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Mathematics is an Art: The Story of a One-Time Course

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Α - alpha, beginning, dreaming, imagining
Λ - lambda, letting, lighting, creating
Ω - omega, ending, finishing, completeing

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Α. IMAGINATION COMES FIRST

To create a work of art, there is that chaotic period beforehand when imagining runs rampant, myriad possibilities cross one's mind, and questions pose themselves one after the other. This has in itself been the theme for some works of art.

There is a drawing by Goya of himself sleeping at his drawing board, with bats, cats, and other monsters of his imagination swirling above him [1, Fig. 172]. At his side is a sketch of Minerva, the goddess of art and science. A similar drawing [1, Fig. 174], originally meant as a frontispiece for his *Caprichos* actually appeared as Capricho 43 and has inscribed on it, "El sueno de la razon produce monstruos" (The sleep of reason produces monsters). In this one, an owl, representing Minerva, proffers Goya a pencil. The imagination dreams up images, irrational and fleeting, providing resources for reason to shape and form before the actualization of a figure on paper. Together, they produce works of art.

The overture of Haydn's oratorio *The Creation* depicts the night of chaos before the first day of creation. This is the time when God the artist lets his imagination run rampant, and he considers the myriad possibilities, like "Should I make space Euclidean or non-Euclidean? And what about the Uncertainty Principle? Might that make things more interesting?" This period ends of course when the chorus breaks in with "Let there be light," which characterizes so well the onset of activity in actually creating a work of art.

α. Imagining Ways to Encourage Mathematical Activity

The idea of mathematics as an art raises a lot of monstrous questions. Most people don't think of math as art. Most math courses do very little to help students think of mathematicians as artists. Is that the way it should be? If a college offers courses in creative writing, in music composition, in painting and in sculpture, why can't it offer one in creative mathematics? Is mathematics really so different from the usual arts that students must wait until their third year in graduate school before being explicitly encouraged to be creative? Is it really the best thing to limit the role of college math students to studying the the great works of the masters (usually without mentioning the masters)? Such imaginings engendered a certain jealousy of colleagues who teach the usual arts, and this jealousy prompted a dream of a course offering students a chance to create some mathematics on their own. Further, this course was imagined to be part of an overall program, which in this dream was called the "Aleph Program." But for such a dream to be realized, support is needed, which suggests a grant, which in turn unleashes more questions of who to turn to and how to actualize on paper what one is dreaming.

λ. Let There Be a Grant Proposal

A grant was requested from the National Science

Foundation under its LOCI (Local Course Improvement) program in 1980. The following is a quote from the narrative part of the application.

> *Man ist was er isst* - Feuerbach.
>
> Feuerbach liked to play with words, while at the same time saying something serious with them. A particular objective of the Aleph Program is to encourage playing around with mathematics while at the same time not forgetting that math has great potential for serious use (if necessity is the mother of invention, then playing around is the father). In fact, its aim is to treat mathematics as a living language, and its goal is to encourage a small number of appropriate students to be fun-loving serious poets in that language. To press this point a bit further, Feuerbach's statement, when actually spoken, is twosided. It is like an element in a two-dimensional space which has a projection on a joking dimension which says "man ist was er ist" (one is what one is), and also a projection on a serious dimension which says "man ist was er isst" (one is what one eats). The Aleph Program likes the joke, and further, to achieve its goal, it takes this serious side, maps it homomorphically to "mind is what it learns," and proposes that quite a different learning process should be provided for minds which are learning a language with the intent of writing poetry than for those who are simply learning it to be fluent enough to function in the marketplace.

ω. The Final Count

Two panels of five persons each evaluated the grant proposal. One person initially gave it a 1, the lowest score possible, deeming it too lighthearted to be taken seriously. Another person gave it a 7, the highest score possible, considering it a gamble, but one definitely worth taking. The other scores fell in between, resulting in an average that actually was big enough for the proposal to be in the group of grants granted.

