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On the Structure of a Class of Archimedean Lattice-Ordered Algebras

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By a *Φ-algebra* A, we mean an archimedean lattice-ordered algebra over the real field R which has an identity element 1 that is a weak order unit. The Φ-algebras constitute the class of the title. It is shown that every Φ-algebra is isomorphic to an algebra of continuous functions on a compact space $\mathcal{X}$ into the two-point compactification of the real line R, each of which is real-valued on an (open) everywhere dense subset of $\mathcal{X}$. Under more restrictive assumptions on A, representations of this sort have long been known. An (incomplete) history of them is given briefly in Section 2.

The compact space in question is the space $\mathcal{M}(A)$ of maximal l-ideals of A with the Stone (= hull-kernel) topology. The subset A^* of bounded elements of A is also a Φ-algebra, and $\mathcal{M}(A^*)$ is homeomorphic to $\mathcal{M}(A)$.

The class of Φ-algebras includes, of course, all lattice-ordered algebras of real-valued functions that contain the constant functions. In addition, it contains the algebra $\mathfrak{B}_0$ of Baire functions modulo null functions, and the algebra $\mathfrak{L}_0$ of Lebesgue measurable functions modulo null functions, on the real line R. It is well known that neither of these is isomorphic (even as a vector-lattice) to any algebra of real-valued functions.

If $M \in \mathcal{M}(A)$, then A/M is a totally ordered integral domain containing R. If $A/M = R$, then M is called a *real* maximal ideal; otherwise it is called *hyper-real*. $\mathcal{R}(A)$ denotes the space of real maximal l-ideals of A. If A is an algebra of real-valued functions, then $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$, but $\mathcal{R}(\mathfrak{B}_0)$, and $\mathcal{R}(\mathfrak{L}_0)$ are empty. If $a \in A$, then $\mathcal{R}(a)$ denotes the set of maximal l-ideals of A such that $M(|a|)$ is not infinitely large. For each $a \in A$, $\mathcal{R}(a)$ is dense in $\mathcal{M}(A)$.

We have summarized the main results of Section 2. In Section 3, we investigate Φ-algebras that are *uniformly closed*, i.e. every Cauchy sequence of elements of A converges in A. It is an easy consequence of

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the Stone-Weierstrass theorem that if A is uniformly closed, then A^* and the algebra $C(\mathcal{M}(A))$ of all continuous real-valued functions on $\mathcal{M}(A)$ are isomorphic. Moreover, if A is uniformly closed, and $a \in A$, then every bounded $f \in C(\mathcal{R}(a))$ has a continuous extension over $\mathcal{M}(A)$. Not every Φ-algebra is a sub-Φ-algebra of a uniformly closed Φ-algebra with the same space of maximal l-ideals.

For any compact space $\mathcal{X}$, let $D(\mathcal{X})$ denote the set of all continuous functions into the two point compactification of R each of which is real on a dense subspace. While $D(\mathcal{X})$ need not always form an algebra, we show that $A = D(\mathcal{M}(A))$ if and only if A is uniformly closed and every element of A is either a divisor of zero or has an inverse.

Consider the Φ-algebra A as a subset of $D(\mathcal{M}(A))$. If $a^{-1}(0) \subset \mathcal{M}(A) \sim \mathcal{R}(b)$ for some $b \in A$ implies that a is contained in no proper l-ideal of A, then A is said to be *closed under l-inversion.* A Φ-algebra A of real-valued functions is said to be *closed under inversion* if every element of A that is contained in no real maximal l-ideal of A is contained in no proper l-ideal of A. The consequences of these postulates, and the relations between them are investigated in Section 4.

In Section 5, we obtain internal characterizations of the algebra $C(\mathcal{Y})$ for several classes of topological spaces. A necessary, but not sufficient condition that a Φ-algebra A be isomorphic to some $C(\mathcal{Y})$ is that A be a uniformly closed algebra of real-valued functions that is closed under inversion. By adding to these conditions we obtain characterizations of $C(\mathcal{Y})$ in case $\mathcal{Y}$ is either Lindelöf, locally compact and σ-compact, extremally disconnected, or discrete.

Φ-algebras are also f-rings in the sense of Birkhoff and Pierce, and we rely on known results on the structure of f-rings given by these authors in [4], and given by D. Johnson in [23]. We also rely heavily on known theorems on the algebraic structure of the ring $C(\mathcal{Y})$. In Section 1, we summarize enough necessary background material to keep this paper reasonably self contained. For more background on $C(\mathcal{Y})$, the reader is referred to [16].

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1. Definitions and preliminary remarks. By a *lattice-ordered ring* $A(+, \cdot, \vee, \wedge)$, we mean a lattice-ordered group that is a ring in which the product of positive elements is positive. If, in addition, A is a (real) vector lattice, then A is called a *lattice-ordered algebra.*

Birkhoff and Pierce have called a lattice-ordered ring A an *f-ring* if, for $a, b, c \in A$, $a \wedge b = 0$ and $c \geqslant 0$ imply $ac \wedge b = ca \wedge b = 0$ ([4]). If A is also a vector lattice, then it is called an *f-algebra.* A lattice-ordered ring A is called *archimedean* if, for each $a \in A$ which is different from 0, the set $\{na: n = \pm 1, \pm 2, \ldots\}$ has no npper bound in A. Birkhoff and Pierce have shown that every archimedean f-ring is commutative. (Indeed, they have shown that associativity is a consequence of the remaining postulates for an archimedean f-ring ([4], Theorem 13, ff.).)

1.1. Let A be a ring of real-valued functions on a set $\mathcal{S}$, under the usual pointwise addition and multiplication. Suppose that for every $f, g \in A$ the function $f \vee g$ defined by $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in \mathcal{S}$, and the function $f \wedge g$ defined by $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in \mathcal{S}$, are in A. Then A is an archimedean f-ring. In particular, the algebra $C(\mathcal{Y})$ of all continuous real-valued functions on a topological space $\mathcal{Y}$, and the subalgebra $C^*(\mathcal{Y})$ of bounded elements of $C(\mathcal{Y})$, are archimedean f-algebras with the same identity element (the constant function $\mathbf{1}$).

1.2. Let $\mathfrak{B}$ denote the set of all Baire functions on the real line R, and let $\mathfrak{L}$ denote the set of all measurable functions on R. Under the usual pointwise operations, these are archimedean f-algebras with identity. Let $\mathfrak{B}_0$ and $\mathfrak{L}_0$ denote, respectively, the f-algebras obtained from $\mathfrak{B}$, respectively $\mathfrak{L}$, by identifying functions that coincide almost everywhere. Then $\mathfrak{B}_0$ and $\mathfrak{L}_0$ are archimedean f-algebras with identity, but neither is isomorphic (even as a vector lattice) to an algebra of real-valued functions. (See [17], and [19].)

1.3. If A is a lattice-ordered ring, then, as usual, we let $A^+ = \{a \in A: a \geqslant 0\}$. For $a \in A$, let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. Then $a^+ \wedge a^- = 0$, and

(i) $a = a^+ - a^-$, and

(ii) $|a| = a^+ + a^-$.

If, in addition, A is an f-ring, then

(iii) $a^2 \geqslant 0$ for each $a \in A$, and

(iv) $|ab| = |a||b|$ for all $a, b \in A$.

For proof, see [4]. (But, note that these authors define $a^- = -(-a) \vee 0$.)

1.4. The kernel of a homomorphism of a lattice-ordered ring A into a lattice-ordered ring B is called an *l-ideal*. (*We assume, of course, that both the ring and the lattice operations are preserved by a homomorphism.*) An l-ideal of A is a ring ideal I which satisfies: $a \in I$, $b \in A$, and $|b| \leqslant |a|$ imply $b \in I$. If A has an identity element, then every proper l-ideal of A is contained in a maximal l-ideal of A.

If A is an f-ring, and M is a maximal l-ideal of A, then A/M is totally ordered. Indeed, A is an f-ring if and only if A is a subdirect union of totally ordered rings ([4], p. 56).

Every maximal ideal, and every prime ideal of a $C(\mathcal{Y})$ is an l-ideal ([16], Chapter 5).

1.5. DEFINITION. *A Φ-algebra is an archimedean f-algebra with identity element* 1.

As remarked above, every Φ-algebra is commutative. The purpose of this paper is to describe the structure of Φ-algebras.

In [23], D. Johnson gave a structure theory for f-rings analogous to the Jacobson theory for abstract rings. We now quote, in the special context of Φ-algebras, some of these results.

An f-ring A is said to be *l-simple* if $A^2 \neq \{0\}$, and if it contains no non-zero proper l-ideals. (Note that every l-simple f-ring is totally ordered.)

1.6. If A is a Φ-algebra, then

(i) the intersection of all maximal l-ideals of A is $\{0\}$,

(ii) every maximal l-ideal M of A is a prime ideal; indeed, A/M is a (totally ordered) l-simple f-algebra without non-zero divisors of zero,

(iii) every prime l-ideal of A is contained in a unique maximal l-ideal of A, and

(iv) if I is an l-ideal of A disjoint from a multiplicative system T of A, then I is contained in a prime l-ideal of A disjoint from T. See [3], Chapter I and II.

