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Ordered ultraconnected rings

M.HENRIKSEN, F.A.SMITH

Dedicated to the memory of Zdeněk Frolík

Abstract. A ring R with identity element 1 is called ultraconnected if for each unital homomorphism ϕ of Z^ω into R, there is an $i < \omega$ such that $\phi(f) = f(i) \cdot 1$ for every $f \in Z^\omega$. Our main result is that if no sum of nonzero squares in R is 0 and R has only trivial idempotents, then R fails to be ultraconnected iff R contains a subring isomorphic to Z^ω/P for some free minimal prime ideal P of Z^ω.

Keywords: Unital homomorphism, ordered ring, connected ring, nonstandard model of Z

Classification: 06F25, 13A17

1. Introduction.

We assume throughout that all rings have an identity element, usually denoted by 1, and let $\mathrm{Hom}(S, R)$ be the set of all unital homomorphisms of S into a ring R. A (not necessarily commutative) ring whose only idempotents are 0 and 1 is said to be *connected.*

In [**BR**], R.Börger and M.Rajagopalan study rings R such that unital homomorphisms from direct products of rings into them are determined by one coordinate. To be more precise, they say a ring R has *property* P_α if for every collection $\{S_\xi\}_{\xi<\alpha}$ of rings and $\phi \in Hom(\Pi_{\beta<\alpha}S_\beta, R)$ there is a $\xi < \alpha$ and a homomorphism $\Psi : S_\xi \to R$ such that $\phi(f) = \Psi(f(\xi)) \cdot 1$ for every $f \in \Pi_{\beta<\alpha}S_\beta$. In this case ϕ is said to be *determined be the coordinate* ξ. Börger and Rajagopalan show there is no loss of generality in assuming that each S_β is the ring of integers, in which case we write $\Pi_{\beta<\alpha}S_\beta$ as Z^α. They also show that R has P_n for n finite if and only if R is connected and no ring has P_α for α a nonmeasurable cardinal. (For further discussion of measurable cardinals see Chapter 12 of [**GJ**]). Given these preliminaries, we rephrase a definition of [**BR**].

Definition 1. A ring R is said to be *ultraconnected* if it satisfies P_ω: that is, for each $\phi \in Hom(Z^\omega, R)$ there is an $n < \omega$ such that $\phi(f) = f(n) \cdot 1$ for every $f \in Z^\omega$.

In [**BR**] a *pot pourri* of results on ultraconnected rings are presented. It is shown, for example, that every connected ring of characteristic 0 of cardinality less than 2^ω, and the real field $\mathfrak{R}$ are ultraconnected, while the p-adic fields, rings of p-adic integers, the complex field and the integers mod p^m for any prime p and positive integer m fail to be ultraconnected. Börger and Rajagopalan pose the problem of characterizing ultraconnected rings.

The main purpose of this article is to characterize connected rings that admit a partial order in which squares are positive and which are not ultraconnected. In

Section 2, Theorem 9 we show that every such ring contains a particular kind of nonstandard model of the integers. These were studied by N.Alling in [A] and using results of A.Dow [D] it is known that all of those nonstandard models of Z are isomorphic if and only if the continuum hypothesis holds.

2. Nonultraconnected rings.

It is shown in [BR] that every ultraconnected ring is connected. Some of the following results apper in [BR] though not always in the same form.

If $A \subset \omega$, let χ_A denote the characteristic function of A and let $\chi_i = \chi_{\{i\}}$ for any $i < \omega$. For any $f \in Z^\omega$, let $Z(f) = f^{-1}(0)$ and let $coz(f) = \omega - Z(f)$. We will denote $\chi_{coz(f)}$ by $\chi(f)$. Clearly χ is multiplicative since $coz(f) \cap coz(g) = coz(fg)$ for any $f, g \in Z^\omega$.

