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The Eigenvalue Set of a Class of Equimodular Matrices

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Two complex $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equimodular if $|a_{ij}| = |b_{ij}|$ for $i, j = 1, 2, \ldots, n$. Let $\Omega(A)$ denote the set of all matrices equimodular with a given matrix A, and let $\mathscr{S}(A)$ denote the set of all eigenvalues of all matrices in $\Omega(A)$. The class $\Omega(A)$ is called regular if it contains only nonsingular matrices.

The purpose of this paper is to examine the general properties of $\mathscr{S}(A)$. Some of these properties are apparent. In particular, $\mathscr{S}(A)$ is clearly symmetric about the origin of the complex plane, for if α is an eigenvalue of $B \in \Omega(A)$, then $\alpha \exp(i\theta)$ is an eigenvalue of $\exp(i\theta)B$. Moreover, if all multiplicities are taken into account, each matrix in $\Omega(A)$ has exactly n eigenvalues, and since the eigenvalues of a matrix are known to vary continuously with its entries (e.g., see [7, pp. 192–194]) and since $\Omega(A)$ is a compact, connected set in complex E^{n^2} space, it follows that $\mathscr{S}(A)$ is the union of at most n closed, connected components. Therefore, $\mathscr{S}(A)$ consists of $r \leqslant n$ closed, annular components centered at the origin. It is clearly possible for some of these rings to be circles, and if A is not regular, the innermost ring is actually a disk. It will be convenient to refer to the components of the complement of $\mathscr{S}(A)$ as gaps in $\mathscr{S}(A)$.

Although the basic structure of $\mathscr{S}(A)$ is rather simple, this set also has many interesting boundary and combinatorial properties, some of which are really quite subtle. For example, the well-known Perron-Frobenius theorem states that the largest nonnegative real boundary point of $\mathscr{S}(A)$ is an eigenvalue of the nonnegative member of $\Omega(A)$. We generalize this result by showing that each real boundary point of $\mathscr{S}(A)$ is actually an eigenvalue of at least one real matrix in $\Omega(A)$. Furthermore, in response to a request by O. Taussky [10], R. S. Varga and

B. W. Levinger [4] have given a complete characterization of $\mathscr{S}(A)$ in terms of their concept of minimal Gerschgorin sets [5, 11]. In this characterization, $\mathscr{S}(A)$ is significantly associated with a set of at most $n + 1$ permutations, and the authors conjecture that this number can be reduced to n. Although our approach is entirely different from that of Varga and Levinger, we also show that $\mathscr{S}(A)$ is related to a set of permutations and then verify that this set contains at most n distinct members.

It will be seen that regular classes play a very important role in our analysis of $\mathscr{S}(A)$, and we ultimately require a complete characterization of such classes. P. Camion and A. Hoffman [2] have shown that $\Omega(A)$ is regular if and only if there exist a permutation matrix P and a positive diagonal matrix D such that PAD is diagonally dominant, but this characterization is not ideally suited to our approach. Therefore, in addition to the results already mentioned, we also derive an alternate characterization of regular classes.

The paper is divided into four sections, the first of which is basically introductory and contains definitions, notational conventions, and well-known results. The second section contains a fundamental theorem on multilinear polynomials which forms the crux of all our main results, and the final two sections are devoted to discussions of the combinatorial and boundary properties of $\mathscr{S}(A)$, respectively.

1. DEFINITIONS, NOTATIONAL CONVENTIONS, AND BASIC RESULTS

We shall denote $n \times n$ complex matrices by upper case English letters, and the identity and null matrices will always be denoted by I and 0, respectively. The matrix whose entries are $|a_{ij}|$ will be denoted by $|A|$, and if $A = |A|$, we shall write $A \geqslant 0$. Similarly, if $x = (x_1, \ldots, x_n)$ is a complex vector, the vector whose ith component is $|x_i|$ will be denoted by $|x|$, and if $x = |x|$, we shall write $x \geqslant 0$. As usual, the Kronecker δ function will be denoted by δ_{ij}.