1.7. A maximal l-ideal of a Φ-algebra A need not be maximal as a ring ideal of A.

For, let R^+ denote the space of nonnegative real numbers, and let A denote the Φ-algebra of all continuous functions on R^+ that are eventually polynomials. That is, $f \in A$ if and only if $f \in C(R^+)$, and there is a $y \in R^+$, and a polynomial p such that $f(x) = p(x)$ for all $x \geqslant y$. It is easily verified that $M = \{f \in A\colon f \text{ is eventually } 0\}$ is a maximal l-ideal of A. Clearly M is not a maximal ring ideal of A.

1.8. A lattice-ordered algebra A is called *complete* (respectively, *σ-complete*) if every (respectively, every countable) bounded subset of A has a least upper bound. Every σ-complete lattice-ordered algebra with identity is archimedean ([4], p. 65).

1.9. We now review some known facts about the Φ-algebra $C(\mathcal{Y})$ of all continuous real-valued functions on a topological space $\mathcal{Y}$.

(i) Every $C(\mathcal{Y})$ is isomorphic to $C(\mathcal{Y}')$ for some completely regular (Hausdorff) space $\mathcal{Y}'$, so, in studying the structure of $C(\mathcal{Y})$, there is no loss of generality in assuming that $\mathcal{Y}$ is completely regular.

A subspace $\mathcal{S}$ of a space $\mathcal{Y}$ is said to be *C^*-imbedded* in $\mathcal{Y}$ if every $f \in C^*(\mathcal{S})$ has an extension $\bar{f} \in C^*(\mathcal{Y})$.

(ii) Every completely regular space $\mathcal{Y}$ is (homeomorphic to) a dense subspace of a compact (Hausdorff) space $\beta\mathcal{Y}$ such that $\mathcal{Y}$ is C^*-imbedded in $\beta\mathcal{Y}$. If $\mathcal{X}$ is a compact space containing $\mathcal{Y}$ as a dense subspace, and

$\mathcal{Y}$ is C^*-imbedded in $\mathcal{X}$, then there is a homeomorphism of $\beta\mathcal{Y}$ onto $\mathcal{X}$ keeping $\mathcal{Y}$ elementwise fixed. $\beta\mathcal{Y}$ is called the *Stone-Čech compactification* of $\mathcal{Y}$.

(iii) Let $\mathcal{Y}$ be a dense subspace of a compact space $\mathcal{X}$. Then, in order that there exist a homeomorphism of $\beta\mathcal{Y}$ onto $\mathcal{X}$ keeping $\mathcal{Y}$ pointwise fixed, it is necessary and sufficient that whenever $f_1, f_2 \in C^*(\mathcal{Y})$ with $f_1^{-1}(0) \cap f_2^{-1}(0) = \emptyset$, then $f_1^{-1}(0)$ and $f_2^{-1}(0)$ have disjoint closures in $\mathcal{X}$.

(iv) If f is a continuous mapping of a completely regular space $\mathcal{Y}$ into a compact space $\mathcal{X}$, then there is a continuous extension $\hat{f}$ of f over $\beta\mathcal{Y}$ into $\mathcal{X}$.

For proofs, see [16], Chapter 6.

1.10. If A is a Φ-algebra, then $A^* = \{a \in A\colon |a| \leqslant \lambda\cdot 1$ for some $\lambda \in R\}$ is also a Φ-algebra. A^* is called the subset of *bounded* elements of A.

1.11. In a vector-lattice A, an element $a \in A^+$ is called a *weak order unit* of A if $b \in A$ and $a \wedge b = 0$ imply $b = 0$, and it is called a *strong order unit* if $b \in A^+$ implies $b \leqslant na$ for some integer n. Clearly the identity element 1 of a Φ-algebra A is a weak order unit, and it is a strong order unit if and only if $A = A^*$.

Indeed, an archimedean lattice-ordered algebra A with identity element 1 is a Φ-algebra if and only if 1 is a weak order unit of A ([4], p. 61).

1.12. A ring A is called *regular*, if for every $a \in A$, there is an $x \in A$ such that $axa = a$. It is easily seen that the examples $\mathfrak{B}$, $\mathfrak{L}$, $\mathfrak{B}_0$, and $\mathfrak{L}_0$ of 1.2 are regular.

2. The representation theorem. If $\mathcal{X}$ is a compact space, let $D(\mathcal{X})$ denote the set of all continuous mappings of $\mathcal{X}$ into the two-point compactification $\gamma R = R \cup \{\pm\infty\}$ of the real field R that are real-valued on an (open) everywhere dense set. The elements of $D(\mathcal{X})$ are called *extended* (*real-valued*) *functions*.

For each $f \in D(\mathcal{X})$, let $\mathcal{R}(f)$ denote the set of points at which f is real-valued, and let $\mathcal{N}(f) = \mathcal{X} \sim \mathcal{R}(f)$.

Let $f, g \in D(\mathcal{X})$ and $\lambda \in R$. Then the functions λf, $f \vee g$, and $f \wedge g$ defined in the usual manner (i.e., pointwise) are in $D(\mathcal{X})$. If there are functions $h, k \in D(\mathcal{X})$ which satisfy

$$h(x) = f(x) + g(x), \qquad k(x) = f(x)\cdot g(x)$$

for each $x \in \mathcal{R}(f) \cap \mathcal{R}(g)$, then h and k are called the *sum* and *product* of f and g, and are denoted $f+g$ and $f\cdot g$. Since $\mathcal{R}(f) \cap \mathcal{R}(g)$ is dense in $\mathcal{X}$, these operations are uniquely defined. However, as the following example shows, $D(\mathcal{X})$ is not, in general, closed under addition and multiplication.

2.1. EXAMPLE. Let $\mathcal{X} = N \cup \{\omega\}$ denote the one point compactification of the discrete space N of positive integers. Let $f_1(x) = x + \sin x$, $f_2(x) = (1/x)\sin x$, $g(x) = -x$ if $x \in N$, and let $f_1(\omega) = \infty$, $f_2(\omega) = 0$, while $g(\omega) = -\infty$. Then f_1, f_2, and $g \in D(\mathcal{X})$, but neither $f_1 + g$ nor $f_2 g$ is defined.

A subset A of $D(\mathcal{X})$ closed nnder all of these operations will be called an *algebra of extended functions* on $\mathcal{X}$. Note that any such A will be archimedean.

2.2. PROPOSITION. *$D(\mathcal{X})$ is an algebra of extended functions if and only if each open, everywhere dense F_σ-set in $\mathcal{X}$ is C^*-imbedded in $\mathcal{X}$.*

Proof. Suppose that each open, everywhere dense F_σ-subset of $\mathcal{X}$ is C^*-imbedded. Then, for $f, g \in D(\mathcal{X})$, $\mathcal{R}(f) \cap \mathcal{R}(g)$ is C^*-imbedded in $\mathcal{X}$. So, by 1.9 (ii), $\mathcal{X} = \beta(\mathcal{R}(f) \cap \mathcal{R}(g))$, whence by 1.9 (iv), $f+g$ and $fg \in D(\mathcal{X})$. It follows that $D(\mathcal{X})$ is an algebra of extended functions.

Conversely, suppose that $\mathcal{S}$ is an open, everywhere dense F_σ-set in $\mathcal{X}$ on which is defined a bounded continuous real-valued function f without a continuous extension over $\mathcal{X}$. Now $\mathcal{X} \sim \mathcal{S}$ is a closed G_δ-set in the compact space $\mathcal{X}$, so there is a $g \in C(X)$ such that $g \geqslant 0$ and $g^{-1}(0) = \mathcal{X} \sim \mathcal{S}$. Since $g^{-1}(0)$ is nowhere dense, $1/g \in D(\mathcal{X})$. The function h defined by

$$h(x) = \begin{cases} \dfrac{1}{g(x)} + f(x), & \text{if } x \in \mathcal{S}, \\ \infty & \text{if } x \notin \mathcal{S} \end{cases}$$

is in $D(\mathcal{X})$. But $h - 1/g \notin D(\mathcal{X})$, since $h(x) - \dfrac{1}{g(x)} = f(x)$ if $x \in \mathcal{S}$.

The condition of 2.2 indicates two large classes of examples of compact spaces $\mathcal{X}$ such that $D(\mathcal{X})$ is an algebra. First, if every closed G_δ in $\mathcal{X}$ has a non-empty interior (e.g., if $\mathcal{X}$ is the one point compactification of an uncountable discrete space), then $D(\mathcal{X}) = C(\mathcal{X})$.

A completely regular space $\mathcal{Y}$ is called an *F-space* if for every $f \in C(\mathcal{Y})$, there is a $k \in C(\mathcal{Y})$ such that $f = k|f|$. If $\mathcal{Y}$ is any locally compact, σ-compact space, then $\beta\mathcal{Y} \sim \mathcal{Y}$ is an F-space. $\mathcal{Y}$ is an F-space if and only if $\mathcal{Y} \sim f^{-1}(0)$ is C^*-imbedded in $\mathcal{Y}$ for every $f \in C(\mathcal{Y})$. (For proofs, see [14], Section 2, or [16], Chapter 14.) Thus the compact F-spaces provide a second class of spaces for which the condition of 2.2 holds.