Lemma 2. *If R is a connected ring and $\phi \in Hom(Z^\omega, R)$ then:*

a) *either $\phi(\chi_n) = 0$ for all $n < \omega$, or there is a unique $i < \omega$ such that $\phi(f) = f(i) \cdot 1$ for all $f \in Z^\omega$, and*

b) *if $f \in Z^\omega$ then $\phi(f) = 0$ if and only if $\phi(|f|) = 0$.*

PROOF : a) Since R is connected and ϕ maps idempotents to idempotents, if $\phi(\chi_i) \neq 0$ for some i then $\phi(\chi_i) = 1$. If $j \neq i$, then $0 = \chi_i\chi_j$, so $\phi(0) = \phi(\chi_i)\phi(\chi_j) = \phi(\chi_j)$, whence i is unique.

If $f \in Z^\omega$ then $f = f\chi_i + f(1 - \chi_i)$ so $\phi(f) = \phi(f)\phi(\chi_i) + \phi(f)\phi(1 - \chi_i) = \phi(f)\phi(\chi_i) = f(i) \cdot 1$

b) Let $f \in Z^\omega$ and $k \in Z^\omega$ be defined by $k(i) = 1$ if $f(i) > 0$, and $k(i) = -1$ if $f(i) \leq 0$. Now $|f| = kf$ and $f = k|f|$, so $f \in \ker\phi$ if and only if $|f| \in \ker\phi$. ∎

Let ΣZ^ω denote the direct sum of ω copies of Z and note that ΣZ^ω is an ideal of Z^ω, which yields the following

Corollary 3. *A connected ring is not ultraconnected if and only if* $\ker\phi \supset \Sigma Z^\omega$ *for some $\phi \in Hom(Z^\omega, R)$.*

If I is an ideal of a ring S, let $E(I)$ denote its set of idempotents. The next theorem relies implicitly, but not explicitly on some of the results in [M]. We being with some definitions.

Definition 4. An ideal I of a commutative ring is said to be *generated by its idempotents* if $E(I)S = I$.

If a is in the commutative ring S, then the *annihilator $A(a)$* of a is given by $A(a) = \{b \in S : ab = 0\}$. If $A(a) = \{0\}$, then a is called a *regular* element of S.

For a commutative ring S, let $Q_{cl}(S)$ denote its classical ring of fractions. That is, $Q_{cl}(S) = \{a/b : a, b \in S \text{ and } A(b) = 0\}$, with the usual addition and multiplication of fractions; see [L, Section 4.6] for background.

Let Q denote the field of rational numbers, and recall that Q^ω is a Von Neumann regular ring; that is for every $f \in Q^\omega$, there is a $g \in Q^\omega$ such that $f^2g = f$. Note also that fg is an idempotent; see [L, Section 3.5].

The next theorem is the principal result of this section.

Theorem 5. *Suppose R is a connected commutative ring and $\phi \in Hom(Z^\omega, R)$ is a surjection. Then the following are equivalent.*

(a) *Ker ϕ is generated by its idempotents*
(b) *$f \in \ker\phi$ implies $\chi(f) \in \ker\phi$*
(c) *ϕ has a surjective extension $\phi^* \in Hom(Q^\omega, Q_{Cl}(R))$*
(d) *Ker ϕ is a minimal prime ideal.*

PROOF : Assume (a) and $f \in \ker\phi$. There is an $e \in E(\ker\phi)$ and a $g \in Z^\omega$ such that $eg = f$, since, as is noted in **[P]** and **[A]**, finitely generated ideals of Z^ω are principal. So $\chi(f) = \chi(eg) = \chi(e)\chi(g)$, whence $\phi(\chi(f)) = \phi(\chi(e))\phi(\chi(g)) = 0$ since $e \in \ker\phi$. Thus (b) holds.

Next, assume (b) and $F \in Q^\omega$, where for each $n < \omega$, $F(n) = f(n)/g(n)$, if $f, g \in Z^\omega$ and $g(n) \neq 0$. We show that $\phi(g)$ is a regular element of $\phi[Z^\omega]$.

