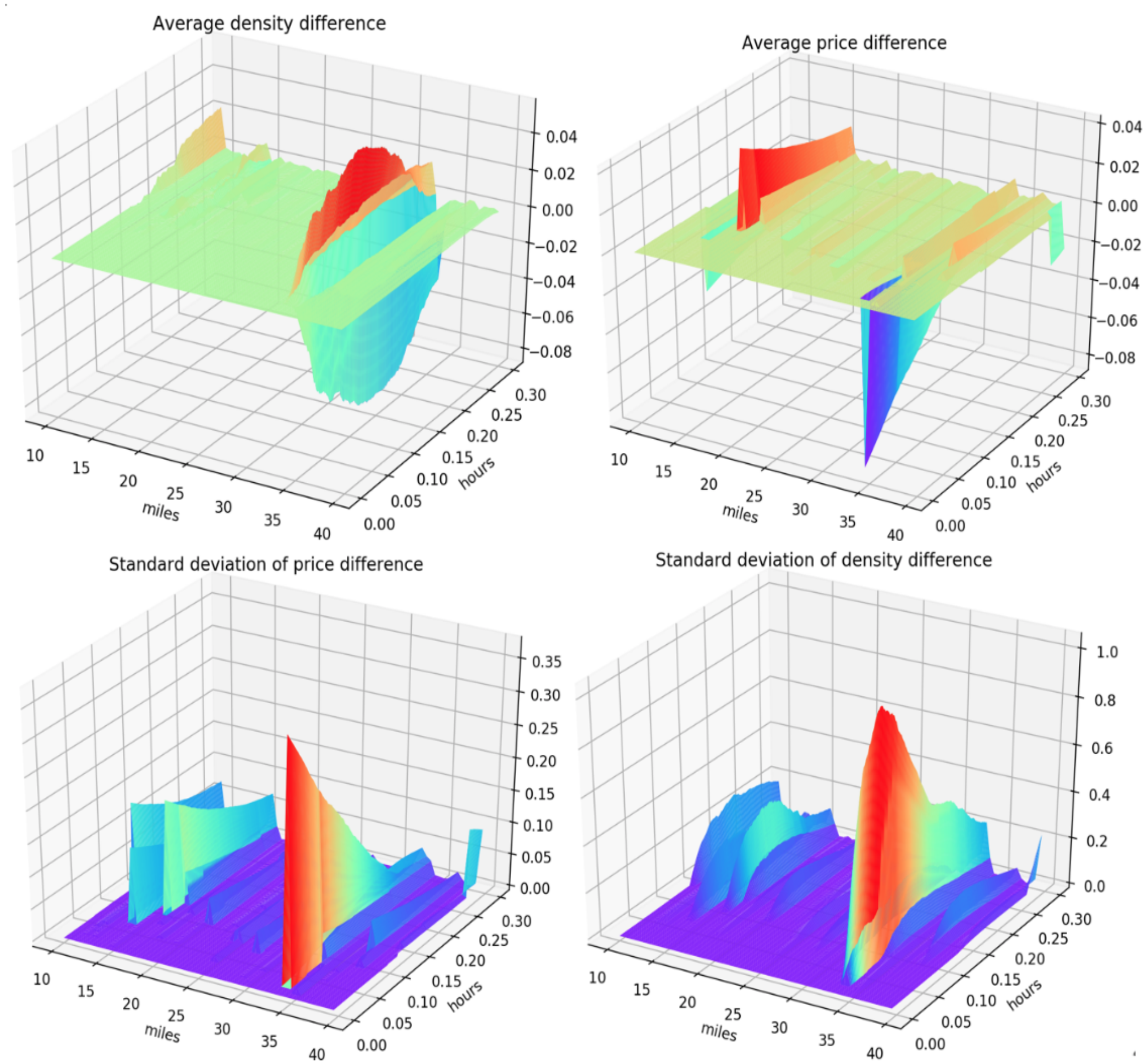

Figure B.1: Inclusion of Transversal Derivatives

Appendix C

Tables of Results

C	**B**	τ	**TimeNormal**	τ/**TimeNormal**	**TotalDistance**
0.1	23.32	0.41	2.01	21%	71283.08
0.2	91.56	0.75	3.92	19%	71325.97
0.3	203.66	1.03	5.78	18%	71359.12
0.4	355.80	1.24	7.55	16%	71391.46
0.5	547.52	1.41	9.27	15%	71411.68
0.6	775.56	1.53	10.92	14%	71361.56
0.7	1038.09	1.59	12.52	13%	71216.58
0.8	1341.05	1.64	14.09	12%	71433.47
0.9	1674.99	1.63	15.61	10%	71372.15
0.95	1887.74	1.62	16.36	10%	71368.85
1.1	2442.63	1.52	18.55	8%	71459.10
1.2	2871.82	1.42	19.96	7%	71469.16
1.3	3326.79	1.28	21.34	6%	71469.58
1.4	3814.43	1.10	22.72	5%	71464.32

Table C.1: Impact of C, initial condition: $Case1$, $k_1 = 0.1$

k_1	**B**	τ	**TimeNormal**	τ**/TimeNormal**	**TotalDistance**
0.1	23.32	0.41	2.01	21%	71283.08
0.2	45.78	0.75	3.92	19%	71325.97
0.3	67.89	1.03	5.78	18%	71359.12
0.4	88.95	1.24	7.55	16%	71391.46
0.5	109.50	1.41	9.27	15%	71411.68
0.6	129.26	1.53	10.92	14%	71361.56
0.7	148.30	1.59	12.52	13%	71216.58
0.8	167.63	1.64	14.09	12%	71433.47
0.9	186.11	1.63	15.61	10%	71372.15
0.95	195.28	1.62	16.36	10%	71368.85
1.1	222.06	1.52	18.55	8%	71459.10
1.2	239.32	1.42	19.96	7%	71469.16
1.3	255.91	1.28	21.34	6%	71469.58