A matrix $P = (p_{ij})$ is called a permutation matrix if its entries are all either 1 or 0 and it has exactly one nonzero entry in each row and column. The matrix A is said to be reducible if there exists a permutation matrix P such that

$$PAP^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},$$

where A_1 and A_3 are square submatrices of A. Otherwise, A is irreducible.

Ostrowski [8, 9] originally defined the concept of M-matrix in terms of the principal minors of a certain type of matrix, but the following equivalent definition (e.g., see [12, p. 85]) is more suitable to our approach.

DEFINITION 1.1. A real $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \leqslant 0$ for all $i \neq j$ is an M-matrix if A is nonsingular and $A^{-1} \geqslant 0$.

It will be convenient to call A a generalized M-matrix if there exists a permutation matrix P such that PA is an M-matrix.

Finally, in terms of our definitions, the Perron-Frobenius theorem takes the following form.

THEOREM 1.1 (*Perron-Frobenius*). *If $A \geqslant 0$ is irreducible and if $\sigma(A)$ denotes the largest nonnegative boundary point of $\mathscr{S}(A)$, then*

(i) *$\sigma(A)$ is an eigenvalue of A of multiplicity one and corresponds to a positive eigenvector;*

(ii) *$\sigma(A)$ is an eigenvalue of $B \in \Omega(A)$ if and only if B has the form $B = DAD^{-1}$, where D is a complex diagonal matrix whose diagonal entries have unit modulus;*

(iii) *if $C - A \geqslant 0$, then $\sigma(C) \geqslant \sigma(A)$ with equality only if $C = A$.*

2. A FUNDAMENTAL THEOREM ON MULTILINEAR POLYNOMIALS

This section is devoted to the proof of a general theorem concerning the coefficients of a multilinear polynomial which does not vanish on the region $|x_i| = 1$. Accordingly, let K_m denote the set of all polynomials in m variables which are of maximal degree one in each variable, and note that each member of K_m can be written in the form $\sum a(S) \prod_{i \in S} x_i$, where the sum is extended over the 2^m subsets of $\{1, 2, \ldots, m\}$.

THEOREM 2.1. *Let p be a member of K_m. If there exists a $q \in K_m$ such that $|p(x)| > |q(x)|$ for all complex vectors x for which $|x_j| = 1, j = 1, 2, \ldots, m$, then*

(i) *one coefficient of p strictly dominates all the other coefficients of p and all those of q*—i.e., *there exists a subset S_0 of $\{1, 2, \ldots, m\}$ such that*

$$|a(S)| < |a(S_0)| \qquad \text{for every } S \neq S_0$$

and

$$|b(S)| < |a(S_0)| \qquad \text{for every } S;$$

(ii) *p does not vanish on the region described by the following conditions:*

$$|x_j| \geqslant 1 \quad \textit{for } j \in S_0; \qquad |x_j| \leqslant 1 \quad \textit{for } j \notin S_0.$$

Proof. To simplify notation, let X_m denote the set of all vectors with m unimodular complex components. The proof of part (i) is by induction on the number m of the variables.

For $m = 0$, the theorem is trivial. In the general case, there exist unique $p_1, p_2 \in K_{m-1}$ such that $p(x) = p_1(u) + x_m p_2(u)$, where $u \in X_{m-1}$. Since $|p(x)| > |q(x)| \geqslant 0$ for all $x \in X_m$, and since X_{m-1} is a connected set, it follows that exactly one of the following situations occurs:

$$|p_1(u)| > |p_2(u)| \qquad \text{for all} \quad u \in X_{m-1}; \tag{2.1}$$

$$|p_2(u)| > |p_1(u)| \qquad \text{for all} \quad u \in X_{m-1}. \tag{2.2}$$

In either case, it follows from the inductive hypothesis that a certain coefficient of p strictly dominates all the other coefficients of p.