To see this, assume $\phi(g)\phi(h) = 0$ for some $h \in Z^\omega$. Then, by (b), $0 = \phi(gh) = \phi(\chi(gh)) = \phi(\chi(g))\phi(\chi(h)) = \phi(\chi(h))$ since $\mathrm{coz}(g) = \omega$. Hence $\chi(h) \in \ker\phi$ as does $h = \chi(h)h$, so $\phi(g)$ is a regular element of $\phi[Z^\omega]$.

If we let $\phi^*(F) = \phi(f)/\phi(g)$, then $\phi^*(F)$ is in $Q_{Cl}(R)$, and it is clear that $\phi^* \in Hom(Q^\omega, Q_{Cl}(R))$ and is a surjection that extends ϕ, so (c) holds.

Finally, let ϕ^* satisfy the conditions of (c). Since Q^ω is a Von Neumann regular ring, if $f \in \ker\phi^*$, there is a $g \in Z^\omega$ such that $f = (fg)f$. As noted above, fg is an idempotent in $\ker\phi^*$. Hence $E(\ker\phi^*)Q^\omega = \ker\phi^*$, so

$$(\dagger) \qquad E(\ker\phi^*)Q^\omega \cap Z^\omega = \ker\phi^* \cap Z^\omega.$$

Clearly $E(\ker\phi^*) = E(\ker\phi)$ and since idempotents assume only values 0 and 1, the left hand side of (†) is equal to $E(\ker\phi)Z^\omega$, while its right hand side is $\ker\phi$. Thus (a) holds, and the equivalence of (a), (b), and (c) has been established.

Assume (b), $fg \in \ker\phi$, and $f \notin \ker\phi$. Since $f = \chi(f)f$, $\ker\phi$ cannot have $\chi(f)$ as an element. Since R is connected, this yields $\phi(\chi(f)) = 1$. Thus, by (b), $0 = \phi(\chi(fg)) = \phi(\chi(f))\phi(\chi(g)) = \phi(\chi(g))$. Thus $\chi(g)$ belongs to $\ker\phi$ as does $g = \chi(g)g$; and we know that $\ker\phi$ is a prime ideal. As is noted in **[GJ]**, to show that $\ker\phi$ is a minimal prime ideal, we need only find, for each $g \in \ker\phi$, an element in $A(g)$ that is not in $\ker\phi$. By (b), $1 - \chi(g)$ plays that role, so $\ker\phi$ is a minimal prime ideal and (d) holds.

Finally, assume (d). Alling shows in Theorem 1.1 of **[A]** that since $\ker\phi$ is a minimal prime ideal of Z^ω, $\mathcal{U} = \{Z(f) : f \in \ker\phi\}$ is an ultrafilter on ω such that $\ker\phi = \{f \in Z^\omega : A(f) \in \mathcal{U}\}$. Since $Z(f) = Z(\chi(A))$, it follows that $\chi(f) \in \ker\phi$ whenever $f \in \ker\phi$. So (b) holds and the proof of the theorem is complete. ∎

In **[A]**, Alling shows that Z^ω/P is a totally ordered integral domain if P is a minimal prime ideal and that it can be ordered in only one way. We generalize these results in what follows.

We include the following definition from **[FGL]**.

Definition 6. A ring R is *formally real* if no sum of nonzero squares is zero. A partially ordered ring $(R, \leq)$ is *quasireal* if $a^2 \geq 0$ for all $a \in R$.

Clearly a formally real ring is reduced (i.e., the only nilpotent element is 0) and has characteristic 0. It is known [**FGL**,Theorem 8.3] that any formally real ring admits a quasireal partial ordering and the one whose positive cone is all sums of squares is the smallest such. Further, any reduced quasireal partially ordered ring is formally real.