Λ. LET THERE BE A CREATIVE MATH COURSE - FALL SEMESTER, 1981

"Para 40-Mathematical Thinking and Writing" is the name of the course, where "Para" is short for "Paracollege." The Paracollege is a small college within St. Olaf College which offers an alternative route, via tutorials and examinations, to a BA degree. It also encourages students and faculty in all disciplines to come up with, and then try out, new ideas. Para 40 was a new idea intended for first year college students, and four of them - Paul Borman, Eric Heppner, Beth Nelson, and Ken Olstad - ventured to sign up for it.

α. Exercising the Imagination

The goal was to produce some mathematical works of art. To start with there was a chaotic period of casting about for ideas with various lures, and then, once there was a strike, there were further questions about how to land it. For several years it had been the custom for a small group of Paracollege students to construct something using an old haymow rope during the first week of school. This year the structure was a "merry-go-round", which was essentially a triangle made of three wooden beams which was hung by means of the rope from an overhanging bough fifty feet above the ground. When the class first met I told them I had no preconceived ideas of what we would end up doing, but that ideas for math systems were lurking everywhere, and they should learn to be on the lookout for them. Since they had helped construct the merry-go-round, I asked if they could imagine some ways to use math in describing it. They had lots of ideas, from the angles of the triangle and the tension in the ropes to its behavior as a pendulum when it was swung back and forth. It was its behavior as a spinning object which interested us most, however, and the succession of the triangle's vertices as they passed by, *A,B,C,A,B,C*.... suggested a counting system in which the passing of three vertices gives the same result as the passing of zero vertices. Modulus 3 arithmetic was thus the result and they checked to see if it had the properties which later on they would learn are subsumed under the name "ring." This system thus had many of the same properties as the system of integers. The fact that 1+2=0 in this system meant that 2=-1, and this prompted Beth to raise the question whether absolute value made any sense here. We were not able to make any sense of absolute value for the modulus 3 system, but at this point Paul remembered he had been interested in the absolute value function in high school, particularly in its similarity to the squaring function, since for ordinary numbers both functions always produced nonnegative values. He knew that interesting things had happened when somebody imagined a number whose square is -1. He accordingly wondered what might happen if he assumed there were a number whose absolute value is -1. That made me wonder too, and at that point I felt we had a strike, namely that we had something on the line worth playing with.

Sensing it would be good to have some experience in embedding a system in a bigger one, I asked the students to generalize what they had done in defining modulus 3 arithmetic to modulus n arithmetic. I then asked them to look specifically at the modulus 2 system and see if they could find a bigger system which had the properties of a ring and which had the modulus 2 system embedded within it. Using exhaustive methods, they painstakingly discovered it can't be embedded in a 3-element system. Continuing these methods, they considered 4-element systems and came up with 4 different ways to do it. In looking at these 4 different systems, with the hope of discovering a better way to arrive at them, they discovered that each could be thought of as adding to 0 and 1 (from the modulus 2 system) an element q and an element $1+q$. The 4 systems were then produced by letting q^2 be in turn each of the 4 elements in the system. Furthermore, the 4 elements of the system can be expressed in the form $x+qy$, for $x, y \in \{0, 1\}$. Also, in looking at the 4 systems, it was quickly discovered that the system with $q^2=0$ and the one with $q^2=1$ were really the same system, since the q and $1+q$ in the first system behave just like the $1+q$ and q respectively in the second system. Thus the idea of "having the same structure," later to be called "isomorphism," first reared its comely head. To test that they had a good method for embedding, I chose the 4-element system where $q^2=1+q$ (some day they'll find out a technical name for it is "GF(2^2)") and asked them to embed it. They of course found 16 different 16-element rings. Furthermore, Beth went through the 16 systems and found that, up to isomorphism, there were only 3 different systems, just as in the previous case. With these exercises completed, I felt we were ready to start the students' first work of art.

