Does Society Need IMO Medalists?

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Synopsis

With a title that sounds provocative but with no intention to embarrass the organizers and participants of the event of IMO (International Mathematical Olympiad), this article should be seen as the sharing of some thoughts on this activity, or more generally on mathematical competitions, by a teacher of mathematics who had once helped in the coaching of the first Hong Kong Team to take part in the 29th IMO held in Canberra in 1988 and in the coordination work of the 35th IMO held in Hong Kong in 1994. The author tries to look at the issue in its educational context and more broadly in its sociocultural context.

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1. Introduction

Does society need IMO Medalists? No, society does not "need" IMO Medalists. Society does not even "need" mathematicians. Does this mean this article will end here? Stopping here and now would amount to an admission of wrong choice of my profession in all these years, so I should go on talking. You will notice that I put the word "need" in quotation marks. Society does not "need" (in quotation marks!) IMO Medalists or mathematicians, but society needs (no quotation mark!) mass transit railway maintenance workers, garbage collectors, street cleaners, plumbers, electricians, etc. Now, perhaps you know what I mean. Let us get back to the IMO.

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[1]This is an expanded version of an invited talk given in the IMO Forum held at the Hong Kong Polytechnic University on July 11, 2016.

After twenty-two years IMO came back to Hong Kong as the host. Hong Kong hosted the 35th IMO in the summer of 1994. I like to mention two Medalists in that particular IMO. One is Maryam Mirzakhani of Iran, who became the first female mathematician to receive a Fields Medal at the International Congress of Mathematicians in 2014. The other one is Subash Ajit Khot of India, who was awarded the Nevanlinna Prize at the same Congress.

2. The "good" and the "bad" of mathematics competitions

In a talk I gave in 2012 with the title "The good, the bad and the pleasure (not pressure) of mathematics competitions", I outlined certain good and bad points of mathematics competitions. Allow me to repeat them here in summary [7, 8].

The "good" points of mathematics competitions are the nurturing of (1) clear and logical presentation, (2) tenacity and assiduity, and (3) "academic sincerity". Moreover, mathematics competitions arouse a passion for mathematics and pique the interest in it.

The "bad" points of mathematics competitions are related to two issues: (1) competition problems versus research, and (2) over-training. We further ask: Is the participant's passion for the subject of mathematics itself genuine? Can the interest of the participant be sustained?

One predominant objection to mathematics competitions is the requirement to work out the problems within a fixed time span, which undermines the intellectual and intrinsic pleasure of doing mathematics. Petar Kenderov points out how this requirement disadvantages those creative youngsters who are "slow workers" [6]. He further points out some important features which are not encouraged in a traditional mathematics competition but which are essential for doing good work in mathematics, including "the ability to formulate questions and pose problems, to generate, evaluate, and reject conjectures, to come up with new and non-standard ideas" [6]. Moreover, he points out that all such activities "require ample thinking time, access to information sources in libraries or the Internet, communication with peers and experts working on similar problems, none of which are allowed in traditional competitions" [6].

In the talk I gave in 2012, I maintained that "pursuing mathematical research is not just to obtain a prescribed answer but to explore a situation in order to understand it as much as one can" and that "it is far more important to be able to raise a good question than to be able to solve a problem set by somebody else who knows the answer already" [8]. In doing research our purpose is to understand the problem in as much depth as possible and by whatever means we can, so we may even change the problem by imposing more conditions or by relaxing the demand in order to make progress. This is, unfortunately, not what a contestant is allowed to do in a mathematics competition!

3. Three examples

Let me first offer three examples, from which I shall gather some lesson to be learnt.

Example 1

The first example is a rather well-known problem that appeared in one IMO. Since I helped with the coaching of the first Hong Kong Team that was sent to take part in the 29th IMO held in Canberra in 1988, naturally I paid some special attention to the questions set in that year. Question 6 of the 29th IMO reads:

"Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that
$$\frac{a^2 + b^2}{ab + 1}$$
is the square of an integer."