Our next result shows that the existence of a homomorphism of Z^ω into a connected formally real ring severely restricts the quasireal orderings.

Theorem 7. *If R is a connected quasireal ring and $\phi \in Hom(Z^\omega, R)$ then*

a) $\ker\phi$ *is a minimal prime ideal, and*

b) $\phi[Z^\omega]$ *is a totally ordered integral domain.*

PROOF : a) Since every integer is a sum of four squares, every nonnegative element of Z^ω is also. Thus, since R is quasireal, ϕ is order preserving.

If $f \in \ker\phi$, *then*$|f| \in \ker\phi$ by lemma 2(b) and since $0 \leq \chi(f) \leq |f|$ and $\phi(|f|) = 0$, we have $\chi(f) \in \ker\phi$. Thus by Theorem 5, $\ker\phi$ is a minimal prime ideal.

b) If $\phi(f) \in \phi(Z^\omega)$, then $\phi(f)^2 = \phi(f^2) = \phi(|f|^2) = \phi(|f|)^2$ so $\phi(f-|f|)\phi(f+|f|) = 0$. Since by (a), $\ker\phi$ is prime, $\phi[Z^\omega] = Z^\omega/\ker\phi$ is an integral domain, and R is quasireal, either $\phi(f) = \phi(|f|) \geq 0$ or $\phi(f) = -\phi(|f|) \leq 0$, and $\phi[Z^\omega]$ is totally ordered. ∎

An integral domain that is elementarily equivalent to Z without being isomorphic to Z is called a *nonstandard model of* Z. For the definition of elementary equivalence and more discussion of nonstandard models of Z, see [**A**], [**CK**] or [**LS**]. As noted in [**A**], if P is a minimal prime ideal of Z^ω containing ΣZ^ω and P' is a prime ideal of Z^ω containing P, then Z^ω/P' and $(Z^\omega/P)/(P'/P)$ are isomorphic, so Z^ω/P' is a homomorphic image of Z^ω/P. For this reason we call Z^ω/P an *ω-maximal nonstandard model of* Z.

The proof of the first part of the next lemma is an exercise, and the second part is shown in [**BR**,1.8].

Lemma 8. *Suppose R is a ring.*

(a) *If R fails to be ultraconnected, then any ring containing R as a subring fails to be ultraconnected.*

(b) *If S ultraconnected, R is connected, and $Hom(R,S)$ is nonempty, then R is ultraconnected.*

The following characterization theorem is an immediate consequence of Theorem 5,7 and Lemma 8.

Theorem 9. *A connected formally real ring fails to be ultraconnected if and only if it contains an ω-maximal nonstandard model of Z as a subring.*

In section 4 of [**A**] a number of properties of nonstandard models of Z are given including finitely generated ideals are principal, and ± 1 are the only invertible elements. In section 5 of [**A**] other properties applicable to ω-maximal nonstandard models of Z are given. If $D(Z)$ is such a model then it has cardinality 2^ω. Recall an ordered set L is a (near) η_1-set if given two countable (nonempty) sets A and B such that $a < b$ for all $a \in A$, $b \in B$ there is $t \in L$ such that $a < t < b$ for all $a \in A$, $b \in B$. Alling shows [**A**,Theorem 5.10] $D(Z)$ is a near η_1 set with no countable cofinal subset whose quotient field $Q(Z)$ is an η_1 set.

Despite this, an integral domain may have a countable cofinal subset and still fail to be ultraconnected.

Example 10. Let $D(Z)$ be as above, and let $R = D(Z)[x]$ denote the ring of polynomials with coefficients in $D(Z)$ lexicographically ordered with leading coefficient dominating. That is, if $p(x) = \Sigma_{r=0}^{n} a_r x^r$ is in R and $a_n \neq 0$, let $p(x) > 0$ if $a_n > 0$ and let $p(x) < 0$ if $a_n < 0$. Then $\{x^n : n < \omega\}$ is a countable cofinal subset of R, while R fails to be ultraconnected by Theorem 9 since it contains $D(Z)$.