λ_1. Let There Be a Number Whose Absolute Value is -1

After the initial creative thrust of bringing a number into being, a next important and fun step is to name it. Paul, whose idea it was, had started taking Russian, and he thought an esoteric symbol, like the Russian tverdy enak, was appropriate for such a strange number. This designation didn't last very long, however, for it was not on the usual typewriter or computer keyboard, and the next symbol used was the dollar sign. Later on (when hyperbolic trigonometry appeared) the symbol was changed to "h," and to keep the same symbol throughout, it will be introduced here. Thus h is the number such that

$$|h| = -1.$$

Since the adjective "imaginary" is used for the number i, the students searched for an appropriate adjective for h and came up with "hallucinatory." The next question was whether the real numbers could be embedded in a bigger system which had h in it. The students guessed, and then showed, that there were rings with elements $x+qy$, for $x, y \in \Re$(eal), where q^2 could equal any element in the ring. They knew that one of these systems was well-known, namely the system of complex numbers, in which case q is given the name "i" and $i^2=-1$. They then wondered if one of these q's could be interpreted as h. The next problem was thus to figure out what element in the ring h^2 might be. The students went off to the library and looked up the properties of the absolute value function for real numbers. One property was $x^2 = |x|^2$. If one assumes this to hold for h, the result is

$$h^2 = |h|^2 = (-1)^2 = 1$$

Having no other argument available for what h^2 might be, other than the aesthetic one that if $|i|=1$, $i^2=-1$ then why not $|h|=-1$, $h^2=1$, we adopted this one. We thus had a system of numbers $x+hy$, for $x, y \in \Re$, which Paul dubbed the "ineptitude numbers." Later on, to parallel the complex number terminology, they became the "perplex numbers."

To get ideas of what to do with these numbers, the students did some checking of what things were done with the complex numbers. One thing was finding inverses. For the complexes, $(x+iy)^{-1}=(x-iy)/(x^2+y^2)$, so that every complex has an inverse except $0+i0=0$, namely the point where $y=x=0$. Using the same method for the perplexes,

$$(x + hy)^{-1} = \frac{x - hy}{x^2 - y^2},$$

so that every perplex has an inverse except $x \pm hx$, namely the points on the lines $y=\pm x$. In time it was found that these numbers without inverses essentially characterize the perplexes. In particular, they divide the x, y plane up into 4 quadrants where x^2-y^2 is positive in the left and right quadrants and negative in the top and bottom quadrants. Further, since absolute value was extended in the complex case by $|x+iy| = (x^2+y^2)^{1/2}$, it was suspected that x^2-y^2 should be involved in the perplex case. For $0+h1=h$, this quantity is $0^2-1^2=-1$. One can't, however, just take the square root of this, since the result would not be -1. Various ways were tried to arrive at this result, and after some time a way was found to do this. One of the criteria was to find a function which would work for both the complexes and perplexes. The function $^{1/\ddot{2}}$, called the "funny square root" by the students, and defined for x real by

$$x^{1/\ddot{2}} = (\operatorname{sgn} x)|x|^{1/2},$$ (sgn x) is the sign function,

satisfied this criterion and was chosen. It is the inverse of the function $\ddot{2}$, called the "funny square," defined for x real by $x^{\ddot{2}}=(\operatorname{sgn} x)|x|^2$. Absolute value is then extended to perplexes by

$$\left|(x+hy)\right| = \left(x^2-y^2\right)^{1/\ddot{2}}.$$

With absolute value defined, one can look at curves of constant absolute value. For the complexes these are circles, and the circle $|x+iy|=1$, namely $x^2+y^2=1$ is of particular interest. By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and since $\cos^2\theta + \sin^2\theta=1$, it follows that $e^{i\theta}$ is on this circle. For the perplexes, the curves of constant absolute value are hyperbolas, and the hyperbola $|x+hy|=1$, namely $x^2-y^2=1$, was guessed to be of particular interest. Looking up the infinite series method for arriving at Euler's formula, Ken was able to come up with the following result,

$$e^{h\theta} = \cosh\theta + h\sinh\theta,$$

and since $\cosh^2\theta-\sinh^2\theta=1$, it follows that $e^{h\theta}$ is on the right hand arm of the hyperbola $x^2-y^2=1$. Although there were lots of ideas of things to do with the perplexes, it was along about this point that the students couldn't refrain from starting a second work of art.