1.4	272.46	1.10	22.72	5%	71464.32

Table C.2: Impact of k_1, initial condition: $Case1$,$C = 0.1$

C	**B**	τ	**TimeNormal**	τ**/TimeNormal**	**TotalDistance**
0.1	23.37	0.40	2.01	20%	71402.70
0.2	91.86	0.74	3.94	19%	71427.31
0.3	203.30	1.00	5.79	17%	71442.30
0.4	356.39	1.21	7.56	16%	71452.37
0.5	621.04	1.32	9.35	14%	71438.76
0.6	777.18	1.47	10.94	13%	71470.51
0.7	1040.34	1.53	12.54	12%	71472.65
0.8	1341.27	1.57	14.10	11%	71443.09
0.9	1699.70	1.54	15.66	10%	71449.45
1	2040.61	1.51	17.11	9%	71434.09
1.05	2277.13	1.47	17.85	8%	71453.40
1.2	2866.74	1.30	19.98	7%	71315.49
1.3	3340.24	1.17	21.39	5%	71418.31
1.4	3811.03	0.98	22.72	4%	71409.73

Table C.3: Impact of C, initial condition: $Case2$, $k_1 = 0.1$

k_1	**B**	τ	**TimeNormal**	τ**/TimeNormal**	**TotalDistance**
0.1	23.37	0.40	2.01	20%	71402.70
0.2	45.93	0.74	3.94	19%	71427.31
0.3	67.77	1.00	5.79	17%	71442.30
0.4	89.10	1.21	7.56	16%	71452.37
0.5	124.21	1.32	9.35	14%	71438.76
0.6	129.53	1.47	10.94	13%	71470.51
0.7	148.62	1.53	12.54	12%	71472.65
0.8	167.66	1.57	14.10	11%	71443.09
0.9	188.86	1.54	15.66	10%	71449.45
1	204.06	1.51	17.11	9%	71434.09
1.05	213.48	1.47	17.85	8%	71453.40
1.2	238.90	1.30	19.98	7%	71315.49
1.3	256.94	1.17	21.39	5%	71418.31
1.4	272.22	0.98	22.72	4%	71409.73