The following result follows immediately from Theorem 5,7, and 9.

Theorem 11. *If R is a connected quasireal ring, then $\phi[Z^\omega]$ is ultraconnected if and only if it has a countable cofinal subset.*

It follows directly from the model-theoretic Corollary 6.12 in [**CK**] that if the continuum hypothesis (CH) holds, then all ω-maximal nonstandard models of Z are isomorphic. The remainder of this section is devoted to showing that (CH) is vital to reaching this conclusion.

By Theorem 5, if $D(Z)$ denotes an ω-maximal nonstandard model of Z and $\phi \in Hom(Z^\omega, D(Z))$ is surjective, then ϕ has an extension $\phi^* \in Hom(Q^\omega, Q(Z))$ where $Q(Z)$ denotes the quotient field of $D(Z)$. Next, we show how to extend ϕ^* to a homomorphism of $\mathfrak{R}^\omega$ onto an appropriately chosen integral domain containing $Q(Z)$. Recall from the above that if $D(Z)$ is an ω-maximal nonstandard model of Z, then there is a free ultrafilter $\mathcal{U}$ on ω such that $D(Z)$ and $Z^\omega/P(\mathcal{U})$ are isomorphic, where $P(\mathcal{U}) = \{f \in Z^\omega : Z(f) \in \mathcal{U}\}$. By Theorem 5, ϕ has an extension $\phi^* \in Hom(Q^\omega, Q(Z))$ and it is clear that $Q(Z)$ and $Q^\omega/P_Q(\mathcal{U})$ are isomorphic, where $P_Q(\mathcal{U}) = \{f \in Q^\omega : Z(f) \in \mathcal{U}\}$. ϕ^*, in turn, has an extension $\Psi \in Hom(\mathfrak{R}^\omega, \mathfrak{R}^\omega/P_\mathfrak{R}(\mathcal{U}))$, where $P_u = \{f \in \mathfrak{R}^\omega : Z(f) \in \mathcal{U}\}$. It is shown in Chapter 13 of [**GJ**] that $\mathfrak{R}^\omega/P_\mathfrak{R}(\mathcal{U})$ is a real closed field of power 2^ω that is an η_1-set, and that any two real closed fields that are η_1-sets of power $\aleph_1$ are isomorphic. So if (CH) holds, then any two such hyperreal fields are isomorphic, and clearly an isomorphism of $Z^\omega/P(\mathcal{U})$ onto $Z^\omega/P(\mathcal{V})$, where $\mathcal{V}$ is a free ultrafilter on ω lifts to an isomorphism of $\mathfrak{R}^\omega/P_\mathfrak{R}(\mathcal{U})$ onto $\mathfrak{R}^\omega/P_\mathfrak{R}(\mathcal{V})$, and an isomorphism between these latter two fields restricts to an isomorphism of $Z^\omega/P(\mathcal{U})$ onto $Z^\omega/P(\mathcal{V})$. In [**D**], A. Dow shows that if (CH) is false, then there are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω such that $\mathfrak{R}^\omega/P_\mathfrak{R}(\mathcal{U})$ and $\mathfrak{R}^\omega/P_\mathfrak{R}(\mathcal{V})$ fail to be isomorphic, in which case $Z^\omega/P(\mathcal{U})$ and $Z^\omega/P(\mathcal{V})$ fail to be isomorphic. Indeed, the latter may be inferred from [**D**] directly. (According to Dow, they are not even similar as ordered sets). Hence we have:

Theorem 12. *Every pair of ω-maximal nonstandard models of Z is isomorphic if and only if (CH) holds.*

In the next and final section of this note, we consider ultraconnected rings that are not formally real.

3. Some remarks about general ultraconnected rings.

We have only a little to add to the results in [**BR**] on connected rings that fail to be formally real. We begin with the following version of Theorem 5 under weaker hypotheses.