A slick solution to this problem, offered by a Bulgarian youngster (Emanouil Atanassov) who received a special prize for it, starts by supposing that

$$k = \frac{a^2 + b^2}{ab + 1}$$

is *not* a perfect square and rewriting the expression in the form

$$a^2 - kab + b^2 = k. \tag{*}$$

Note that for any integral solution (a, b) of (*) we have $ab > 0$ since k is not a perfect square. Let (a, b) be an integral solution of (*) with $a, b > 0$ and $a + b$ *smallest*. We shall produce from it another integral solution (a', b) of (*) with $a', b > 0$ and $a' + b < a + b$, which is a contradiction. To obtain such a solution (a', b) we regard (*) as a quadratic equation with a positive root a and another root a'; this way we see that $a + a' = kb$ and $aa' = b^2 - k$. Hence a' is also an integer and (a', b) is an integral solution for (*). Since $a'b > 0$ and $b > 0$, we have $a' > 0$, and

$$a' = \frac{b^2 - k}{a} \leq \frac{b^2 - 1}{a} \leq \frac{a^2 - 1}{a} < a.$$

Slick as the proof is, it also invites a couple of queries. (1) What makes one suspect that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer? (2) The argument should hinge crucially upon the condition that k is not a perfect square. In the proof this condition seems to have slipped in casually so that one does not see what really goes wrong if k is not a perfect square. More pertinently, this proof by contradiction has *not explained* why

$$\frac{a^2 + b^2}{ab + 1}$$

must be a perfect square, even though it *confirms* that it is so.

In contrast let us look at a much less elegant solution, which is my own attempt. When I first heard of the problem, I was on a trip in Europe and had a "false insight" by putting $a = N^3$ and $b = N$ so that

$$a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1).$$

Under the impression that any integral solution (a, b, k) of

$$k = \frac{a^2 + b^2}{ab + 1}$$

is of the form (N^3, N, N^2), I formulated a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ another equality

$$\left(a - (3b^2 - 3b + 1)\right)^2 + (b-1)^2 = (k - (2b - 1))\left(\left(a - (3b^2 - 3b + 1)\right)(b - 1) + 1\right).$$

Were I able to achieve that, I could have reduced b in steps of one until I got down to the equation

$$k = \frac{a^2 + 1}{a + 1}$$

for which $a = k = 1$. By reversing steps I would have solved the problem. I tried to carry out this strategy while I was travelling on a train, but to no avail. Upon returning home I could resort to systematic brute-force checking and look for some actual solutions, resulting in a (partial) list shown below.

a :	**1**	**8**	**27**	30	**64**	112	**125**	**216**	240	**343**	418	**512**	...
b :	**1**	**2**	**3**	8	**4**	30	**5**	**6**	27	**7**	112	**8**	...
k :	**1**	**4**	**9**	4	**16**	4	**25**	**36**	9	**49**	4	**64**	...

Then I saw that my ill-fated strategy was doomed to failure, because there are solutions other than those of the form (N^3, N, N^2). However, not all was lost. When I stared at the pattern, I noticed that for a fixed k, the solutions could be obtained recursively as (a_i, b_i, k_i) with

$$
\begin{aligned}
a_{i+1} &= a_i k_i - b_i; \\
b_{i+1} &= a_i; \\
k_{i+1} &= k_i = k.
\end{aligned}
$$

It remained to carry out the verification. Once that was done, all became clear. There is a set of "basic solutions" of the form (N^3, N, N^2), with $N = 1, 2, , 3, \cdots$. All other solutions are obtained from a basic solution recursively as described above. In particular,

$$k = \frac{a^2 + b^2}{ab + 1}$$

is the square of an integer. I feel that through this process, I understand the phenomenon much more than if I had just read the slick proof.