A completely regular space $\mathcal{Y}$ is called *extremally disconnected* (respectively, *basically disconnected*) if the closure of every open set (respectively, every open set of the form $\mathcal{Y} \sim f^{-1}(0)$ for some $f \in C(\mathcal{Y})$) is open. Every basically disconnected space is an F-space. That $D(\mathcal{X})$ is an algebra in case $\mathcal{X}$ is basically disconnected has long been known (cf., e.g., [26]). If $\mathcal{Y}$ is completely regular, then $C(\mathcal{Y})$ is σ-complete (respectively, complete) if and only if $\mathcal{Y}$ is basically (respectively, extremally) disconnected. This statement remains true if "$C(\mathcal{Y})$" is replaced by "$C^*(\mathcal{Y})$". It follows that $\mathcal{Y}$ is basically or extremally disconnected if and only if $\beta\mathcal{Y}$ is ([14], Section 8, [16], Chapter 6).

Let A denote a Φ-algebra, and let $\mathcal{M}(A)$ denote the set of maximal l-ideals of A. The *Stone topology* on $\mathcal{M}(A)$ is defined in the following way. For any $\mathcal{S} \subset \mathcal{M}(A)$, the *kernel* $k(\mathcal{S})$ of $\mathcal{S}$ is $\bigcap\{M\colon M \in \mathcal{S}\}$ (where it is understood that $k(\emptyset) = A$). If I is an l-ideal of A, the *hull* $h(I)$ of I is $\{M \in \mathcal{M}(A)\colon M \supset I\}$. A subset $\mathcal{S}$ of $\mathcal{M}(A)$ is said to be *closed* if $\mathcal{S} = h(k(\mathcal{S}))$.

It is readily verified that with this definition of closed set, $\mathcal{M}(A)$ becomes a T_1-space such that every open covering has a finite subcovering. These assertions can be verified by examining [22], [12], or the more abstract formulation given in [2]. Unless otherwise stated, $\mathcal{M}(A)$ will denote the topological space defined above. Note that the sets

$$\mathcal{M}(a) = \{M \in \mathcal{M}(A)\colon\ a \in M\}$$

for $a \in A$, form a base for the closed sets in $\mathcal{M}(A)$.

The main result of this section is the following representation theorem.

2.3. THEOREM. *Every Φ-algebra A is isomorphic to an algebra $\bar{A}$ of extended functions on $\mathcal{M}(A)$. Moreover,*

(i) *$\mathcal{M}(A)$ is a compact space (in particular, it is a Hausdorff space), and*

(ii) *if $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint closed subsets of $\mathcal{M}(A)$, then there is an $\bar{a} \in \bar{A}$ such that $\bar{a}[\mathcal{S}_1] = 0$, $\bar{a}[\mathcal{S}_2] = 1$, and $0 \leqslant \bar{a} \leqslant 1$.*

Proof. If $a \in A$, and $M \in \mathcal{M}(A)$, let $M(a)$ denote the image of a under the natural isomorphism of A onto A/M. With each $a \in A$, we associate a function $\bar{a}$ on $\mathcal{M}(A)$ into γR as follows. If $a \in A^+$, let

$$\bar{a}(M) = \inf\{\lambda \in R\colon\ M(a) \leqslant \lambda\}$$

(where $\inf\emptyset$ is understood to be $+\infty$). If $a \in A$ is arbitrary, let

$$\bar{a}(M) = \overline{a^+}(M) - \overline{a^-}(M)\,.$$

Since $a^+ \wedge a^- = 0$, either $M(a^+) = 0$ or $M(a^-) = 0$, so $\bar{a}$ is well defined.

Let $\bar{A}$ denote the collection of all functions $\bar{a}$, for $a \in A$. The next two observations are easily verified.

(1) If 1 denotes the identity element of A, then $\bar{1}$ is the constant function $\mathbf{1}$, and $\overline{\lambda\cdot 1} = \lambda$ for all $\lambda \in R$.

(2) For each $a \in A$, and $\lambda \in R$, $\overline{(a+\lambda)} = \bar{a} + \lambda$ and $\overline{\lambda a} = \lambda\bar{a}$.

For each $a \in A$, the set $\{M \in \mathcal{M}(A)\colon M(a) > 0\} = \{M \in \mathcal{M}(A)\colon M(a^+) > 0\} = \{M \in \mathcal{M}(A)\colon\ a^+ \notin M\}$ is a basic open set in $\mathcal{M}(A)$. We use this fact to demonstrate continuity of $\bar{a}$ at each point M_0 of $\mathcal{M}(A)$. We may assume that $\bar{a}(M_0) \geqslant 0$.

Suppose first that $\bar{a}(M_0) = +\infty$. Then, for each $\lambda \in R$, the set $\{M \in \mathcal{M}(A)\colon \bar{a}(M) > \lambda\}$ contains the open neighborhood

$$\{M \in \mathcal{M}(A)\colon M(a) > \lambda+1\} = \{M \in \mathcal{M}(A)\colon M(a-\lambda-1) > 0\}$$

of M_0. Hence $\bar{a}$ is continuous at M_0.

If $\bar{a}(M_0) = \lambda \in R$, then for each real $\varepsilon > 0$, the set $\{M \in \mathcal{M}(A)\colon \lambda-\varepsilon < \bar{a}(M) < \lambda+\varepsilon\}$ contains the open neighborhood

$$\{M \in \mathcal{M}(A)\colon \lambda-\varepsilon/2 < M(a) < \lambda+\varepsilon/2\}$$
$$= \{M \in \mathcal{M}(A)\colon M(a-\lambda+\varepsilon/2) > 0\} \cap \{M \in \mathcal{M}(A)\colon M(-a+\lambda+\varepsilon/2) > 0\}$$

of M_0. Thus, we have proved.

(3) For each $a \in A$, $\bar{a}$ is a continuous mapping of $\mathcal{M}(A)$ into γR.

Now let M_1 and M_2 be distinct maximal l-ideals of A, and choose a positive element a in M_1 but not in M_2. Then, by 1.6 (ii), since A/M_2 is an l-simple f-algebra, there is a $b \in A^+$ such that $M_2(ab) \geqslant 1$. Let $c = ab \wedge 1$. Then $\bar{c}(M_1) = 0$, and $\bar{c}(M_2) = 1$. So, by (3),

$$\{M \in \mathcal{M}(A)\colon \bar{c}(M) < \tfrac{1}{3}\} \quad \text{and} \quad \{M \in \mathcal{M}(A)\colon \bar{c}(M) > \tfrac{2}{3}\}$$

are disjoint open neighborhoods of M_1, respectively M_2. Hence $\mathcal{M}(A)$ is a Hausdorff space. Indeed, as remarked above, $\mathcal{M}(A)$ is compact. Thus (1) has been established.

Now (ii) holds when $\mathcal{S}_1$ and $\mathcal{S}_2$ each consist of a single point. A standard compactness argument may be used to extend this first to the case in which $\mathcal{S}_1$ consists of a single point and $\mathcal{S}_2$ is arbitrary, and then to the general case.

For each $a \in A$, let $\mathcal{R}(\bar{a}) = \{M \in \mathcal{M}(A)\colon |\bar{a}(M)| \neq \infty\}$. We will show that $\mathcal{R}(\bar{a})$ is dense in $\mathcal{M}(A)$. For, suppose that $b \in A^+$, and $M(b) = 0$ for all $M \in \mathcal{R}(\bar{a})$. Then, for $n = 1, 2, \ldots,$ $M(n(b\wedge 1)) = 0$ if $M \in \mathcal{R}(\bar{a})$, and $M(n(b\wedge 1)) \leqslant M(|a|)$ if $M \notin \mathcal{R}(\bar{a})$, so $n(b\wedge 1) \leqslant |a|$. Thus, since A is archimedean, $b\wedge 1 = 0$. But, by 1.11, 1 is a weak order unit of A, so $b = 0$. Thus, each $\bar{a} \in \bar{A}$ is a continuous function on $\mathcal{M}(A)$ into γR that is real-valued on a dense subset. Hence

(4) $\bar{A} \subset D(\mathcal{M}(A))$.

We now define operations on $\bar{A}$ by inducing those of $D(\mathcal{M}(A))$ on it, and proceed to show that the mapping $a \to \bar{a}$ is an isomorphism of A onto $\bar{A}$.

Suppose that $a, b \in A$, and let $M \in \mathcal{R}(\bar{a}) \cap \mathcal{R}(\bar{b})$. It is easily verified that $(\bar{a}+\bar{b})(M) = \overline{(a+b)}(M)$ and that $\bar{a}\bar{b}(M) = \overline{ab}(M)$. Since $\overline{a+b}$, and $\overline{ab} \in \bar{A} \subset D(\mathcal{M}(A))$, $\bar{a}+\bar{b}$ and $\bar{a}\bar{b}$ exist and are in $\bar{A}$.