Theorem 13. *If R is connected and $\phi \in Hom(Z^\omega, R)$ then there is a unique minimal prime ideal contained in* $\ker\phi$.

PROOF : If $e^2 = e \in Z^\omega$, then $\phi(e) = 0$ if an only if $\phi(1-e) = 1$, so it is clear that $u = \{Z(e) : \phi(e) = 0\}$ is an ultrafilter on ω, and by Theorem 5, $P(\mathcal{U}) = \{f \in Z^\omega : Z(f) \in u\}$ is a minimal prime ideal of Z^ω contained in $\ker\phi$. Since distinct minimal prime ideals of Z^ω are contained in distinct maximal ideals of Z^ω [**A**,Proposition 8.1] then uniqueness follows. ∎

We do not known if the converse of Theorem 13 holds. In particular we do not know whether Z^ω/I is connected for every I containing a prime ideal.

The next result can be deduced easily from results in [**BR**] but is not stated explicitly therein.

Theorem 14.

a) *If $m > 1$ is an integer, then there is a unital surjective homomorphism ϕ of Z^ω to Z/mZ such that $\Sigma Z^\omega \subset Ker\phi$.*
b) *No ring of finite characteristic is ultraconnected.*

PROOF : Let $\mathcal{U}$ be a free ultrafilter on ω. For $0 \le i \le m-1$, let $\mathcal{V}_i(f) = \{n < \omega : f(n) \equiv i \mod m\}$. Clearly, $\{\mathcal{V}_0(f), \dots, \mathcal{V}_{m-1}(f)\}$ is a partition of ω. Since $\mathcal{U}$ is an ultrafilter, $\mathcal{V}_i(f) \in \mathcal{U}$ for exactly one i. Define $\phi : Z^\omega \to Z_m$ by letting $\phi(f) \equiv i \mod m$ if $\mathcal{V}_i(f) \in \mathcal{U}$. Clearly, ϕ is a unital homomorphism of Z^ω onto Z/mZ whose kernel contains ΣZ^ω since $\mathcal{U}$ is free. Thus (a) holds.

(b) It follows immediately from (a) that for any prime $p, Z/pZ$ fails to be ultraconnected. Since every ring of finite characteristic contains an isomorphic copy of Z/pZ for some prime p, the conclusion follows by Lemma 8(a). ∎

An immediate consequence of Theorem 14(a) is that the kernel of a homomorphism of Z^ω onto a connected ring need not be a prime ideal (e.g., there is such a homomorphism of Z^ω onto $Z/4Z$).

In [**BR**], the authors show, using an inverse limit argument, that for any prime p, the ring Z_p of p-adic integres fails to be ultraconnected and use this to show that the complex field C is not ultraconnected. For, Z_p is contained in its quotient field Q_p, the field of p-adic numbers, and C contains every field of characteristic 0 and cardinality $\le 2^\omega$. Actually, to see that C is not ultraconnected, it suffices to produce an integral domain D of cardinality 2^ω and characteristic 0, and $\phi \in Hom(Z^\omega, D)$ such that $\Sigma Z^\omega \subset \ker\phi$. Such a D may be obtained by noting that ΣZ^ω is an ideal of Z^ω containing no constant function; and hence is contained in a prime ideal P containing no constant function. (See [**GJ**, Chap. 0]). Thus $Z^\omega/P = D$ is the required integral domain.

Clearly, an integral domain fails to be ultraconnected if and only if it is a homomorphic image of an ω-maximal nonstandard model of Z. There is a rich variety of such rings as may be seen be examining [**A**] or [**P**].

It would be interesting to obtain an internal characterization of ultraconnected rings, even in the formally real case.

In [LLS], R.Levy, P.Loustanau, and J.Shapiro made a thorough study of the prime ideals of Z^{α}, where α is an infinite cardinal. It should be of value in future studies of ultraconnected rings.

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