Assume for the moment that situation (2.1) occurs, and write $q(x) = q_1(u) + x_m q_2(u)$. For each fixed u, we have the following inequalities:

$$|q_1(u)| + |q_2(u)| < |p_1(u)| + |p_2(u)|$$

and

$$|p_1(u)| - |p_2(u)| > \big||q_1(u)| - |q_2(u)|\big|.$$

Adding these two inequalities, we obtain

$$|p_1(u)| > \max(|q_1(u)|, |q_2(u)|),$$

and since this inequality holds for each $u \in X_{m-1}$, it follows from the inductive hypothesis that the dominant coefficient of p also dominates all the coefficients of both q_1 and q_2 and hence of q. Furthermore, it follows from inequality (2.1) that this dominant term must be found in p_1 rather than p_2. An entirely analogous argument shows that if situation (2.2) occurs, then p has a dominant coefficient which is found in p_2 and which dominates all the coefficients of q.

Now, for $j = 0, 1, \ldots, m$, let T_j denote the set of all vectors z such that $|z_i| = 1$ if $i > j$; and for $i \leqslant j$, $|z_i| \geqslant 1$ if $i \in S_0$ and $|z_i| \leqslant 1$ if $i \notin S_0$. We shall prove part (ii) of the theorem by induction on j.

Since $|p(x)| > |q(x)| \geqslant 0$ for all $x \in X_m$, the theorem certainly holds for T_0. To prove the general case, choose $p_3, p_4 \in K_{m-1}$ so that if $x \in X_m$, then $p(x) = p_3(t) + x_j p_4(t)$ for $t \in X_{m-1}$. Assume for the moment that $j \in S_0$. Then the dominant coefficient of p is found in p_4, and as before, it follows that $|p_4(t)| > |p_3(t)|$ for all $t \in X_{m-1}$. If w is a member of T_{j-1}, we can write $w = (y, w_j)$ where $y = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_m)$. By the inductive hypothesis, $p(w) \neq 0$ for all $w \in T_{j-1}$, and since T_{j-1} is a connected set, it follows that $|p_4(y)| > |p_3(y)|$ for all such y. If z is a member of T_j, we have $p(z) = p_3(y) + z_j p_4(y)$, where y is a vector of the type just described. Therefore, $|z_j| \geqslant 1$ since $j \in S_0$, and it follows that $p(z)$ cannot be zero. The proof for $j \notin S_0$ is entirely analogous to the one just given, and we conclude that $p(z) \neq 0$ for all $z \in T_j$, as desired.

3. COMBINATORIAL PROPERTIES OF THE SET $\mathscr{S}(A)$

We now apply Theorem 2.1 to a set of r matrices $\{A_1, \ldots, A_r\}$, where $A_k = (a_{ij}^{(k)})$. For each fixed k, let $x_{ij}^{(k)}$ be n^2 independent complex variables, and denote by $B_k(x)$ the (variable) matrix whose entries are $a_{ij}^{(k)} x_{ij}^{(k)}$. It is the fact that $F(x) = \det \sum_{k=1}^{r} B_k(x)$ is a member of K_{rn^2} which enables us to use Theorem 2.1 as a tool in investigating the properties of regular classes and the set $\mathscr{S}(A)$.

Several new terms must be defined before Theorem 2.1 can be effectively applied to sets of matrices. First of all, let $\Omega(A_1, \ldots, A_r)$ denote the set of all matrices of the form $\sum_{k=1}^{r} C_k$, where $C_k \in \Omega(A_k)$—i.e., $\Omega(A_1, \ldots, A_r)$ is the direct sum of the equimodular classes $\Omega(A_1), \Omega(A_2), \ldots, \Omega(A_r)$. If $\Omega(A_1, \ldots, A_r)$ contains only nonsingular matrices, it will be called a regular class. Second, the ordered r-tuple $(P_1, \ldots, P_r)$ of matrices will be called a partition of the permutation matrix P if $\sum_{k=1}^{r} P_k = P$ and the entries $p_{ij}^{(k)}$ are all either 0 or 1.