If $a \in A$ is such that $\bar{a} = 0$, then for each $M \in \mathcal{M}(a)$, $M(a) = 0$ or $|M(a)|$ is infinitely small. Hence $M(n|a|) \leqslant M(1)$ for each positive integer

n, and for all $M \in \mathcal{M}(A)$. Since A is archimedean, $|a| = 0$, whence $a = 0$. Thus:

(5) $\bar{a} = 0$ implies $a = 0$.

The lattice operations induced on $\bar{A}$ by $D(\mathcal{M}(A))$ yield the usual pointwise order on $\bar{A}$. Hence our proof of Theorem 2.3 will be completed as soon as we show that

(6) $a \in A^+$ if and only if $\bar{a} \in \bar{A}^+$.

If $\bar{a} \geqslant 0$, then $\bar{a}^-(M) = 0$ for each $M \in \mathcal{M}(A)$, so $a^- = 0$ by (5). Conversely, if $a \geqslant 0$, then clearly $\bar{a} \geqslant 0$.

This completes the proof of Theorem 2.3.

There are a large number of representation theorems similar to Theorem 2.3. The earliest seems to be due to M. H. Stone, and requires that A be (conditionally) σ-complete as a lattice ([34], [35]). Similar theorems were obtained by Dieudonne ([7], [8]), Nakano ([31]) and Yosida ([37]). Representations of A as a vector lattice abound; Birkhoff's book [3], Chapter 15, and the latter's paper with Pierce, [4], contain several such references. Particular care has been given by Kadison ([24]), and Kakutani ([25]) in case A has a strong order unit. The work of Fell and Kelley ([10]), Kantorovič, Pinsker, and Vulih ([26]), Shirota ([33]), and Vulih ([36]) also deserve mention. Representations of a different sort have been obtained by Goffman ([18]) and Olmstead ([32]).

The theories closest to the present work seem to be those of Domračeva ([9]) and Zawadowski ([37]). These authors do not rely on completeness assumptions. On the other hand, they do not work with objects readily identified as Φ-algebras, and it does not seem possible to apply their work directly to Theorem 2.3 or to the sequel. Hence a fresh exposition seems in order.

Henceforth, we will identify, whenever it is convenient to do so, the Φ-algebra A with the isomorphic algebra $\bar{A} \subset D(\mathcal{M}(A))$ of extended functions obtained from Theorem 2.3.

Recall that A^* denotes the set of bounded elements of A. An l-ideal I of A or A^* is called *fixed* if there is an $M \in \mathcal{M}(A)$ such that $a \in I$ implies $a(M) = 0$.

2.4. Lemma. *If I is a proper l-ideal of A or A^*, then I is fixed.*

Proof. Since every proper l-ideal of A is a subset of a maximal l-ideal of A, the lemma is immediate for A.

If I is an l-ideal of A^* that is not fixed, then for every $M \in \mathcal{M}(A)$, there is an $a_M \in I$ such that $a_M(M) > 0$. Since $\mathcal{M}(A)$ is compact, a finite number of the open sets $\mathcal{U}_M = \{M' \in \mathcal{M}(A)\colon a_M(M') > 0\}$ cover $\mathcal{M}(A)$, say $\mathcal{U}_{M_1}, \dots, \mathcal{U}_{M_k}$. Then $a = |a_{M_1}| + \dots + |a_{M_k}| \in I$, and there is a real number $\lambda > 0$ such that $a \geqslant \lambda \cdot 1$. Then $1 \leqslant (1/\lambda) a \in I$, whence I is not proper.

Now suppose that the Φ-algebra A is given to us explicitly as an algebra of extended functions on a compact space $\mathcal{X}$ such that $\mathcal{X} = \mathcal{M}(A)$. The following proposition describes the maximal l-ideals of A in terms of this representation. It generalizes a result obtained by Gelfand and Kolmogoroff in case $A = C(\mathcal{Y})$ for some completely regular space $\mathcal{Y}$.

2.5. Theorem. *A subset M of A is a maximal l-ideal of A if and only if there is a unique $x \in \mathcal{M}(A)$ such that*

$$M = M_x = \{a \in A\colon\ (ab)(x) = 0 \ \textit{for all} \ b \in A\}.$$

Proof. Clearly M_x, thus defined, is an l-ideal of A. If $c \notin M_x$, then there is a $d \in A$ such that $|cd|(x) \geqslant 1$. Let $\mathcal{U}$ denote a closed neighborhood of x disjoint from $(cd)^{-1}(0)$. By Theorem 2.3 (ii), there is an $a \in A^+$ such that $a[\mathcal{U}] = 0$, and $a[(cd)^{-1}(0)] = 1$. Since $\mathcal{R}(b)$ is dense in $\mathcal{M}(A)$ for every $b \in A$, we know that $a \in M_x$. But there is a $\lambda \in R$ such that $\lambda(a + |cd|) \geqslant 1$. Hence M_x and c together generate A. Thus, M_x is a maximal l-ideal.

That every maximal l-ideal of A takes this form follows from Lemma 2.4. The uniqueness of x is an immediate consequence of Theorem 2.3 (ii).

If $x \in \mathcal{M}(A)$ and $a(x) = 0$, then $(ab)(x) = 0$ for all $b \in A^*$. Thus, we have

2.6. Corollary. *A subset M^* of A^* is a maximal l-ideal of A^* if and only if there is a unique $x \in \mathcal{M}(A)$ such that*

$$M^* = M_x^* = \{a \in A^*\colon\ a(x) = 0\}.$$

If M is a maximal l-ideal of A, then the totally ordered algebra A/M contains R as a subfield. M is called *real* or *hyper-real* according as $A/M = R$ or A/M contains R properly.

If $x \in \mathcal{M}(A)$, then the mapping $a \to a(x)$ is clearly a homomorphism of A^* onto R. Hence, we have

2.7. Corollary. *Every maximal l-ideal of A^* is real.*

The *weak topology* for $\mathcal{M}(A)$ induced by the elements of A^* is the smallest topology for $\mathcal{M}(A)$ in which all of the functions in A^* are continuous. An immediate consequence of part (ii) of Theorem 2.3 is that the Stone topology for $\mathcal{M}(A)$ coincides with the weak topology induced by the bounded elements of A. Similarly, the Stone topology for $\mathcal{M}(A^*)$ is the weak topology induced by all of the elements of A^*.

By 2.5 and 2.6, there is a one-to-one correspondence $M \leftrightarrow M^*$ between $\mathcal{M}(A)$ and $\mathcal{M}(A^*)$. We show that this is a homeomorphism by showing that, for $a \in A^*$, the value of the function $a \in D(\mathcal{M}(A))$ at M is the same as the value of the function $\bar{a}$ at M^*, where $a \to \bar{a}$ denotes the representation of A^* as an algebra of extended functions on $\mathcal{M}(A^*)$. Now, by 2.7,

M^* is a real maximal l-ideal of A^*, so $\bar{a}(M^*) = M^*(a) = r \in R$. Thus, $a-r \in M^*$. Since M^* is the unique maximal l-ideal of A^* containing the prime l-ideal $M \cap A^*$ (see 1.6 (iii)), $a-r \in M^*$ if and only if $a-r$ is infinitely small modulo $M \cap A^*$, hence if and only if $a-r$ is infinitely small modulo M (in A). Thus, $\bar{a}(M^*) = r$ if and only if $a(M) = r$. Hence, we have established.

2.8. COROLLARY. *$\mathcal{M}(A)$ and $\mathcal{M}(A^*)$ are homeomorphic.*

That $\mathcal{M}(A)$ and $\mathcal{M}(A^*)$ are homeomorphic in case A is the ring of all continuous functions on a completely regular space $\mathcal{Y}$ was shown by Gelfand and Kolmogoroff in [11]. (See also [15].) Indeed, in this case they are homeomorphic to $\beta\mathcal{Y}$. In case A is σ-complete and regular, Corollary 2.8 was obtained by Brainerd in [5].

If $x \in \mathcal{M}(A)$, let

$$N_x = \{a \in A\colon\ a \text{ vanishes on a neighborhood of } x\}.$$

If $a, b \in N_x$, then it is clear that $a-b \in N_x$, and if $c \in A$, and $|c| \leqslant |a|$, then $c \in N_x$. Thus, to show that N_x is an l-ideal of A, we must show that $ad \in N_x$ for all $d \in A$. There is an open neighborhood $\mathcal{U}$ of x on which a vanishes. Clearly $(ad)(y) = 0$ for all $y \in \mathcal{R}(d) \cap \mathcal{U}$. But $\mathcal{R}(d)$ is dense in $\mathcal{M}(A)$, so $(ad)(z) = 0$ for each $z \in \mathcal{U}$. Hence, $ad \in N_x$. Thus, we have

2.9. *If A is a Φ-algebra, then for each $x \in \mathcal{M}(A)$, N_x is an l-ideal, and every l-ideal of A containing N_x is in the unique maximal l-ideal M_x.*

We conclude this section with a theorem concerning prime l-ideals.