λ_2. Let There Be a Nonreal Number Whose Square is Zero

In the language of elements q, $q^2=-1$ gave the complexes and $q^2=1$ the perplexes. Eric in particular was interested in the case $q^2=0$. The original symbol selected for this case was "Z" and it was called "Zarph," but later on (when linear trigonometry appeared) the symbol was changed to "ℓ" and it was called "ludicrous." Thus ℓ is the number such that

$$\ell^2 = 0 \cdot$$

In no time at all the following results appeared.

$$\left(x+\ell y\right)^{-1} = \frac{\left(x-\ell y\right)}{x^2},\ \left|x+\ell y\right| = \left(x^2\right)^{1/\ddot{2}} = |x|,$$
$$e^{\ell\theta} = 1+\ell\theta.$$

Numbers on the line $x=0$ have no inverses. Curves of constant absolute value are vertical lines, and $|x+\ell y|=1$, namely the lines $x^2=1$ are of particular interest, with $e^{\ell\theta}$ being on the line $x=1$. Because we weren't aware of any known trigonometric functions being defined on this line, we defined the "linear cosine" and "linear sine" to be

$$\cos\ell\theta = 1,\ \sin\ell\theta = \theta.$$

With this much done, there was curiosity about other q-systems, and, instead of treating them individually, Paul wondered if we couldn't produce them en masse. A somewhat more technical term for this would be "to generalize," and a generalization, somewhat like a gallery show with all the works on the same theme, might itself be considered a work of art.

λ_3. Let There Be a Number System for Every Point in the Real Plane

For the point (a, b) in the real plane, $\Re_q$ is the *q-extension of the reals*, or simply the *q-number system*, where

$$\Re_q = \{x+qy \mid x,y \in \Re\},$$ with $q^2= a+qb$.

At this point "q" stood for "quixotic," namely "romantic without regard to practicality." Later, it was realized it could stand for "quadratic," since the parameters a, b define a unique quadratic form zz^* for each number system.

It is convenient in this section to always let $z=x+qy$ and $z'=x'+qy$. Explicit definitions of *addition* and *multiplication* then are

$$z + z' = (x + x') + q(y + y'),$$
$$zz' = (xx' + ayy') + q(xy' + yx' + byy').$$

For zz' to be real, $xy'+yx'+byy'=0$, i.e. $x'=(-y/y')(x+by)$. A choice which motivates the next definition then is $y'=-y$, $x'=x+by$. The *q-conjugate* of z is z^*, where

$$z^* = (x + by) + q(-y), \text{ so that } zz^* = x^2 + bxy - ay^2.$$

The following properties were checked to hold for the q-conjugate.

$$(z + z')^* = z^* + z'^*,\ (zz') = z^* z'^*,\ z^{**} = z.$$

The *multiplicative inverse*, if it exists, is

$$z^{-1} = \frac{z^*}{zz^*}$$

It accordingly fails to exist when $zz^*=0$, namely when $x^2+bxy-ay^2=0$. If $y=0$ then $x=0$ as expected. If $y\neq 0$, dividing by y^2 gives

$$\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) - a = 0, \text{ so that}$$
$$\frac{x}{y} = \frac{1}{2}\left(-b \pm \left(b^2 + 4a\right)^{\frac{1}{2}}\right) = \frac{1}{c_\pm},$$

which describes 2 lines, $y=c\pm x$, with slopes c_+ and c_- This provides three different cases. If $b^2+4a>0$, the 2 lines lie in the x, y plane, and they can be any two nonhorizontal lines passing through the origin. If $b^2+4a=0$, the 2 lines merge into 1 with slope $c_+=c_-=-2/b$. If $b^2+4a<0$, the slopes are no longer real, so the 2 lines "leave the x, y plane," intersecting it only at the origin. From our earlier experience with finite systems, students expected that, up to isomorphism, there might be only three systems among all the q-extensions. With a great deal of effort, Eric succeeded in showing that all the q-extensions with $b^2+4a>0$ were (ring)-isomorphic to the h-extension. Using the same technique it soon followed that all with $b^2+4a=0$ were isomorphic to the ℓ-extension and those with $b^2+4a<0$ were isomorphic to the i-extension. This however did not diminish a continued interest in all the q-extensions. The *q-absolute value of z* is

$$|z| = (zz^*)^{1/2}, \text{ with properties } |zz'| = |z||z'|,\ |z^*| = |z|,$$
$$|z^{-1}| = |z|^{-1}.$$