The notion of a partition of a permutation matrix provides a convenient means for describing the coefficients of the function $F(x) = \det \sum_{k=1}^{r} B_k(x)$, where the $B_k(x)$ are the variable matrices defined above. In fact, we have

$$F(x) = \sum \pm \prod (a_{ij}^{(k)} x_{ij}^{(k)}),$$

where the products are extended over all triples (i, j, k) for which $p_{ij}^{(k)} = 1$ and the sum is extended over all partitions of all permutation matrices. The coefficients $\pm \prod a_{ij}^{(k)}$ of $F(x)$ are appropriately called the partitioned generalized diagonal products of the matrices A_k. In terms of this new notation, Theorem 2.1 takes the following form.

THEOREM 3.1. *If $\Omega(A_1, \ldots, A_r)$ is a regular class, there exists a unique permutation matrix P and a unique partition $(P_1, \ldots, P_r)$ of P such that*

(i) *the partitioned generalized diagonal product associated with the partition $(P_1, \ldots, P_r)$ strictly dominates all others;*

(ii) *any matrix $C = (c_{ij})$ whose elements satisfy the following conditions is nonsingular:*

$$|c_{ij}| \geqslant |a_{ij}^{(k)}| - \sum_{\sigma \neq k} |a_{ij}^{(\sigma)}| \qquad \text{for} \quad p_{ij}^{(k)} = 1;$$

$$|c_{ij}| \leqslant \sum_{k=1}^{r} |a_{ij}^{(k)}| \qquad \text{if} \quad p_{ij} = 0.$$

Since $\lambda \notin \mathscr{S}(A)$ if and only if $\Omega(A, \lambda I)$ is regular, we shall be particularly interested in the special case where $\{A_1, \ldots, A_r\}$ is $\{A, \lambda I\}$. In this case, Theorem 3.1 yields the following theorem.

THEOREM 3.2. *Let λ_1 and λ_2 be the nonnegative boundary points of a gap in $\mathscr{S}(A)$. If λ is any number in the open interval (λ_1, λ_2), there exists a unique partition (Q, R) of a unique permutation matrix P such that*

$$\text{(i)} \qquad \left(\prod_{q_{ij}=1} |a_{ij}|\right)\left(\prod_{r_{ij}=1} \lambda\delta_{ij}\right) > \left(\prod_{s_{ij}=1} |a_{ij}|\right)\left(\prod_{t_{ij}=1} \lambda\delta_{ij}\right) \tag{3.1}$$

where (S, T) is any other partition of a permutation matrix.

(ii) *The partition (Q, R) does not depend on the choice of λ from the interval (λ_1, λ_2).*

(iii) *The matrix $C = (c_{ij})$ is nonsingular if*

$$\begin{aligned} |c_{ij}| &\geqslant \big||a_{ij}| - \lambda\delta_{ij}\big|, \qquad p_{ij} = 1, \\ |c_{ij}| &\leqslant |a_{ij}| + \lambda\delta_{ij}, \qquad p_{ij} = 0. \end{aligned} \tag{3.2}$$

Proof. Since $\Omega(A, \lambda I)$ is regular, parts (i) and (iii) are immediate consequences of Theorem 3.1. Therefore, an inequality such as (3.1) must hold for each λ in (λ_1, λ_2), and since the terms in (3.1) are continuous functions of λ, it follows that as λ varies over the connected set (λ_1, λ_2), the dominant partitioned generalized diagonal product is always associated with the partition (Q, R).

We shall refer to the matrix P and the partition (Q, R) which appear in Theorem 3.2 as the permutation matrix and the partition associated with the gap (λ_1, λ_2), respectively. Since there exist at most $n + 1$ gaps in $\mathscr{S}(A)$, it is clear that the set of all permutation matrices associated with gaps in $\mathscr{S}(A)$ contains at most $n+1$ members. Using inequality (3.1), we now prove that this set contains at most n different members, a result which establishes the conjecture of Varga and Levinger mentioned in the introduction.