2.10. THEOREM. *Let A be a Φ-algebra and let P be a prime l-ideal of A. Then there is a unique $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$. Moreover, N_x is the intersection of all the prime l-ideals containing it.*

Proof. By 1.6 (iii), P is contained in a unique maximal l-ideal M_x of A. If $a \in N_x$, then there is an open neighborhood $\mathcal{U}$ of x on which it vanishes. By Theorem 2.3 (ii), there is a $b \in A$ such that $b(x) = 1$ and $b[\mathcal{M}(A) \sim \mathcal{U}] = 0$. Then $ab = 0 \in P$. Since $b \notin M_x$, $b \notin P$. So, since P is prime, $a \in P$. Hence $N_x \subset P$.

To prove the last statement, suppose $a \in M_x$, and $a \notin N_x$. Then no power of a is in N_x. Hence $\{a, a^2, \ldots, a^n, \ldots\}$ is a multiplicative system disjoint from N_x. By 1.6 (iv), there is a prime l-ideal P of A containing N_x and not containing a.

We remark, finally, that the first part of Theorem 2.10 can be inferred from results of Gillman given in [12].

3. Uniformly closed Φ-algebras. A sequence $\{a_n\colon n = 1, 2, \ldots\}$ of elements of a Φ-algebra A is a *Cauchy sequence* if for each real $\varepsilon > 0$, there is a positive integer n_0 such that $|a_n - a_m| < \varepsilon$ whenever $n, m \geqslant n_0$. Taking $\varepsilon = 1$, we obtain

3.1. *If* $\{a_n: n = 1, 2, \ldots\}$ *is a Cauchy sequence in a* Φ*-algebra* A *then there is a positive integer* n_0 *such that* $\mathcal{R}(a_n) = \mathcal{R}(a_{n_0})$ *for* $n \geqslant n_0$.

A sequence $\{a_n: n = 1, 2, \ldots\}$ of elements of A is said to *converge* to $a \in A$ if for each real $\varepsilon > 0$, there is a positive integer n_0 such that $|a_n - a| < \varepsilon$ for $n \geqslant n_0$. *If* $\{a_n: n = 1, 2, \ldots\}$ *converges to both* a *and* b *in* A, *then* $a = b$. For, given any real $\varepsilon > 0$, there is an integer n_0 such that $|a_n - a| < \varepsilon/2$ and $|a_n - b| < \varepsilon/2$ for $n \geqslant n_0$, so $|a-b| \leqslant |a - a_n| + |a_n - b| < \varepsilon$. Hence, since A is archimedean, $a = b$.

A Φ-algebra A is said to be *uniformly closed* if every Cauchy sequence in A converges in A. If A is an algebra of real-valued functions, then this notion coincides with the usual notion of being closed under uniform convergence.

If A is a uniformly closed Φ-algebra, then A^*, considered as a subset of $D(\mathcal{M}(A))$, is by Theorem 2.3, a uniformly closed algebra of continuous real-valued functions on a compact space. Moreover, it contains the constant functions and separates points. Hence, by the Stone-Weierstrass theorem, we have:

3.2. *If* A *is a uniformly closed* Φ*-algebra, then* A^* *and* $C(\mathcal{M}(A))$ *are isomorphic.*

A Φ-algebra A is said to be *closed under bounded inversion* if, for $a \in A$, $a \geqslant 1$ implies $1/a \in A$. Thus, by 3.2, we have:

3.3. *Every uniformly closed* Φ*-algebra is closed under bounded inversion.*

For any $a \in A$, we denote the smallest l-ideal containing a by $\langle a \rangle$.

Suppose that A is closed under bounded inversion, and $a \in A$ is such that $\langle a \rangle = A$. Then there is a $b \in A$ such that $|ab| \geqslant 1$. Thus, $1/|ab| \in A$, so $1/|a| \in A$. Thus $1/|a|^2 = 1/a^2 \in A$, whence $1/a \in A$. So we have proved

3.4. *If* A *is a* Φ*-algebra closed under bounded inversion, then for* $a \in A$, $\langle a \rangle = A$ *if and only if* $1/a \in A$.

3.5. Lemma. *If* A *is a* Φ*-algebra such that* A^* *is uniformly closed, then* $\mathcal{R}(a)$ *is* C^**-imbedded in* $\mathcal{M}(A)$ *for each* $a \in A$. *Thus* $\mathcal{M}(A) = \beta\mathcal{R}(a)$.

Proof. By 3.2, we may identify A^* with $C(\mathcal{M}(A))$. Let $a \in A$. Since $\mathcal{R}(a) = \mathcal{R}(a^2 \vee 1)$, we may assume that $a \geqslant 1$. Let $g \in C^*(\mathcal{R}(a))$. Let $f(x) = g(x)/a(x)$ if $x \in \mathcal{R}(a)$, and $f(x) = 0$ if $x \in \mathcal{N}(a)$. Then, since g is bounded, $f \in C(\mathcal{M}(A)) = A^*$. Thus fa is the desired continuous extension of g over $\mathcal{M}(A)$.

With the aid of Lemma 3.5, we are now able to produce an example of a Φ-algebra that cannot be imbedded in a uniformly closed Φ-algebra with the same space of maximal l-ideals.

3.6. Example. Let R^+ denote the space of non-negative real numbers with its usual topology, and, as in 1.7, let A denote the Φ-algebra of all continuous real-valued functions on R^+ that are eventually polynomials, i.e. if $f \in A$, then $f \in C(R^+)$, and there is an $x \in R^+$ and a polynomial p such that $f(y) = p(y)$ for all $y \geqslant x$. It is easily verified that every maximal l-ideal of A either takes the form $M_x = \{f \in A\colon f(x) = 0\}$, for $x \in R^+$, or the form $M_\omega = \{f \in A\colon$ there is an $x \in R^+$ such that $y \geqslant x$ implies $f(y) = 0\}$. Thus $\mathcal{M}(A)$ is homeomorphic with the one-point compactification $aR^+ = R^+ \cup \{\omega\}$ of R^+.

If it were possible to imbed A as a subalgebra of a uniformly closed Φ-algebra B such that $\mathcal{M}(B) = \mathcal{M}(A)$, then, by Lemma 3.5, for each $a \in A$, $\mathcal{R}(a)$ would be C^*-imbedded in $\mathcal{M}(A)$. But this is not the case if a is a non-constant polynomial in A.

A subset S of a partially ordered set T is called *order-convex* if $a, b \in S$, and $x \in T$ with $a \leqslant x \leqslant b$, imply $x \in S$.

3.7. LEMMA. *For a Φ-algebra A, the following are equivalent.*

(i) *A is uniformly closed.*

(ii) *A^* is uniformly closed.*

(iii) *A is (isomorphic with) an order-convex subset of $D(\mathcal{M}(A))$.*

(iv) *A^* is (isomorphic with) an order-convex subset of $D(\mathcal{M}(A))$.*

Proof. It is obvious that (i) implies (ii) and (iii) implies (iv). By 3.2, it is clear that (ii) and (iv) are equivalent. Next, we show that (ii) implies (iii).

First consider $a \in A$, and $g \in D(\mathcal{M}(A))$ such that $1 \leqslant g \leqslant a$. On $\mathcal{R}(a)$, g/a is a bounded continuous function. By Lemma 3.5, it has a continuous extension $f \in C(\mathcal{M}(A)) = A^*$. Since $g(x) = f(x)a(x)$ for x in the dense subset $\mathcal{R}(a)$ of A, $g = fa \in A$.

Now suppose that $a, b \in A$, $g \in D(\mathcal{M}(A))$, and $a \leqslant g \leqslant b$. Then $g^+ \leqslant |g| \leqslant |a|+|b|$, so $1 \leqslant g^+ +1 \leqslant |a|+|b|+1 \in A$. Thus, the argument above shows that $g^+ +1 \in A$. Hence, $g^+ \in A$, and similarly $g^- \in A$. Hence $g = g^+ - g^- \in A$.

Finally, we show that (ii) implies (i). If $\{a_n\colon n = 1, 2, \ldots\}$ is a Cauchy sequence in A, then there is a positive integer n_0 such that $|a_n - a_{n_0}| < 1$ if $n \geqslant n_0$. Then, the sequence $\{a_{n_0+k} - a_{n_0}\colon k = 1, 2, \ldots\}$ is a Cauchy sequence in A^*, which converges, by hypothesis, to some $b \in A^*$. Thus, $\{a_n\colon n = 1, 2, \ldots\}$ converges to $b + a_{n_0} \in A$.

The next result of this section shows that the lattice structure of a uniformly closed Φ-algebra is uniquely determined by its algebraic structure. That is, all of the axioms for uniformly closed Φ-algebras could be rephrased in terms of the algebraic operations alone.

3.8. THEOREM. *If A is a uniformly closed Φ-algebra, then $a \in A^+$ if and only if $a = b^2$ for some $b \in A$.*

Proof. Let $a \in A^+$. Then a is a non-negative extended function on $\mathcal{M}(A)$, so $a^{1/2} \in D(\mathcal{M}(A))$. Now $0 \leqslant a^{1/2} \leqslant (a+1)^{1/2} \leqslant a+1 \in A$. Thus, by Lemma 3.7, $a^{1/2} \in A$.

For the converse, recall that squares are positive in any Φ-algebra (1.3 (i)).