Table C.4: Impact of k_1, initial condition: $Case2$, $C = 0.1$

C	B	tau	TimeNormal	tau/TimeNormal	TotalDistance
0.1	30.40	0.26	1.70	15%	71484.31
0.15	66.90	0.33	2.50	13%	71484.19
0.2	116.87	0.37	3.28	11%	71482.96
0.25	181.07	0.36	4.03	9%	71481.55
0.3	255.09	0.33	4.75	7%	71483.82
0.35	342.20	0.25	5.46	5%	71482.21
0.4	440.56	0.15	6.14	2%	71482.07
0.45	547.44	0.02	6.79	0%	71476.21
0.5	666.10	-0.14	7.42	-2%	71478.81
0.55	793.76	-0.33	8.04	-4%	71479.82
0.61	963.12	-0.60	8.76	-7%	71475.72
0.65	1078.32	-0.80	9.21	-9%	71476.81
0.7	1228.77	-1.05	9.76	-11%	71463.73
0.75	1392.50	-1.34	10.31	-13%	71475.37

Table C.5: Impact of C, initial condition: $Case3$, $k_1 = 0.1$

k1	B	tau	TimeNormal	tau/TimeNormal	TotalDistance
0.1	30.40	0.26	1.70	15%	71484.31
0.15	44.60	0.33	2.50	13%	71484.19
0.2	58.43	0.37	3.28	11%	71482.96
0.25	72.43	0.36	4.03	9%	71481.55
0.3	85.03	0.33	4.75	7%	71483.82
0.35	97.77	0.25	5.46	5%	71482.21
0.4	110.14	0.15	6.14	2%	71482.07
0.45	121.65	0.02	6.79	0%	71476.21
0.5	133.22	-0.14	7.42	-2%	71478.81
0.55	144.32	-0.33	8.04	-4%	71479.82
0.61	157.89	-0.60	8.76	-7%	71475.72
0.65	165.90	-0.80	9.21	-9%	71476.81
0.7	175.54	-1.05	9.76	-11%	71463.73
0.75	185.67	-1.34	10.31	-13%	71475.37

Table C.6: Impact of k_1, initial condition: $Case3$, $C = 0.1$

C	B	τ	TimeNormal	τ/TimeNormal	TotalDistance
0.1	30.29	0.26	1.69	15%	71484.35
0.15	68.41	0.32	2.50	13%	71481.48
0.2	118.52	0.35	3.27	11%	71482.32
0.25	179.57	0.35	4.02	9%	71483.67
0.3	254.61	0.31	4.74	6%	71482.58
0.35	340.80	0.23	5.44	4%	71476.08
0.4	439.05	0.13	6.12	2%	71480.88
0.45	545.26	0.00	6.76	0%	71478.73
0.5	665.31	-0.17	7.40	-2%	71476.80
0.55	797.95	-0.37	8.03	-5%	71477.28
0.6	930.24	-0.58	8.61	-7%	71477.07
0.65	1077.36	-0.83	9.20	-9%	71475.78
0.7	1229.19	-1.09	9.75	-11%	71390.47
0.75	1488.98	-1.43	10.38	-14%	71479.84

Table C.7: Impact of C, initial condition: $Case4$, $k_1 = 0.1$

k_1	**B**	τ	**TimeNormal**	τ**/TimeNormal**	**TotalDistance**
0.1	30.29	0.26	1.69	15%	71484.35
0.15	45.61	0.32	2.50	13%	71481.48
0.2	59.26	0.35	3.27	11%	71482.32
0.25	71.83	0.35	4.02	9%	71483.67
0.3	84.87	0.31	4.74	6%	71482.58
0.35	97.37	0.23	5.44	4%	71476.08
0.4	109.76	0.13	6.12	2%	71480.88
0.45	121.17	0.00	6.76	0%	71478.73
0.5	133.06	-0.17	7.40	-2%	71476.80
0.55	145.08	-0.37	8.03	-5%	71477.28
0.6	155.04	-0.58	8.61	-7%	71477.07
0.65	165.75	-0.83	9.20	-9%	71475.78
0.7	175.60	-1.09	9.75	-11%	71390.47
0.75	198.53	-1.43	10.38	-14%	71479.84