A curve of particular interest is

$$|z| = 1 \text{ namely}$$
$$x^2 + bxy - ay^2 = (x + \frac{1}{2}by)^2 - \frac{1}{4}(b^2 + 4a)y^2 =$$
$$(x - y/c_+)(x - y/c_-) = 1$$

It always passes through the numbers 1 and -1. If $b^2+4a<0$ it is an ellipse. If $b^2+4a=0$ it is two parallel lines with slope $-2/b$. If $b^2+4a>0$ it is a hyperbola with asymptotes being the two lines $y=c_\pm x$, namely the lines $|z|=0$. In this last case $|z|$ can also be negative and the curve $|z|$ same asymptotes as $|z|=1$, but "orthogonal" to it.

In order to generalize the concept of unit circle from the complex case, it seemed that $|z|=1$ and $|z|=-1$ should both be included, which prompted the next definition. The absolute value of the q-absolute value of z, called the *radial function of z* is

$$p(z) = \|z\|, \text{ with properties } p(zz') = p(z)p(z'),$$
$$p(z^*) = p(z),\ p(z^{-1}) = (p(z))^{-1},\ p(p(z)) = p(z).$$

The curve $p(z)=1$ is the *q-unit curve*, and the set of points on this curve is

$$U = \{z \mid p(z) = 1\}.$$

It follows that if z, z' are in U so is their product, as well as their inverses. The system $U(\cdot)$ consisting of U together with multiplication is then a group. For $b=0$, $c_\pm=\pm a^{-1/2}$, and there is symmetry about both the x and y axes. Fig.1 indicates how these change with the parameters a and $c_\pm$. As b varies from zero, the axial symmetry is lost and the curve gets more and more skewed.

In order to generalize trigonometry now from the complex case, angles have to be assigned in some way to points on the unit curve. When $q=i$ we know that $e^{i\theta}$, $0<\theta<2\pi$, describes the points on the unit curve, and θ can be interpreted as arc length. Similarly, when $q=\ell$, then $e^{\ell\theta}$, $\theta \in \Re$, describes the righthand line of the unit curve, and θ again can be interpreted as arc length. When $q=h$, then $e^{h\theta}$, $\theta \in \Re$, describes the points on the unit curve in the righthand quadrant, but θ cannot be interpreted as arc length in this case. Looking up information about hyperbolic trigonometry, the students found that θ can be interpreted as twice the area swept out by a ray from the origin as its outer point moves from e^{h0} to $e^{h\theta}$. Since this interpretation also works for the i and ℓ cases, it suggests a way to generalize to q for that continuous part of the unit curve passing through $z=1$. The fact that $e^{q(\theta+\theta')}=e^{q\theta}e^{q\theta'}$ holds for the cases $q = i,\ \ell,\ h$, however, suggests a more general approach, since e^q acts like an isomorphism, with which the students had already had some experience.

To assign a unique angle to each element in U and to allow for addition of angles, let $\Phi(+)$ be an isomorphic copy of $U(\cdot)$. The elements of Φ are *angles*. The *angle function* α is an isomorphism from $U(\cdot)$ to $\Phi(+)$, so that, for $z,\ z' \in U$,

$$\alpha(zz') = \alpha(z) + \alpha(z').$$

In particular, then, for 0 the additive identity of $\Phi(+)$,

$$\alpha(1) = 0.$$

Further, specify a particular element $\mu \in U$ as the *unit measure for angles* and define an element $1 \in \Phi$ by

$$\alpha(\mu) = 1.$$

Then let μ be the inverse mapping α^{-1} by means of the following notation, for $\phi \in \Phi$,

$$\mu^\phi = \alpha^{-1}(\phi), \text{ so that } \mu^{\alpha(z)} = z.$$

It follows that

$$\mu^{\phi+\phi'} = \mu^\phi \mu^{\phi'},\ \mu^0 = 1,\ \mu^1 = \mu.$$

For $q=i,\ \ell,\ h$, the choice for μ is of course e^q. But e^q for the general case is on the unit curve only for $b=0$. To remedy this for the general case, we set

$$\mu = e^{-(b/2)}e^q.$$

Now we can define the *q-cosine*, *q-sine*, and *q-tangent* as functions cosq, sinq, and tanq from Φ to $\Re$ by

$$\cos\mathrm{q}\,\alpha(z) = x,\ \sin\mathrm{q}\,\alpha(z) = y,\ \tan\mathrm{q}\,\alpha(z) = \frac{y}{x}.$$