Example 2

Problem 5 in the 21$^{\text{st}}$ Hong Kong Primary School Mathematics Competition held in May of 2010 says:

> "Using only a ruler draw a triangle ABC on the A3-size paper so that AB is of length 20 cm., angle BAC is of measure 45°, and angle ABC is of measure 60°. Find the shortest distance from C to AB correct to one decimal place (see Figure 1)."
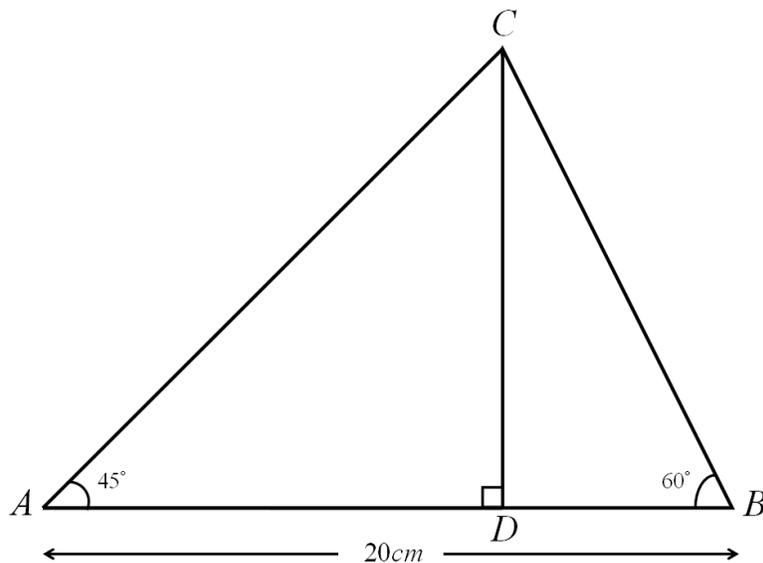

Figure 1: Triangle ABC given in Problem 5.

There are various ways to do this problem, which was probably originally set to see if a primary school pupil knows how to make use of paper-folding to arrive at the required triangle, then makes use of paper-folding again to get the perpendicular to the base and measures it by the ruler.

Some "early starters (who jump the gun)" tried to solve it by using trigonometry, which is not normally taught in primary school. They even knew about the Law of Sines! But they were stumped when they came to an angle of measure other than 30, 45 or 60 degrees!

Here is a clever solution which does not rely on secondary school mathematics. We extend the figure to a larger right triangle (see Figure 2) and borrow from the wisdom of ancient Chinese mathematicians.
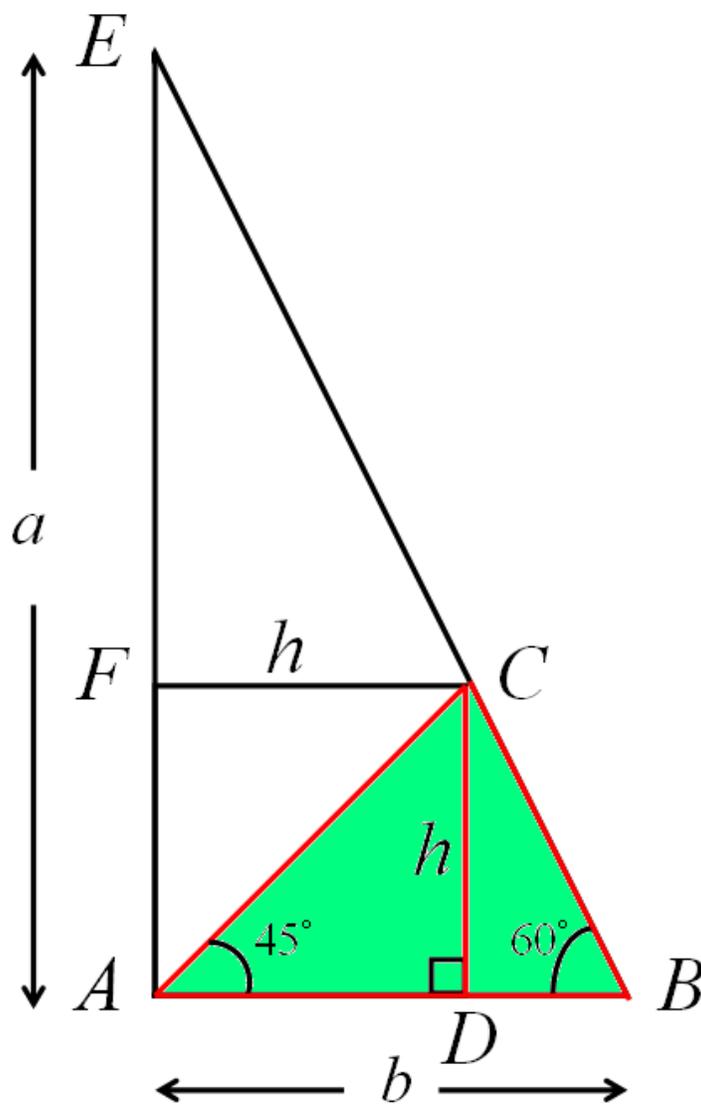