We close this section with the following characterization theorem. Note first that *an element a of a Φ-algebra A of extended functions is a divisor of zero if and only if $a^{-1}(0)$ has a non-empty interior*. For, if the latter holds there is an $x \in a^{-1}(0)$, and an open neighborhood $\mathcal{U}$ of x on which a vanishes. By Theorem 2.3, there is a $b \in A$ such that $b(x) = 1$, and $b[X \sim \mathcal{U}] = 0$. Clearly $ab = 0$. The converse is obvious.

3.9. Theorem. *A Φ-algebra A is isomorphic to $D(\mathcal{X})$ for some compact space $\mathcal{X}$ if and only if*

(i) *A is uniformly closed, and*

(ii) *if $a \in A$, then either a is a divisor of zero or $\langle a \rangle = A$.*

Proof. Suppose first that (i) and (ii) hold. If $f \in D(\mathcal{M}(A))$, and $f \geqslant 1$, then by 3.2, $g = 1/f \in A^*$. Now, $g^{-1}(0) = \mathcal{N}(f)$ is nowhere dense, so by (ii), $\langle g \rangle = A$. Then, by 3.4, $1/g = f \in A$.

If h is any element of $D(\mathcal{M}(A))$, the above shows that $h^+ + 1$ and $h^- + 1$ are in A. Hence $h = (h^+ + 1) - (h^- + 1) \in A$. Thus $A = D(\mathcal{M}(A))$.

Conversely, if $A = D(\mathcal{X})$ for some compact space $\mathcal{X}$, then clearly $\mathcal{X} = \mathcal{M}(A)$, and A is uniformly closed. If $a \in A$, and $ab = 0$ implies $b = 0$, then $a^{-1}(0)$ is nowhere dense, so $1/a \in D(\mathcal{M}(A)) = A$. Thus (ii) holds.

If A is a regular ring (1.12), then for every $a \in A$, there is a $c \in A$ such that $a(ac-1) = 0$. Thus (ii) holds. Hence we have:

3.10. Corollary. *If A is a uniformly closed, regular Φ-algebra, then $A = D(\mathcal{M}(A))$.*

Corollary 3.10 shows that if A is the ring $\mathfrak{L}$ of Lebesgue measurable functions on R, the ring $\mathfrak{B}$ of Baire functions on R, or the rings $\mathfrak{L}_0$ or $\mathfrak{B}_0$ obtained by reducing these rings modulo the ideal of null functions (see 1.2), then $A = D(\mathcal{M}(A))$.

4. Algebras of real-valued functions.

If a is an element of a Φ-algebra A, let

$$\mathcal{Z}(a) = \{M \in \mathcal{M}(A) \colon \ a(M) = 0\}.$$

Thus $M \in \mathcal{Z}(a)$ if and only if $M(|a|)$ is infinitely small or zero. Hence, if M is real, $a(M) = 0$ implies $a \in M$.

Let $\mathcal{R}(A)$ denote the subspace of real maximal ideals of A. That is,

$$\mathcal{R}(A) = \bigcap \{\mathcal{R}(a) \colon \ a \in A\}.$$

In this section, we will consider Φ-algebras A which satisfy one or more of the following restrictions.

4.1. *A Φ-algebra A is said to be closed under l-inversion if, for $a, b \in A$, $\mathcal{Z}(a) \subset \mathcal{N}(b)$ implies $\langle a \rangle = A$.*

4.2. *A Φ-algebra A is called an algebra of real-valued functions if* $\bigcap\{M\colon\ M \in \mathcal{R}(A)\} = \{0\}$.

4.3. *A Φ-algebra A of real-valued functions is said to be closed under inversion if, for $a \in A$, $\mathcal{Z}(a) \cap \mathcal{R}(A) = \emptyset$ implies $\langle a\rangle = A$.*

Condition 4.1 makes sense, of course, even if A is not an algebra of real-valued functions. It holds, in particular, if $D(\mathcal{M}(A))$ is an algebra and $A = D(\mathcal{M}(A))$, and hence it holds in the Φ-algebras $\mathfrak{L}_0$ and $\mathfrak{B}_0$ of 1.2 by Corollary 3.10.

Note that the condition of 4.2 states that $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$, so that A is, in fact, an algebra of (continuous) real-valued functions on $\mathcal{R}(A)$. As mentioned earlier, not every Φ-algebra is an algebra of real-valued functions; it may be that $\mathcal{R}(A) = \emptyset$. This is, indeed, the case if $A = \mathfrak{L}_0$ or $A = \mathfrak{B}_0$.

By 3.3 and 3.4, in a uniformly closed Φ-algebra A, $\langle a\rangle = A$ if and only if $1/a \in A$.

4.4. *A uniformly closed Φ-algebra A is closed under inversion (respectively, l-inversion) if and only if, for $a \in A$, $\mathcal{Z}(a) \cap \mathcal{R}(A) = \emptyset$ (respectively, $\mathcal{Z}(a) \subset \mathcal{N}(b)$ for some $b \in A$) implies $1/a \in A$.*

It is clear that every Φ-algebra of real-valued functions closed under inversion is closed under l-inversion. That the converse is not true will be shown by an example at the end of this section. Next, we give an example of a uniformly closed Φ-algebra of real-valued functions that is not closed under either type of inversion.

4.5. EXAMPLE. Let $A = \{f \in C(R^+)\colon \lim_{x\to\infty} f(x)e^{-ax} = 0$ for all real $a > 0\}$. It is easily verified that A is a uniformly closed Φ-algebra. Since A^* and $C^*(R^+)$ are isomorphic, $\mathcal{M}(A) = \beta R^+$. The function g such that $g(x) = e^{-x}$ for all $x \in R^+$ is in A. Moreover $\mathcal{Z}(g) = \mathcal{N}(f) = \beta R^+ \sim R^+$, where $f(x) = x$ for all $x \in R^+$. However $1/g \notin A$.

In case $A = C(\mathcal{Y})$ for some completely regular space $\mathcal{Y}$, the following result is due to Gelfand and Kolmogoroff ([11]). (See, also [15].) For this special case, it is equivalent to Theorem 2.5.

4.6. THEOREM. *If A is a Φ-algebra of real-valued functions which is closed under inversion, then for each $x \in \mathcal{M}(A)$,*

$$M_x = \left\{a \in A\colon\ x \in \left(\mathcal{Z}(a) \cap \mathcal{R}(A)\right)^{-}\right\}.$$

Proof. For $a \in A^+$, let $\mathcal{Z} = \mathcal{Z}(a) \cap \mathcal{R}(A)$, and suppose that $x \notin \mathcal{Z}^-$. Then we may choose a closed neighborhood $\mathcal{U}$ of x disjoint from $\mathcal{Z}^-$. By Theorem 2.3, there is a $b \in A^+$ such that $b[\mathcal{U}] = 0$, and $b[\mathcal{Z}^-] = 1$. Now, since $\mathcal{Z}(a+b) \cap \mathcal{R}(A) = \emptyset$ and A is closed under inversion, there is a $c \in A$ such that $(a+b)c \geqslant 1$. Since $b[\mathcal{U}] = 0$, $(ac)(y) \geqslant 1$ for all $y \in \mathcal{U} \cap \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$, this means that $(ac)(x) \geqslant 1$.

Thus, by Theorem 2.5, $a \notin M_x$. Since a is in a l-ideal of A if and only if $|a|$ is, we have shown that $M_x \subset \{a \in A: x \in (\mathcal{Z}(a) \cap \mathcal{R}(A))^{-}\}$.

Conversely, if $x \in (\mathcal{Z}(a) \cap \mathcal{R}(A))^{-}$, then every neighborhood of x contains points of $\mathcal{R}(A)$ at which a vanishes. Thus, in every neighborhood of x, there are points at which ab vanishes for any $b \in A$. Hence, by Theorem 2.5, $a \in M_x$. This completes the proof of the Theorem 4.6.

It is easily seen that closure under inversion is also necessary for this description of the maximal l-ideals of A. For, if $a \in A$ is such that $\mathcal{Z}(a) \cap \mathcal{R}(A) = \emptyset$, and $\langle a \rangle \neq A$, then a is contained in some maximal l-ideal M_x of A, and no such description of M_x is available.

We close this section with an example of a uniformly closed Φ-algebra of real-valued functions that is closed under l-inversion, but is not closed under inversion.

4.7. Example. Let N denote the discrete space of positive integers, and let $\mathcal{Y}$ denote any locally compact, σ-compact space that is not compact. Let $\mathcal{T} = N \times \mathcal{Y}$, and, for each $n \in N$, let $\mathcal{L}_n = \{n\} \times \mathcal{Y}$. Let $A = \{f \in D(\beta\mathcal{T})$: $f|T$ is real-valued, and there is an $n_f \in N$ such that $m \geqslant n_f$ implies $f|\mathcal{L}_m$ is bounded$\}$.

Thus, if $f \in A$, then f is real-valued on all but finitely many of the spaces $\mathcal{L}_n^{-}$. It is easily seen that A is a Φ-algebra such that A^* and $C^*(\mathcal{T})$ are isomorphic, so $\mathcal{M}(A) = \beta\mathcal{T}$. Thus, by Lemma 3.7,

(1) A is a uniformly closed Φ-algebra with $\mathcal{M}(A) = \beta\mathcal{T}$.