The *q-Euler formula* then is

$$\mu^{\alpha(z)} = z = x + qy = \cos\mathrm{q}\,\alpha(z) + q\sin\mathrm{q}\,\alpha(z)$$

From this, in the usual way, the angle addition formulas are found to be

$$\cos\mathrm{q}(\phi + \phi') = \cos\mathrm{q}\,\phi\cos\mathrm{q}\,\phi' + a\sin\mathrm{q}\,\phi\sin\mathrm{q}\,\phi',$$

$$\sin\mathrm{q}(\phi + \phi') = \sin\mathrm{q}\,\phi\cos\mathrm{q}\,\phi' + \cos\mathrm{q}\,\phi\sin\mathrm{q}\,\phi' + b\sin\mathrm{q}\,\phi\sin\mathrm{q}\,\phi',$$

$$\tan\mathrm{q}(\phi + \phi') = \frac{\tan\mathrm{q}\,\phi + \tan\mathrm{q}\,\phi' + b\tan\mathrm{q}\,\phi\tan\mathrm{q}\,\phi'}{1 + a\tan\mathrm{q}\,\phi\tan\mathrm{q}\,\phi'}.$$

Generalization of the usual trigonometric identities follows in similar fashion.

Finally, to have q-polar coordinates for all z with $p(z)\neq 0$, note that

$$p[z/p(z)] = p(z)/p[p(z)] = p(z)/p(z) = 1,$$

so $z/p(z) \in U$. The *extension of the angle function* α to $z \in \Re_q$ is then defined by

$$\alpha(z) = \alpha[z/p(z)].$$

Thus, for $p(z)\neq 0$,

$$z = p(z)[z/p(z)] = p(z)\mu^{\alpha[z/p(z)]} = p(z)\mu^{\alpha(z)}.$$

It took a while for me to realize that q-number systems were not new, but have been around since the time of Cauchy, since they can be considered as $\Re[x]/(x^2\text{-}bx\text{-}a)$, namely as the quotient ring of the ring of polynomials $\Re[x]$ modulo the polynomial $x^2\text{-}bx\text{-}a$. The q-definitions of absolute value and angle, however, were, to the best of my knowledge, new, and it is they which make possible the many q-generalizations from classical results in areas such as trigonometry, geometry, and physics.

One idea generates more ideas and as we worked during the semester ideas of things to do came faster than we could do them. For example, we wondered how many of all the different things done with the complexes might have q-analogues. We concentrated on q-trigonometry, but what about q-geometry, q-analysis, and q-Hilbert space? We also wondered about embedding q-number systems in bigger systems. Can we have i and h in the same system, and maybe ℓ as well? We did check out that there are q-extensions if one replaces the real field by an arbitrary field, and that each extension has its own trigonometry, but then we wondered what happens if one starts with a ring which isn't a field. We also wondered if the ℓ-number system would be good for doing calculus, since ℓ can be interpreted as an infinitesimal. This has since been done by some other students, and in that context the numbers came to be called "ethereal numbers" [2]. Other creative work on q-geometry has since been done by students, and [3] is a byproduct of that.

With the increasing degree of generalization and attendant abstractness in our work, the students did show various signs of restlessness. Already with the perplexes Eric and Ken were saying things like "Okay, we've invented a number system, but so what, what good is it?" I was somewhat taken aback. I thought the beauty and elegance of the perplex number system was ample reward for our large investment of time and energy. Thinking of it as a work of abstract art with no need to describe something in the real world was somewhat soothing for them, but not really satisfying. In varying degrees I think the students' aesthetic instincts leaned toward representational art, so I urged them to be on the lookout for possible applications of our q-number systems.