THEOREM 3.3. *The set of all permutation matrices associated with gaps in $\mathscr{S}(A)$ contains at most n different members.*

Proof. Assume there are $n+1$ gaps in $\mathscr{S}(A)$. First of all, note that if a gap is associated with the partition (Q, R), then the dominant variable partitioned generalized diagonal product which appears on the left side of inequality (3.1) may be written in the form $K\lambda^r$, where r is the number of nonzero diagonal entries of R. Call this number the characteristic of the gap.

If $\sigma \geqslant 0$ and $\rho \geqslant 0$ are contained in gaps associated with different permutation matrices, it follows from inequality (3.1) that there exist constants K_1, K_2, r_1, and r_2 such that

$$K_1\sigma^{r_1} > K_2\sigma^{r_2} \qquad \text{and} \qquad K_2\rho^{r_1} > K_1\rho^{r_2}$$

and it is clear from these inequalities that $r_1 \neq r_2$. Therefore, two gaps which are associated with different permutation matrices cannot have the same characteristic, and it follows that at least two of the gaps in $\mathscr{S}(A)$ must be associated with the same permutation matrix unless each of the integers $0, 1, \ldots, n$ is the characteristic of exactly one gap in $\mathscr{S}(A)$. However, since I is the only permutation matrix which can be associated with a gap whose characteristic is $n-1$ or n, we conclude that in any case at least two of the gaps in $\mathscr{S}(A)$ must be associated with the same permutation matrix, and this completes the proof of the theorem.

The set of permutation matrices associated with gaps in $\mathscr{S}(A)$ has several interesting properties in addition to the one described in Theorem 3.3. First of all, the unbounded gap in $\mathscr{S}(A)$ is always associated with the partition $(0, I)$ of the identity matrix since for large λ, the variable partitioned generalized product λ^n dominates all others. Moreover, for essentially the same reason, every gap in $\mathscr{S}(A)$ which is further from the origin than a gap associated with I must also be associated with I. While it is clearly possible for several gaps to be associated with the same permutation matrix, it can be shown [1, pp. 47–71] that no two gaps can be associated with the same partition, but the proof of this result is too long and complicated to be included in the present paper.

4. BOUNDARY PROPERTIES OF $\mathscr{S}(A)$

In this section, we generalize part of the Perron-Frobenius theorem by showing that each real boundary point of $\mathscr{S}(A)$ is actually an eigenvalue of at least one real matrix in $\Omega(A)$. It should not be too surprising that our approach is similar to that used in the study of M-matrices, for we have already shown that each gap is associated with a collection of regular classes, and it is clear that the concept of matrix regularity is closely related to the more familiar strong nonsingularity concepts of diagonal dominance and M-matrix. In fact, a byproduct of our results is a complete characterization of regular classes in terms of generalized M-matrices. We begin with the following useful definition.

DEFINITION 4.1. A real matrix is said to be of type $\mathscr{T}$ if in each row it has at most one positive term. If $A = (a_{ij})$ is of type $\mathscr{T}$, we denote by $\mathscr{D}(A)$ the set of all complex matrices $B = (b_{ij})$ for which

$$|b_{ij}| \geqslant a_{ij} \qquad \text{if} \quad a_{ij} > 0,$$

$$|b_{ij}| \leqslant -a_{ij} \qquad \text{if} \quad a_{ij} \leqslant 0.$$

Our first goal is to show that if A is of type $\mathscr{T}$, then every matrix in $\mathscr{D}(A)$ is nonsingular if and only if A^{-1} exists and $A^{-1} \geqslant 0$. In particular, this criterion can be used to determine whether or not a matrix of type $\mathscr{T}$ is regular. The following lemma is the key to the proof of this result.