Since $\mathcal{Y}$ is locally compact and σ-compact, there is an $h \in C(\beta\mathcal{Y})$ that never vanishes on $\mathcal{Y}$ such that $h[\beta\mathcal{Y} \sim \mathcal{Y}] = 0$. Observe, also, that for each $n \in N$, $\mathcal{L}_n^{-}$ and $\beta\mathcal{Y}$ are homeomorphic.

We wish to show that if $p \in \beta\mathcal{T} \sim \mathcal{T}$, then $M_p \notin \mathcal{R}(A)$. Suppose first that there is an $m \in N$ such that $p \in \mathcal{L}_m^{-}$. Define the function f on $\mathcal{T}$ by letting $f(n, y) = 1/h(y)$ if $n = m$, $f(n, y) = 0$ if $n \neq m$, for all $y \in \mathcal{Y}$. By 1.9 (iv), f has a continuous extension $\hat{f}$ over $\beta\mathcal{T}$ into γR. Clearly $\hat{f} \in A$, and $\hat{f}(p) = \infty$.

If, for every $n \in N$, $p \notin \mathcal{L}_n^{-}$, then every neighborhood of p meets infinitely many of the spaces $\mathcal{L}_n$. Thus, if $g(n, y) = n$ for all $n \in N$, $y \in \mathcal{Y}$, then the continuous extension $\hat{g}$ of g over $\beta\mathcal{T}$ into γR is such that $\hat{g}(p) = \infty$. Since $\hat{g} \in A$, we have:

(2) $\mathcal{R}(A) = \mathcal{T}$.

As observed above, we also have:

(3) If $a \in A$, then $\{n \in N: \mathcal{N}(a) \cap \mathcal{L}_n^{-} \neq \emptyset\}$ is finite.

Now, if $a, b \in A$ are such that $\mathcal{Z}(a) \subset \mathcal{N}(b)$, then there is an $n_a \in N$ such that $m \geqslant n_a$ implies $\mathcal{Z}(a) \cap \mathcal{L}_m^{-} = \emptyset$. Hence $m \geqslant n_a$ implies that $a|\mathcal{L}_m$ is bounded away from 0, so $1/a \in A$. Thus,

(4) A is closed under l-inversion.

Finally, we observe that A is not closed under inversion. The function k defined by letting $k(n, y) = h(y)$ for all $n \in N$, $y \in \mathcal{Y}$ has a continuous extension $\bar{k}$ over $\beta\mathcal{T}$. Clearly $\bar{k} \in A^*$. Now, $\mathcal{Z}(\bar{k}) \cap \mathcal{R}(A) = \emptyset$, but $1/k \notin A$.

5. Some internal characterizations of $C(\mathcal{Y})$. In this section, the algebra $C(\mathcal{Y})$ is characterized among the class of Φ-algebras for several classes of topological spaces $\mathcal{Y}$ by means of *internal* properties of Φ-algebras. In each case, one of the requirements is uniform closure, so, in view of Theorem 3.8, the characterizations are, in reality, purely algebraic.

In case $\mathcal{Y}$ is compact, the celebrated Stone-Weierstrass theorem provides an internal characterization of $C(\mathcal{Y})$. In this case, a characterization of $C(\mathcal{Y})$ as a ring was provided by McKnight in 1953 ([30]), and it was improved by Kohls in 1957 ([28]). Characterizations of $C(\mathcal{Y})$ in the general (completely regular) case were provided by Anderson and Blair in 1959 ([1]), both as a ring, and as a lattice-ordered ring. These characterizations, however, are *external* in nature. In each case, one must examine a large class of extensions of the algebra in question in order to determine if this is a $C(\mathcal{Y})$. The demand that the characterization be internal seems to make the problem more difficult.

The assumptions that are common to most of our results are that the Φ-algebra A be a uniformly closed algebra of real-valued functions that is closed under inversion. Obviously, each of these conditions is necessary. Isbell has supplied an example of a Φ-algebra A satisfying all of these conditions that is not isomorphic to $C(\mathcal{Y})$ for any completely regular $\mathcal{Y}$ ([21], p. 108). Below, we give a few other such examples, which, we believe are simpler in character. Note that if a Φ-algebra A is isomorphic to some $C(\mathcal{Y})$, then it is isomorphic to $C(\mathcal{R}(A))$.

5.1. EXAMPLE. Consider the Φ-algebra $\mathfrak{B}$ of Baire functions on the real line. (See 1.2.) It is an algebra of real-valued functions and, since $\mathfrak{B}$ is closed under point-wise convergence, it is uniformly closed. Let M be a real maximal l-ideal of $\mathfrak{B}$. Now, $C(R)$ is a subalgebra of $\mathfrak{B}$, so

$$\frac{C(R)}{M \cap C(R)} \cong \frac{C(R)+M}{M} \subset \frac{\mathfrak{B}}{M} \cong R.$$

Since the left-hand member contains R, we must have $M \cap C(R)$ a real maximal ideal of $C(R)$. Hence ([16], Chapter 5) there is an $x \in R$ such that $M \cap C(R) = \{f \in C(R): f(x) = 0\}$. There is a $k \in C(R)$ such that $k^{-1}(0) = \{x\}$. Thus, if $g \in \mathfrak{B}$, and $g(x) \neq 0$, then $|k|+|g|$ is a positive element of $\mathfrak{B}$ that vanishes nowhere, and hence has an inverse. So, $M = M_x = \{f \in \mathfrak{B}: f(x) = 0\}$.

We have shown that $\mathcal{R}(\mathfrak{B})$ consists precisely of the l-ideals M_x, $x \in R$. Since $\mathfrak{B}$ contains all characteristic functions of one-point subsets of R, the Stone topology on $\mathcal{R}(\mathfrak{B})$ is discrete. Hence $\mathcal{R}(\mathfrak{B})$ is homeomorphic to the space of real numbers with the discrete topology. It is now clear that $\mathfrak{B}$ is closed under inversion.

Not only is $\mathfrak{B}$ not isomorphic to $C(\mathcal{R}(\mathfrak{B}))$, but card $C(\mathcal{R}(\mathfrak{B})) = 2^c$, while card $\mathfrak{B} = c$ ([19]).

The argument just given applies verbatim to the Φ-algebra of all functions in any Baire class, except that the latter need not be closed under point-wise convergence. $\mathfrak{B}$, however, has the advantage that is both σ-complete and regular (1.12).

A similar argument shows that the Φ-algebra $\mathfrak{L}$ of all measurable functions on R is not isomorphic to a full algebra of continuous functions. In this case, however $\mathfrak{L}$ and $C(\mathcal{R}(\mathfrak{L}))$ have the same cardinal number. Note that $\mathfrak{L}$ is also regular and σ-complete (1.12).

For a Φ-algebra A of real-valued functions that is closed under inversion, a necessary and sufficient condition that A be isomorphic to $C(\mathcal{R}(A))$ is that $\mathcal{M}(A) = \beta(\mathcal{R}(A))$. In fact, we may weaken this condition slightly.

5.2. Lemma. *A Φ-algebra A is isomorphic to $C(\mathcal{Y})$ for some completely regular space $\mathcal{Y}$ if and only if*

(i) *A is an algebra of real-valued functions,*

(ii) *A is uniformly closed,*

(iii) *A is closed under inversion, and*

(iv) *if $f \in C(\mathcal{R}(A))$, then there is an $a \in A$ such that $f^{-1}(0) = \mathcal{Z}(a) \cap \cap \mathcal{R}(A)$.*

Proof. These conditions are obviously necessary. To prove sufficiency we show first that $\mathcal{M}(A) = \beta\mathcal{R}(A)$. Now, by 1.9 (iii), this is true if and only if whenever $f_1, f_2 \in C(\mathcal{R}(A))$, and $f_1^{-1}(0)$ and $f_2^{-1}(0)$ are disjoint, then $f_1^{-1}(0)$ and $f_2^{-1}(0)$ have disjoint closures in $\mathcal{M}(A)$. By (iv), there are elements $a_i \in A$ such that $f_i^{-1}(0) = \mathcal{Z}(a_i) \cap \mathcal{R}(A)$, for $i = 1, 2$. Now $\mathcal{Z}(a_1^2+a_2^2) \cap \cap \mathcal{R}(A) = \emptyset$, so by (iii), there is a $b \in A$ such that $b(a_1^2+a_2^2) = 1$. Now $a_1^2 b[f_1^{-1}(0)] = 0$, and $a_1^2 b[f_2^{-1}(0)] = 1$. Hence, since $a_1^2 b$ is continuous on $\mathcal{M}(A)$, $f_1^{-1}(0)$ and $f_2^{-1}(0)$ have disjoint closures in $\mathcal{M}(A)$.