ω. Finally, an Application

Toward the end of the semester the results obtained by the class were presented at one of the math department's weekly colloquia. One of the figures shown was the unit curve for the perplexes with its hyperbolas and their $|z|=0$ asymptotes. Lynn Steen commented that these asymptotes reminded him of the light cone in special relativity theory. That was just the hint needed for an application, and excitement rose when Taylor and Wheeler's book *Space-Time Physics* was discovered to use hyperbolic trig functions in an elegant way to express the equations of special relativity [4]. The perplexes suddenly became a very natural language for this theory of physics [5]. Generalization, however, had become a habit of thought, and if the perplexes corresponded to one kind of physics, then each q-number system should correspond to a specific kind of physics. The following interpretation of $\Re_q$ will be called *q-physics*, and in this section let $z \in \Re_q$ be $z=t+qy$, where t is interpreted as a time coordinate and y as a position coordinate which is a function of time. Then z, as a function of time, traces out a curve called a *world line*. Differentially speaking, $dz=dt+qdy$. Let $d\tau$ be the radial coordinate of dz, called the *proper time*, and when $d\tau\neq 0$, let 0 be the angular coordinate of dz, called the velocity parameter. Thus,

$$dz = dt + qdy = d\tau\mu^{\phi} = d\tau(\cos\mathrm{q}\,\phi + q\sin\mathrm{q}\,\phi),$$

giving,

$$dt = d\tau\cos\mathrm{q}\,\phi,\ \ dy = d\tau\sin\mathrm{q}\,\phi, \text{ so that}$$

$$v = \frac{dy}{dt} = \frac{\sin\mathrm{q}\,\phi}{\cos\mathrm{q}\,\phi} = \tan\mathrm{q}\,\phi,$$

where v is the *velocity*. For *rest mass* m_0, the *mass m* and *momentum p* are defined as

$$m = m_0\cos\mathrm{q}\,\phi,\ \ p = m_0\sin\mathrm{q}\,\phi = mv.$$

The "light cone" is determined by the lines $y=c_{\pm}t$, so that the slopes $c_{\pm}$ are interpreted as the velocities of light in the positive and negative directions along the y-axis. For b=0, the "light cone" is symmetric with regard to the two directions, i.e. $c_\pm=\pm a^{-1/2}$, and for the specific cases $q=h$, ℓ, i one has respectively $c_\pm=\pm 1$, $\pm\infty$, $\mp i$.

The elements μ^ϕ are the *q-Lorentz transformations*, such that z and z' are related by

$$z = z'\,\mu^\phi,$$

where the reference frame for z' moves relative to that for z with velocity $v=\tan q\,\phi$. A transformation μ^ϕ followed by $\mu^{\phi'}$, results in $\mu^\phi\mu^{\phi'} = \mu^{(\phi+\phi')}$. For the corresponding velocities,

$$v = \tan q\,\phi,\;\; v' = \tan q\,\phi',\;\; v'' = \tan q(\phi+\phi'),$$

the *addition rule for velocities*, from the addition formula for tangents, is

$$v'' = \frac{v + v' + bvv'}{1 + avv'} = \frac{v(1 - v'/c_+) + v'(1 - v/c_-)}{1 - vv'/c_+c_-}.$$

If $v'=c_+$, then $v''=c_+$, and if $v=c_-$, then $v''=c_-$. Because of the symmetry for v and v', if either one is $c_\pm$, then $v''=c_\pm$. One thus has the interesting result, that, *for each q-physics, the speed of light in a given direction will be the same in every inertial frame*.

Ω. THE FINALE

A lot had happened in the course. There was a surge of mathematical creativity, culminating in q-number systems. Then the wish to find a use for these systems was fulfilled by means of q-physics. On the last day of class some literary creativity was sparked when Eric came up with an idea for a story.