Table C.8: Impact of k_1, initial condition: $Case4$, $C = 0.1$

C	B	τ	TimeNormal	τ/TimeNormal	TotalDistance
0.05	4.27	0.05	0.32	17%	59867.33
0.1	16.95	0.10	0.64	16%	59871.42
0.15	38.06	0.14	0.96	15%	59874.34
0.2	67.12	0.18	1.27	14%	59874.60
0.25	104.00	0.21	1.57	13%	59879.44
0.3	148.30	0.23	1.87	12%	59881.85
0.35	200.72	0.24	2.17	11%	59731.91
0.4	260.08	0.25	2.46	10%	59872.92
0.45	327.28	0.26	2.75	9%	59886.83
0.5	402.16	0.25	3.03	8%	59885.81
0.55	482.47	0.24	3.31	7%	59842.82
0.6	570.92	0.23	3.59	6%	59846.96
0.65	665.33	0.21	3.86	5%	59888.45
0.7	765.96	0.18	4.12	4%	59891.59

Table C.9: Impact of C, initial condition: $Case5$, $k_1 = 0.1$

k_1	**B**	τ	**TimeNormal**	τ/**TimeNormal**	**TotalDistance**
0.05	8.54	0.05	0.32	17%	59867.33
0.1	16.95	0.10	0.64	16%	59871.42
0.15	25.37	0.14	0.96	15%	59874.34
0.2	33.56	0.18	1.27	14%	59874.60
0.25	41.60	0.21	1.57	13%	59879.44
0.3	49.43	0.23	1.87	12%	59881.85
0.35	57.35	0.24	2.17	11%	59731.91
0.4	65.02	0.25	2.46	10%	59872.92
0.45	72.73	0.26	2.75	9%	59886.83
0.5	80.43	0.25	3.03	8%	59885.81
0.55	87.72	0.24	3.31	7%	59842.82
0.6	95.15	0.23	3.59	6%	59846.96
0.65	102.36	0.21	3.86	5%	59888.45
0.7	109.42	0.18	4.12	4%	59891.59

Table C.10: Impact of k_1, initial condition: $Case5$, $C = 0.1$

C	**B**	**tau**	**TimeNormal**	**tau/TimeNormal**	**TotalDistance**
0.05	6.90	0.13	0.70	19%	71432.56
0.1	27.14	0.23	1.37	17%	71437.72
0.15	60.76	0.30	2.04	15%	71442.64
0.2	106.19	0.35	2.67	13%	71446.31
0.25	164.01	0.37	3.28	11%	71449.68
0.3	233.51	0.36	3.88	9%	71452.72
0.35	314.37	0.33	4.47	7%	71454.13
0.4	405.13	0.27	5.03	5%	71393.86
0.45	506.24	0.19	5.58	3%	71454.32
0.5	618.91	0.08	6.11	1%	71461.18
0.55	738.25	-0.04	6.64	-1%	71457.00
0.6	868.53	-0.19	7.15	-3%	71463.64
0.65	997.36	-0.34	7.62	-4%	71465.14
0.7	1153.33	-0.54	8.12	-7%	71454.07

Table C.11: Impact of C, initial condition: $Case6$, $k_1 = 0.1$

k_1	**B**	τ	**TimeNormal**	τ/**TimeNormal**	**TotalDistance**
0.05	13.80	0.13	0.70	19%	71432.56
0.1	27.14	0.23	1.37	17%	71437.72
0.15	40.51	0.30	2.04	15%	71442.64
0.2	53.09	0.35	2.67	13%	71446.31
0.25	65.60	0.37	3.28	11%	71449.68
0.3	77.84	0.36	3.88	9%	71452.72
0.35	89.82	0.33	4.47	7%	71454.13
0.4	101.28	0.27	5.03	5%	71393.86
0.45	112.50	0.19	5.58	3%	71454.32
0.5	123.78	0.08	6.11	1%	71461.18
0.55	134.23	-0.04	6.64	-1%	71457.00
0.6	144.76	-0.19	7.15	-3%	71463.64
0.65	153.44	-0.34	7.62	-4%	71465.14
0.7	164.76	-0.54	8.12	-7%	71454.07