By 3.2, to show that A is isomorphic to $C(\mathcal{R}(A))$, it suffices to show that if $1 \leqslant g \in C(\mathcal{R}(A))$, then there is an $a \in A$ such that $g = a|\mathcal{R}(A)$. Now $1/g \in C^*(\mathcal{R}(A))$, so, by the above, it has an extension $b \in A^*$. But $\mathcal{Z}(b) \cap \mathcal{R}(A) = \emptyset$, so by (iii), $a = 1/b \in A$. Clearly $a|\mathcal{R}(A) = g$.

The authors are indebted to M. Jerison for the following lemma. Recall that a Hausdorff space $\mathcal{Y}$ is called a *Lindelöf* space if every open cover of $\mathcal{Y}$ has a countable subcover.

5.3. Lemma. *If $\mathcal{Y}$ is a Lindelöf space contained in a (compact) space $\mathcal{X}$, then, for every $f \in C(\mathcal{Y})$, there is an $a \in C(\mathcal{X})$ such that $f^{-1}(0) = a^{-1}(0) \cap \mathcal{Y}$.*

Proof. For each $y \in \mathcal{Y} \sim f^{-1}(0)$, there is an $a_y \in C(\mathcal{X})$ such that $a_y(y) = 1$, $|a_y| \leqslant 1$, and $a_y[f^{-1}(0)] = 0$. Let $\mathcal{U}_y = \{y' \in \mathcal{Y}: a_y(y') > \frac{1}{2}\}$. Then $\{\mathcal{U}_y: y \in \mathcal{Y} \sim f^{-1}(0)\}$ is an open covering of $\mathcal{Y} \sim f^{-1}(0)$.

Now $\mathcal{Y} \sim f^{-1}(0)$ is an F_σ-subset of the Lindelöf space $\mathcal{Y}$, and hence is a Lindelöf space. So, there exist countably many elements $y_1, y_2, \ldots, y_n, \ldots$ of $\mathcal{Y}$ such that $\mathcal{Y} \sim f^{-1}(0) \subset \bigcup \{\mathcal{U}_{y_n};\ n = 1, 2, \ldots\}$. Thus, if $a = \sum_{n=1}^{\infty} \frac{1}{2^n} |a_{y_n}|$, then $f^{-1}(0) = a^{-1}(0) \cap \mathcal{Y}$.

The hypothesis that $\mathcal{Y}$ be a Lindelöf space in Lemma 5.3 cannot be deleted. In particular, if $\mathcal{Y}$ is an uncountable discrete space, and $\mathcal{X}$ is its one-point compactification, then the conclusion of Lemma 5.3 need not hold.

We are now ready to give our first characterization.

5.4. THEOREM. *A Φ-algebra A is isomorphic to $C(\mathcal{Y})$ for some Lindelöf space $\mathcal{Y}$ if and only if*

(i) *A is an algebra of real-valued functions,*

(ii) *A is uniformly closed,*

(iii) *A is closed under inversion, and*

(iv) *if $\{a_\alpha: \alpha \in \Gamma\}$ is a collection of elements of A such that for each $M \in \mathcal{R}(A)$, there is an $\alpha \in \Gamma$ with $a_\alpha \notin M$, then there is a countable subset $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ of Γ such that $\{a_{\alpha_i}: i = 1, 2, \ldots\}$ has this property.*

Proof. Condition (iv) states that every open cover of $\mathcal{R}(A)$ by basic open sets of the form $(\mathcal{M}(A) \sim \mathcal{Z}(a)) \cap \mathcal{R}(A)$, $a \in A$, has a countable subcover. Thus, (iv) is equivalent to the statement that $\mathcal{R}(A)$ is a Lindelöf space. Hence the theorem follows from Lemmas 5.3 and 5.2.

In case $\mathcal{Y}$ is locally compact and σ-compact, we have a somewhat simpler characterization of $C(\mathcal{Y})$, but we cannot claim that it is original. It differs only superficially from a result of Isbell, [21], Lemma 1.18. While we could prove our theorem by reducing it to his, it seems easier to give a direct proof.

5.5. THEOREM. *A Φ-algebra A is isomorphic to $C(\mathcal{Y})$ for $\mathcal{Y}$ locally compact and σ-compact if and only if*

(i) *A is an algebra of real-valued functions,*

(ii) *A is uniformly closed,*

(iii) *A is closed under l-inversion and*

(iv) *there is an $h \in A$ such that $\mathcal{R}(A) = \mathcal{R}(h)$.*

Proof. These conditions are obviously necessary. If $a \in A$ is such that $\mathcal{Z}(a) \cap \mathcal{R}(A) = \emptyset$, then, by (iv), $\mathcal{Z}(a) \subset \mathcal{N}(h)$, whence by (iii), $1/a \in A$. Thus, in the presence of (iv), closure under l-inversion implies closure under inversion. By Lemma 3.5, $\mathcal{R}(A) = \mathcal{R}(h)$ is C^*-imbedded in $\mathcal{M}(A)$, so $\mathcal{M}(A) = \beta\mathcal{R}(A)$. Thus, by Lemma 5.2, A is isomorphic to $C(\mathcal{R}(A))$.

We require an additional fact about extremally disconnected spaces. (See the discussion following Proposition 2.2.) Every dense subspace of an extremally disconnected space is C^*-imbedded ([29]). Hence, by Lemma 5.2, we have

5.6. THEOREM. *A Φ-algebra A is isomorphic to $C(\mathcal{Y})$ for some extremally disconnected space $\mathcal{Y}$ if and only if*

(i) *A is an algebra of real-valued functions,*
(ii) *A is uniformly closed,*
(iii) *A is closed under inversion, and*
(iv) *A is complete.*

The algebra $\mathfrak{B}$ of Baire functions of Example 5.1 shows that we cannot replace (iv) above by the requirement that A be σ-complete.

By using Theorem 3.9, we may replace condition (iii) above by requirement that every element of A be either a divisor of zero or have an inverse. For, in this case, we may conclude that $A = D(\mathcal{M}(A))$, and that $\beta(\mathcal{R}(A)) = \mathcal{M}(A)$. This change does not, however, either weaken or strengthen the hypothesis of this theorem.

By (1.8), we could also delete the requirement that A be archimedean.

An infinite cardinal number $\mathfrak{m}$ is said to be *nonmeasurable* if there is no countably additive measure on a set of power $\mathfrak{m}$ giving points measure 0, the whole set measure 1, and assuming only the values 0 and 1. In 1930, Ulam showed that $\mathfrak{m}$ is nonmeasurable unless $\mathfrak{m}$ is strongly inaccessible from $\aleph_0$. Moreover, it is consistent with the axioms of set theory to rejeet the existence of such cardinal numbers. For a thorough discussion of nonmeasurable cardinals, see [16], Chapter 12, where it is shown that if $\mathfrak{m}$ is nonmeasurable, so is $2^{\mathfrak{m}}$. From this, we may derive

5.7. THEOREM. *Let A be a Φ-algebra of nonmeasurable power. Then A is isomorphic to $C(\mathcal{Y})$ for some discrete space $\mathcal{Y}$ if and only if*

(i) *A is an algebra of real-valued functions,*
(ii) *A is uniformly closed,*
(iii) *A is complete, and*
(iv) *A is regular.*

Proof. By Corollary 3.10, (ii) and (iv) imply that $A = D(\mathcal{M}(A))$. By (iii), $\mathcal{M}(A)$ is extremally disconnected. Hence, as remarked above, (i) implies that $\beta\mathcal{R}(A) = \mathcal{M}(A)$. Thus A and $C(\mathcal{R}(A))$ are isomorphic. Since A is regular, $\mathcal{R}(A)$ is a P-space (i.e. every G_δ is open; see [13]). But Isbell has shown that every extremally disconnected P-space of nonmeasurable power is discrete (see [20]; [16], Chapter 12). Also, card $\mathcal{R}(A) \leqslant 2^{\mathfrak{m}}$, where $\mathfrak{m} =$ card A. This completes the proof of the theorem.

Example 5.1 shows that (iii) above cannot be replaced by the requirement that A be σ-complete. A characterization of $C(\mathcal{Y})$ among the class of regular σ-complete Φ-algebras was obtained by Brainerd [6].

Our last theorem is a simple application of Theorem 5.5.

5.8. THEOREM. *Let A be a Φ-algebra that is uniformly closed and closed under l-inversion. If $h \in A$, let $B_h = \{f \in A\colon \mathcal{R}(f) \subset \mathcal{R}(h)\}$. Then B_h and $C(\mathcal{R}(h))$ are isomorphic.*

Proof. It is clear that B_h is a Φ-algebra that is uniformly closed and closed under l-inversion, indeed, $B_h^* = A^*$. Hence $\mathcal{M}(B_h) = \mathcal{M}(A)$, and $\mathcal{R}(B_h^*) = \mathcal{R}(h)$. Since $h \in B_h$, the theorem follows from Theorem 5.5.

We have been unable to obtain an internal characterization of $C(\mathcal{Y})$ in the general case. By now, it is evident that the heart of the difficulty lies in our lack of ability to find an internal equivalent of condition (iv) of Lemma 5.2.

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Added in proof. J. E. Kist has pointed out that, in the presence of completeness, the hypothesis that A be uniformly closed in Theorems 5.6 and 5.7 is redundant. (See, e.g. [31], p. 30.)