α. IMAGINATIVE CONJECTURES

All the different q-physics provided much puzzlement, especially when $b\neq 0$ and the light speed is faster in one direction than another. To try and get a better understanding, we first went back to the special cases of $q=h$, ℓ, i and restricted attention to angles which correspond to traveling forward in time at speeds less than the speed of light. A tabulation of some results is given in Figure 1.

h-physics is special relativity physics with units chosen so that light speed is unity. The distinguishing characteristic here is that m increases with $|v|$. The $e^{h\phi}$, $\phi\in\Re$, are the classic Lorentz transformations. It was rather a surprise to discover that ℓ-physics is Newtonian physics, where m is independent of $|v|$. That seemed to make sense, however, for an infinite light speed. The $e^{\ell\phi}$, $\phi\in\Re$, are the Galilean transformations. i-physics, as far as we know, was a physics no one had thought of before. It is, in a sense, the complement of h-physics in that m here decreases as $|v|$ increases, actually approaching zero as $|v|$ approaches infinity. The $e^{i\phi}$, $-\pi/2<\phi<\pi/2$, thus have no name originating from physics. They are of course the ordinary Euclidean rotations through the angle ϕ. That $c_\pm=\mp i$ was intriguing, and we puzzled over what that could mean. Might light traverse a real distance in an imaginary time, or an imaginary distance in a real time? Whatever it might mean, we ended up conjecturing that we as sentient beings could only perceive light if it traversed a real distance in real time. We then broadened our scope some and considered all the cases where b=0, so now there is a q-physics for each value of a, namely for each point on the real line. This was something to ponder over. The centerpoint,

q	$c_\pm$	v	m	p	ϕ
h	± 1	$\tanh\phi$	$m_0\cosh\phi$	$m_0\sinh\phi$	$-\infty<\phi<\infty$
ℓ	$\pm\infty$	$\tan\ell\,\phi=\phi$	$m_0\cos\ell\,\phi=m_0$	$m_0\sin\ell\,\phi=m_0\phi$	$-\infty<\phi<\infty$
i	$\mp i$	$\tan\phi$	$m_0\cos\phi$	$m_0\sin\phi$	$-\pi/2<\phi<\pi/2$

Figure 1
A tabulation of some results.

with $c_{\pm}=\pm\infty$, is Newtonian. Things get more and more relativistic as light speed decreases to the right. To the left we conjectured perceptual darkness, and at light speeds of zero, i.e $a=\pm\infty$, we further conjectured no motion, assuming nothing can travel faster than light. Eric then came up with his conjecture that the real line is a time line. This stimulated more ideas. This time line needn't be coordinated with ordinary clock times. Clock times speed up and slow down relative to each other and thus also to this time line. But such a time line means that the kind of physics changes with time. There is a beginning, call it "alpha," at $-\infty$, and there is an end, call it "omega," at ∞. In addition, there is a very special centerpoint, let's call it "lambda," corresponding with the dictum "Let there be light." This cosmological scenario accordingly suggests a little story.

λ. Let There Be a Story About a Universe

In the beginning there was nothing. The speed of light was nothing, and since no thing can move faster than light, there was no motion. Then the speed of light edged away from zero, but it was imaginary, and there were also slight motions, but they too were imaginary. Nevertheless, this imaginary commotion continued to build, gaining momentum, accelerating more and more into a phantastic frenzy, moving off toward imaginary infinity, at which point, suddenly

there was light !

It was the big bang of a white hole exploding into reality the imagining which had gone on before. Now there was real light, light traveling instantaneously from one place to another, with motions to match, and the motion was Newtonian. And the light and motion interacted, bringing forth day and night, earth and firmament, planetary merry-go-rounds and galactic vortices. Then the speed of light decreased slightly from infinity, motion took on a tinge of relativity, and complexity evolved unabatedly. At one point motion became genuine Einsteinian, but the speed of light waned inexorably, and as aeons passed, everything slowed down more and more, heading off into a black hole where finally everything came to a stop, including light, motion, and even time, so that the end was like the beginning.

ω. A Timeless Finish

In a relativistic universe clock times slow to a stop for things which reach light speed, and when the speed of light is zero, everything travels at light speed, so at the omega point clock time is frozen. The time line may keep on going, but when it reaches the omega point, the universe, as a work of art, is finished. Eric's story, like Haydn's oratorio, is itself a work of art, which, at one level, tells how another work of art, a universe, was made. At another level, however, it is an analogy which can be interpreted as depicting the creative process involved in making any work of art, with the initial chaotic period devoted to the imagination, the point of inspiration when imagined things start becoming real, and the final moment when time becomes frozen for that particular work of art and it is finished (at which point it can be put on a record, or hung on a wall, or put on a bookshelf, and serve as a resource for other imaginations to draw upon).

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