Figure 2: Extended triangle in Problem 5.

Problem 15 in Chapter 9 of the ancient Chinese mathematical classics Jiuzhang Suanshu (九章算術 *The Nine Chapters of the Mathematical Art*), which was compiled between the first century B.C.E. and the first century C.E., asks for the side of an inscribed square in a given right triangle, which is equal to $ab/(a+b)$. In the figure below (see Figure 3) there is a "proof without words" of this neat relationship, using the method of "dissect-and-re-assemble" credited to the 3rd-century Chinese mathematician LIU Hui (劉徽). Simply equate the area of the two separate rectangles on the left and on the right [2].
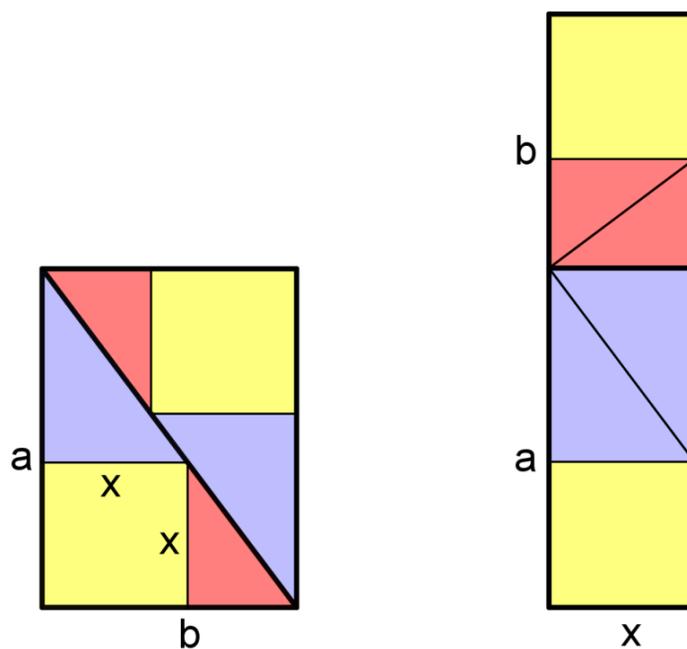

Figure 3: Proof by the method of "dissect-and-re-assemble".

Applying this neat relationship to the large right angle in Figure 2, we can calculate the height of the original triangle. However, this clever solution would not work if the measures of the two base angles are arbitrary, while the not-so-clever "dry" method which relies on the Law of Sines still works well.

Example 3

There is a well-known anecdote about the famous mathematician John von Neumann (1903-1957). A friend of von Neumann once gave him a problem to solve. Two cyclists A and B at a distance 20 miles apart were approaching each other, each going at a speed of 10 miles per hour. A bee flew back and forth between A and B at a speed of 15 miles per hour, starting with A and back to A after meeting B, then back to B after meeting A, and so on. By the time the two cyclists met, how far had the bee travelled? In a flash von Neumann gave the answer—15 miles. His friend responded by saying that von Neumann must have already known the trick so that he gave the answer so fast. His friend had in mind the slick solution to this quickie, namely, that the cyclists met after one hour so that within that one hour the bee had travelled 15 miles. To his friend's astonishment von Neumann said that he knew no trick but simply summed an infinite series!