LEMMA 4.1. *If A is of type $\mathscr{T}$, and if $B \in \mathscr{D}(A)$, then $A|x| \leqslant |Bx|$ for every x.*

Proof. For fixed i, denote by J the set $\{j | a_{ij} > 0\}$ and by J' the complementary set. If $x = (x_1, \ldots, x_n)$, we have

$$\sum_j a_{ij}|x_j| = \sum_J + \sum_{J'} \leqslant \sum_J |b_{ij}x_j| - \sum_{J'} |b_{ij}|\,|x_j|$$

and since J has at most one term,

$$\sum_j a_{ij}|x_j| \leqslant \left| \sum_J b_{ij}x_j \right| - \sum_{J'} |b_{ij}|\,|x_j| \leqslant \left| \sum_j b_{ij}x_j \right|.$$

Therefore, $A|x| \leqslant |Bx|$, as desired.

THEOREM 4.1. *If A is of type $\mathscr{T}$, then every member of $\mathscr{D}(A)$ is nonsingular if and only if A is nonsignular and $A^{-1} \geqslant 0$.*

Proof. Assume that $A^{-1} \geqslant 0$, and let B be an arbitrary member of $\mathscr{D}(A)$. If $By = 0$, it follows from Lemma 4.1 that $A|y| \leqslant 0$. Let $A|y| = z$. Then $|y| = A^{-1}z \leqslant 0$ since $A^{-1} \geqslant 0$ and $z \leqslant 0$, and we conclude that $y = 0$. Therefore, B is nonsingular.

Conversely, assume that every matrix in $\mathscr{D}(A)$ is nonsingular, and let x be a column vector of A^{-1}. Let $y = |x| - x$. Since $Ax = |Ax| \geqslant A|x|$, we have $Ay \leqslant 0$, and it follows that $By = 0$, where $B = (b_{ij})$ is that member of $\mathscr{D}(A)$ whose elements satisfy the following conditions:

$$b_{ij} = \begin{cases} 0 & \text{if for } m \neq j, \quad a_{im} > 0 \quad \text{and} \quad y_m = 0, \\ a_{ij} - y_j^{-1} \sum_k a_{ik} y_k & \text{if} \quad a_{ij} > 0 \quad \text{and} \quad y_j \neq 0, \\ a_{ij} & \text{otherwise.} \end{cases}$$

Since we have assumed that every member of $\mathscr{D}(A)$ is nonsingular, it follows that $y = 0$. Therefore, $x = |x| \geqslant 0$, and $A^{-1} \geqslant 0$, as desired.

Theorems 4.1 and 3.2 can now be combined to yield necessary and sufficient conditions for a number λ to be in the complement of $\mathscr{S}(A)$.

THEOREM 4.2. *For $A \geqslant 0$ and $\lambda \geqslant 0$, $\lambda \notin \mathscr{S}(A)$ if and only if there exists a permutation matrix P such that the following matrix $B = (b_{ij})$ is nonsingular and $B^{-1} \geqslant 0$:*

$$b_{ij} = \begin{cases} |a_{ij} - \lambda\delta_{ij}| & \text{if} \quad p_{ij} = 1, \\ - a_{ij} - \lambda\delta_{ij} & \text{if} \quad p_{ij} = 0. \end{cases}$$

Proof. If $\lambda \notin \mathscr{S}(A)$, the existence of P and the fact that $B^{-1} \geqslant 0$ follow from part (iii) of Theorem 3.2 and from Theorem 4.1, respectively.

Conversely, let λ be a number for which there exist matrices P and B which satisfy the conditions of the theorem, and let C be an arbitrary member of $\Omega(A)$. Then, for all i, j we have

$$|a_{ij} - \lambda\delta_{ij}| \leqslant |c_{ij} - \lambda\delta_{ij}| \leqslant a_{ij} + \lambda\delta_{ij},$$

which implies that $C - \lambda I \in \mathscr{D}(B)$. By hypothesis, $C - \lambda I$ is nonsingular, and it follows that $\lambda \notin \mathscr{S}(A)$.

In particular, $\Omega(A)$ is regular if and only if $0 \notin \mathscr{S}(A)$, and the following characterization of regular classes is an immediate consequence of Theorem 4.2.

COROLLARY 4.1. *A class $\Omega(A)$ is regular if and only if it contains a generalized M-matrix.*

Using Theorem 4.2, we can now prove the main result of this section.