Table C.12: Impact of k_1, initial condition: $Case6$, $C = 0.1$

Time	Total Number of Cars		Time	Total Number of Cars		Time	Total Number of Cars	
	Case 7	Case 8		Case 7	Case 8		Case 7	Case 8
0.000	8408.140	8408.140	0.102	8408.140	8408.368	0.202	8408.140	8409.843
0.002	8408.140	8408.140	0.104	8408.140	8408.378	0.204	8408.140	8409.881
0.004	8408.140	8408.140	0.106	8408.140	8408.392	0.206	8408.140	8409.915
0.006	8408.140	8408.140	0.108	8408.140	8408.398	0.208	8408.140	8409.952
0.008	8408.140	8408.140	0.110	8408.140	8408.417	0.210	8408.140	8409.988
0.010	8408.140	8408.140	0.112	8408.140	8408.434	0.212	8408.140	8410.043
0.012	8408.140	8408.140	0.114	8408.140	8408.447	0.214	8408.140	8410.091
0.014	8408.140	8408.140	0.116	8408.140	8408.485	0.216	8408.140	8410.147
0.016	8408.140	8408.140	0.118	8408.140	8408.502	0.218	8408.140	8410.192
0.018	8408.140	8408.140	0.120	8408.140	8408.538	0.220	8408.140	8410.241
0.020	8408.140	8408.140	0.122	8408.140	8408.552	0.222	8408.140	8410.278
0.022	8408.140	8408.140	0.124	8408.140	8408.578	0.224	8408.140	8410.324
0.024	8408.140	8408.140	0.126	8408.140	8408.590	0.226	8408.140	8410.356
0.026	8408.140	8408.140	0.128	8408.140	8408.607	0.228	8408.140	8410.402
0.028	8408.140	8408.140	0.130	8408.140	8408.629	0.230	8408.140	8410.445
0.030	8408.140	8408.140	0.132	8408.140	8408.652	0.232	8408.140	8410.496
0.032	8408.140	8408.140	0.134	8408.140	8408.670	0.234	8408.140	8410.544
0.034	8408.140	8408.140	0.136	8408.140	8408.702	0.236	8408.140	8410.605
0.036	8408.140	8408.140	0.138	8408.140	8408.724	0.238	8408.140	8410.654
0.038	8408.140	8408.140	0.140	8408.140	8408.757	0.240	8408.140	8410.716
0.040	8408.140	8408.140	0.142	8408.140	8408.790	0.242	8408.140	8410.753
0.042	8408.140	8408.140	0.144	8408.140	8408.820	0.244	8408.140	8410.805
0.044	8408.140	8408.140	0.146	8408.140	8408.848	0.246	8408.139	8410.846
0.046	8408.140	8408.140	0.148	8408.140	8408.870	0.248	8408.139	8410.888
0.048	8408.140	8408.141	0.150	8408.140	8408.887	0.250	8408.139	8410.946