For me this anecdote is very instructive. (1) Different people may have different ways to go about solving a mathematical problem. There is no point in forcing everybody to solve it in just the same way you solved it. Likewise, there is no point in forcing everybody to learn mathematics in just the same way you learned it. (2) Both methods of solution have their separate merits. The first method of calculating when the cyclists met is slick and captures the key point of the problem. The other method of summing an infinite series, which is slower (but not for von Neumann!) and is seemingly more cumbersome and not as clever, goes about solving the problem in a systematic manner. It indicates patience, determination, down-to-earth approach and meticulous care. Besides, it can help to consolidate some basic skills and nurture in a student a good working habit.

4. Two approaches in the teaching and learning of mathematics

What do we see from these three examples given in Section 3? These examples make me think that there are two approaches in doing mathematics. To give a military analogue one is like positional warfare and the other guerrilla warfare [8]. The first approach, which has been going on in the classrooms of most schools and universities, is to present the subject in a systematically organized and carefully designed format supplemented with exercises and problems. The other approach, which goes on more predominantly in

the training for mathematics competitions, is to confront students with various kinds of problems and train them to look for points of attack, thereby accumulating a host of tricks and strategies. Each approach has its merits and the two approaches indeed supplement and complement each other. Each approach calls for day-to-day preparation and solid basic knowledge. Just as in positional warfare flexibility and spontaneity are called for, while in guerrilla warfare careful prior preparation and groundwork are needed, in the teaching and learning of mathematics we should not just teach tricks and strategies to solve special type of problems or just spend time on explaining the general theory and working on problems that are amenable to routine means. We should let the two approaches supplement and complement each other in our classrooms. In the biography of the famous Chinese general and national hero of the Southern Song Dynasty, Yue Fei (岳飛 1103-1142) we find the description: "Setting up the battle formation is the routine of the art of war. Manoeuvring the battle formation skillfully rest solely with the mind. [陣而後戰，兵法之常。運用之妙，存乎一心。]"

Sometimes the first approach may look quite plain and dull, compared with the excitement acquired from solving competition problems by the second approach. However, we should not overlook the significance of this seemingly bland approach, which can cover more general situations and turns out to be much more powerful than an ad hoc method which, slick as it is, solves only a special case. Of course, it is true that frequently a clever ad hoc method can develop into a powerful general method or can become a part of a larger picture. A classic case in point is the development of calculus in history. In ancient times, only masters in mathematics could calculate the area and volume of certain geometric objects, to name just a couple of them, Archimedes (c. 287 B.C.E. - c. 212 B.C.E.) and LIU Hui (劉徽 3rd century). In hindsight their formula for the area of a circle, $A = (1/2) \times C \times r$, embodies the essence of the Fundamental Theorem of Calculus. With the development of calculus since the seventeenth and the eighteenth centuries, today even an average school pupil who has learnt calculus will be able to handle what only great mathematicians of the past could have resolved.

Since many mathematics competitions aim at testing the contestants' ability in problem solving rather than their acquaintance with specific subject content knowledge, the problems are set in some general areas which can be made comprehensible to youngsters of that age group, independent of different school syllabi in different countries and regions. That would cover topics

in elementary number theory, algebra, combinatorics, sequences, inequalities, functional equations, plane and solid geometry and the like. Gradually the term "Olympiad mathematics" is coined to refer to this conglomeration of topics. One question that I usually ponder over is this: why can't this type of so-called "Olympiad mathematics" be made good use of in the school classroom as well? If one aim of mathematics education is to let students know what the subject is about and to arouse their interest in it, then interesting non-routine problems should be able to play their part well when used to supplement the day-to-day teaching and learning. Making use of "Olympiad mathematics" in the classroom does not mean a direct transplant of competition problems to the classroom, but rather making use of the kind of topics, the spirit and the way the question is designed and formulated, even if the confine is to be within the official syllabus. Good questions would benefit the learners. Designing good questions is a challenging task for the teachers. In this respect mathematics competitions can benefit teachers as well in upgrading themselves when they try to make use of the competition problems to enrich the learning experience of their students.