THEOREM 4.3. *Each real boundary point of $\mathscr{S}(A)$ is an eigenvalue of at least one real matrix in $\Omega(A)$.*

Proof. Assume $A \geqslant 0$, and let λ_1 and λ_2 be the nonnegative boundary points of a gap in $\mathscr{S}(A)$ which is associated with the permutation matrix P. Let $B_z = (b_{ij}(z))$ be the following variable matrix:

$$b_{ij}(z) = \begin{cases} |a_{ij} - z\delta_{ij}| & \text{if} \quad p_{ij} = 1, \\ -a_{ij} - z\delta_{ij} & \text{if} \quad p_{ij} = 0. \end{cases}$$

We prove the theorem by showing that B_{λ_1} is singular. The explicit form of a real matrix in $\Omega(A)$ which has λ_1 as an eigenvalue will be given in Corollary 4.2.

It follows from Theorem 4.2 that $B_\lambda^{-1} \geqslant 0$ for all λ in the open interval (λ_1, λ_2), and since the entries of B_λ^{-1} are continuous functions of λ, we conclude that $B_{\lambda_1}^{-1} \geqslant 0$ if B_{λ_1} is nonsingular. However, since $\lambda_1 \in \mathscr{S}(A)$, it follows from Theorem 4.2 that $B_{\lambda_1}^{-1}$ cannot be nonnegative if B_{λ_1} is nonsingular, and we conclude that B_{λ_1} is singular, as desired.

COROLLARY 4.2. *If $\lambda \geqslant 0$ is a boundary point of a gap in $\mathscr{S}(A)$ which is associated with the partition (Q, R) of the permutation matrix P, then λ is an eigenvalue of $C = (c_{ij})$ where*

$$c_{kk} = \begin{cases} |a_{kk}| & \text{if} \quad p_{kk} = 1, \\ -|a_{kk}| & \text{if} \quad p_{kk} = 0, \end{cases}$$

and for $i \neq j$

$$c_{ij} = \begin{cases} |a_{ij}| & \text{if} \quad p_{ij} = 1 \quad \text{or if} \quad r_{ii} = 1, \\ -|a_{ij}| & \text{if} \quad p_{ij} = 0 \quad \text{and} \quad r_{ii} = 0. \end{cases}$$

COROLLARY 4.3 (*Perron-Frobenius*). *If $A \geqslant 0$, the largest nonnegative boundary point of $\mathscr{S}(A)$ is an eigenvalue of A.*

Proof. Since the unbounded gap in $\mathscr{S}(A)$ is associated with the partition $(0, I)$ of I, the matrix C of Corollary 4.2 is nonnegative.

Theorems 3.3, 4.2, and 4.3 give a fairly clear idea of the main features of $\mathscr{S}(A)$. First of all, $\mathscr{S}(A)$ consists of a set of $r \leqslant n$ annular components separated by r or $r + 1$ gaps, depending on whether or not $\Omega(A)$ is regular. Each of these gaps is significantly associated with a unique partition of a unique permutation matrix, and the set of all permutation matrices associated with gaps in $\mathscr{S}(A)$ has at most n different members, at least one of which is the identity matrix. Finally, each real boundary point of $\mathscr{S}(A)$ is actually an eigenvalue of a real matrix in $\Omega(A)$ which may be determined by the conditions given in Corollary 4.2 once the gap partitions have been found.

In conclusion, the author would like to thank Professors John Todd and H. F. Bohnenblust of the California Institute of Technology for their many helpful suggestions during the preparation of this paper. Professor Bohnenblust has observed that Theorem 3.1 implies that the set of all regular matrix pairs, regarded as a set in complex E^{2n^2} space, consists of $2^n n!$ open connected components, each of which contains exactly one partition of a permutation matrix. He has also pointed out to the author that the results of this paper can probably be fruitfully extended and generalized in several directions. In particular, it seems clear that Theorem 3.1 must play a key role in any study of the properties of regular equimodular classes.

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