Table C.13: Conservation of cars

Time	Total Number of Cars		Time	Total Number of Cars		Time	Total Number of Cars	
	Case 7	Case 8		Case 7	Case 8		Case 7	Case 8
0.050	8408.140	8408.141	0.152	8408.140	8408.932	0.252	8408.139	8411.005
0.052	8408.140	8408.143	0.154	8408.140	8408.955	0.254	8408.138	8411.074
0.054	8408.140	8408.143	0.156	8408.140	8409.003	0.256	8408.138	8411.131
0.056	8408.140	8408.144	0.158	8408.140	8409.023	0.258	8408.137	8411.187
0.058	8408.140	8408.149	0.160	8408.140	8409.060	0.260	8408.136	8411.240
0.060	8408.140	8408.150	0.162	8408.140	8409.076	0.262	8408.135	8411.294
0.062	8408.140	8408.152	0.164	8408.140	8409.109	0.264	8408.132	8411.338
0.064	8408.140	8408.156	0.166	8408.140	8409.135	0.266	8408.130	8411.386
0.066	8408.140	8408.157	0.168	8408.140	8409.172	0.268	8408.127	8411.425
0.068	8408.140	8408.168	0.170	8408.140	8409.210	0.270	8408.122	8411.476
0.070	8408.140	8408.169	0.172	8408.140	8409.253	0.272	8408.116	8411.519
0.072	8408.140	8408.175	0.174	8408.140	8409.294	0.274	8408.107	8411.569
0.074	8408.140	8408.192	0.176	8408.140	8409.341	0.276	8408.097	8411.625
0.076	8408.140	8408.205	0.178	8408.140	8409.371	0.278	8408.083	8411.674
0.078	8408.140	8408.215	0.180	8408.140	8409.412	0.280	8408.068	8411.715
0.080	8408.140	8408.223	0.182	8408.140	8409.436	0.282	8408.049	8411.749
0.082	8408.140	8408.228	0.184	8408.140	8409.468	0.284	8408.025	8411.770
0.084	8408.140	8408.232	0.186	8408.140	8409.497	0.286	8407.993	8411.784
0.086	8408.140	8408.235	0.188	8408.140	8409.537	0.288	8407.955	8411.793
0.088	8408.140	8408.237	0.190	8408.140	8409.583	0.290	8407.907	8411.799
0.090	8408.140	8408.262	0.192	8408.140	8409.629	0.292	8407.847	8411.794
0.092	8408.140	8408.297	0.194	8408.140	8409.676	0.294	8407.783	8411.796
0.094	8408.140	8408.310	0.196	8408.140	8409.723	0.296	8407.697	8411.769
0.096	8408.140	8408.336	0.198	8408.140	8409.764			
0.098	8408.140	8408.346	0.200	8408.140	8409.805			

Table C.14: Conservation of cars

a(b=1)	Corr(price,speed)	b(a=1)	Corr(price,speed)
0.5	1.66	0.5	1.91
1.0	1.56	1.0	1.58
1.5	1.47	1.5	1.38
2.0	1.39	2.0	1.26
2.5	1.31	2.5	1.17
3.0	1.23	3.0	1.10
3.5	1.16	3.5	1.05
4.0	1.09	4.0	1.01
4.5	1.03	4.5	0.98
5.0	0.97	5.0	0.96
5.5	0.92	5.5	0.93
6.0	0.87	6.0	0.92
6.5	0.82	6.5	0.90
7.0	0.78	7.0	0.89
7.5	0.74	7.5	0.88
8.0	0.70	8.0	0.88
8.5	0.66	8.5	0.87
9.0	0.63	9.0	0.87
9.5	0.60	9.5	0.87

Table C.15: Relation between $Corr(price, speed)$ and a/b

	Number of cars in Lanes				Cars that change lanes					
Time	Lane 1	Lane 2	Lane 3	Lane 4	1 to 2	2 to 1	2 to 3	3 to 2	3 to 4	4 to 3
5	87	103	119	98	3	4	0	1	0	0
10	99	114	131	100	1	3	3	1	1	0
15	68	95	121	116	0	4	1	4	0	0
20	65	96	101	104	0	5	2	5	0	0
25	65	73	95	95	0	2	0	3	0	0
30	70	89	98	100	0	3	1	6	0	0
35	63	88	107	103	1	2	1	5	1	0
40	66	91	108	107	1	3	0	2	0	0
45	73	85	112	10	0	5	0	6	1	1
50	65	100	115	107	0	7	0	4	0	0
55	71	86	117	99	0	2	0	4	0	1
60	76	97	106	97	0	8	0	4	0	0