5. Friends of mathematics

Let us get back to the question: Does society need IMO Medalists? No, society does not "need" IMO Medalists. Society does not even "need" mathematicians. But society needs (again, no quotation mark!) "friends of mathematics". A "friend of mathematics" may not know a lot of mathematics but would understand well what mathematics is about and appreciate well the role of mathematics in the modern world.

The mathematician Paul Halmos once said, "It saddens me that educated people don't even know that my subject exists" [3]. Allen Hammond, editor of *Science*, once described mathematics as "the invisible culture" [4]. On the other hand, perhaps it is a blessing to remain not that visible! Not long ago I read in the news (Associated Press, May 7, 2016): "Ivy League Professor Doing Math Equation on Flight Mistaken for Terrorist". An American Airline passenger seated next to Guido Menzio of University of Pennsylvania suspected the unfamiliar writings of the professor were a code for a bomb. It led to Professor Menzio being taken away from the plane to be interrogated! We really need "friends of mathematics".

In ancient China the third-century mathematician LIU Hui (劉徽) said, "The subject [mathematics] is not particularly difficult by using methods transmitted from generation to generation, like the compasses [gui] and gnomon [ju] in measurement, which are comprehensible to most people. However, nowadays enthusiasts for mathematics are few, and many scholars, much erudite as they are, are not necessarily cognizant of the subject." [[[九數⋯]至於以法相傳，亦猶規矩度量可得而共，非特難為也。當今好之者寡，故世雖多通才達學，而未必能綜於此耳。] [2] Why is it like that?

Here is a passage taken from a book: "Central to my argument is the idea that ***** distinguished by a self-conscious attention to its own ***** language. Its claim to function as art derives from its peculiar concern with its own materials and their formal patterning, aside from any considerations about its audience or its social use" [5] Can you guess what the missing words are?

This passage is taken from the book *Who Needs Classical Music? Cultural Choice and Musical Value* by Julian Johnson [5]. The missing words are "classical music" and "musical". However, the passage would ring equally true if "classical music" is replaced by "mathematics"! In the same book the author says, "... that it [meaning classical music] relates to the immediacy of everyday life but not immediately. That is to say, it takes aspects of our immediate experience and reworks them, reflecting them back in altered form. In this way, it creates for itself a distance from the everyday while preserving a relation to it" [5]. Mathematics is also like that. This explains why it is not easy to bring mathematics to the general public. To become a "friend of mathematics" one needs to be brought up from school days onward in an environment where mathematics is not only enjoyable but also makes good sense. In the preface to an undergraduate textbook the authors John Baylis and Rod Haggarty remark, "The professional mathematician will be familiar with the idea that entertainment and serious intent are not incompatible: the problem for us is to ensure that our readers will enjoy the entertainment but not miss the mathematical point" [1].

6. Epilogue

My good friend, Tony Gardiner, an experienced four-time UK IMO team leader, once commented that I should not blame the negative aspects of mathematics competitions on the competition itself. He went on to enlighten me on one point, namely, a mathematics competition should be seen as just

the tip of a very large, more interesting, iceberg, for it should provide an incentive for each country to establish a pyramid of activities for masses of interested students [7]. It would be to the benefit of all to think about what other activities besides mathematics competitions can be organized to go along with it. These may include the setting up of a mathematics club or publishing a magazine to let interested youngsters share their enthusiasm and their ideas, organizing a problem session, holding contests in doing projects at various levels and to various depth, writing book reports and essays, producing cartoons, videos, softwares, toys, games, puzzles, ...

Finally the question boils down to one in an even more general context: Does society need ME? We frequently hear about the cliché "No one is indispensable!" But please bear in mind that everyone has his or her worth and can do his or her part to make this world a better place to live in. An IMO Medalist is no exception!

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