Table C.16: Flux data from Cal-Trans live camera 1

	Number of cars in Lanes				Cars that change lanes					
Time	Lane 1	Lane 2	Lane 3	Lane 4	1 to 2	2 to 1	2 to 3	3 to 2	3 to 4	4 to 3
5	91	96	104	52	1	1	3	8	1	2
10	88	90	105	70	1	2	2	2	1	1
15	92	103	115	84	3	3	0	4	3	1
20	91	111	115	74	0	0	1	4	0	3
25	81	96	113	89	0	3	2	1	0	1
30	94	11	103	77	1	2	2	3	1	0
35	88	98	105	63	0	6	0	4	1	1
40	100	100	96	66	0	4	1	4	2	0
45	95	113	116	94	2	3	2	2	1	1
50	105	114	125	79	1	4	2	1	0	1
55	94	102	100	75	0	1	2	2	0	1
60	91	105	97	64	2	6	1	0	2	0
65	105	119	105	82	1	5	2	1	0	0
70	100	106	110	91	0	2	3	1	1	0
75	94	120	106	92	1	4	1	4	2	1
80	87	89	118	94	3	2	2	3	0	0
85	98	110	117	72	1	0	0	2	0	0
90	90	99	105	75	1	5	4	3	0	0
95	96	121	118	73	0	1	1	2	2	0
100	103	120	129	95	1	3	3	3	0	0
105	105	120	125	82	1	1	2	2	0	0
110	91	103	118	80	0	4	0	1	1	0
115	108	123	124	91	2	1	1	2	0	0
120	102	110	123	84	1	2	1	2	0	0

Table C.17: Flux data from Cal-Trans live camera 2

	Longitudinal flux				Number of cars that change lanes					
Time	Lane 1	Lane 2	Lane 3	Lane 4	1 to 2	2 to 1	2 to 3	3 to 2	3 to 4	4 to 3
5	103	113	138	100	5	6	2	1	1	0
10	92	89	99	53	5	2	3	0	1	1
15	82	82	93	53	8	6	7	1	0	0
20	94	91	102	70	2	3	2	1	1	0
25	85	97	101	74	3	0	1	1	1	1
30	103	94	109	84	7	5	3	1	2	1
35	98	105	122	88	3	3	2	1	0	0
40	93	107	117	88	10	1	7	1	1	0
45	102	110	130	105	6	1	7	0	3	0
50	106	122	137	110	9	3	11	2	4	0
55	105	106	123	100	7	1	4	1	1	0
60	107	121	155	124	7	3	4	5	0	0

Table C.18: Flux data from Cal-Trans live camera 3

	Longitudinal flux				Number of cars that change lanes					
Time	Lane 1	Lane 2	Lane 3	Lane 4	1 to 2	2 to 1	2 to 3	3 to 2	3 to 4	4 to 3
5	105	111	118	104	1	3	4	0	0	0
10	120	103	127	106	0	3	4	2	1	0
15	101	113	111	105	5	1	5	0	1	1
20	103	105	129	104	4	3	4	1	1	0
25	83	96	105	94	7	4	5	1	1	0
30	102	105	123	109	4	1	6	3	0	0
35	90	109	101	104	3	3	3	4	0	0
40	97	109	117	116	10	2	3	0	2	0
45	98	115	123	94	1	1	4	0	3	0
50	103	124	133	125	4	1	3	4	0	2
55	111	106	124	104	3	2	5	0	0	0
60	110	125	128	120	3	2	1	1	3	0
65	98	111	119	104	3	4	4	3	3	0
70	93	109	115	107	3	2	5	1	3	0
75	98	108	108	107	5	1	3	3	0	2
80	118	135	142	120	6	1	8	3	0	3
85	105	112	129	109	6	0	1	3	0	3
90	103	126	127	107	9	2	2	1	0	0
95	103	118	124	114	6	4	2	2	1	0
100	97	124	132	109	8	4	5	2	0	0
105	103	111	124	126	3	0	2	1	2	4
110	100	110	129	129	8	2	5	2	2	1
115	99	120	135	130	1	5	8	2	2	1
120	111	125	151	132	5	3	6	2	4	1

Table C.19: Flux data from Cal-Trans live camera 4