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SOME SUFFICIENT CONDITIONS FOR THE JACOBSON RADICAL OF A COMMUTATIVE RING WITH IDENTITY TO CONTAIN A PRIME IDEAL

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1. Introduction

Throughout, the word «ring» will abbreviate the phrase «commutative ring with identity element 1» unless the contrary is stated explicitly. An ideal I of a ring R is called *pseudoprime* if $ab = 0$ implies a or b is in I. This term was introduced by C. Kohls and L. Gillman who observed that if I contains a prime ideal, then I is pseudoprime, but, in general, the converse need not hold. In [9 p. 233], M. Larsen, W. Lewis, and R. Shores ask if whenever the Jacobson radical J(R) of an arthmetical ring is pseudoprime, it follows that J(R) contains a prime ideal?

In Section 2, I answer this question affirmatively. Indeed, if R is arithmetical and J(R) is pseudoprime, then the set N(R) of nilpotent elements of R is a prime ideal (Corollary 9). Along the way, necessary and sufficient conditions for J(R) to contain a prime ideal are obtained.

In Section 3, I show that a class of rings introduced by A. Bouvier [1] are characterized by the property that every minimal prime ideal of R is contained in J(R). The remainder of the section is devoted to rings with pseudoprime Jacobson radical that satisfy

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a variety of chain conditions. In particular, it is shown that if R is a Noetherian multiplication ring with pseudoprime Jacobson radical J(R), then J(R) contains a unique minimal prime ideal (Theorem 20), but there is a Noetherian semiprime ring R such that J(R) is pseudoprime and fails to contain a prime ideal (Example 21).

2. The ideal mI and pseudoprime ideals

As in [5], if I is an ideal of a ring R, let

$$mI = \cup \{\mathcal{A}(1 - i) : i \in I\}$$

where $\mathcal{A}(a) = \{x \in R : ax = 0\}$. In [5], the following assertions are proved.

1. Lemma (*Jenkins-McKnight*) *If* I *and* K *are ideals of a ring* R *and* I $\subset$ K, *then*

(a) mI *is an ideal of* R *contained in* I

(b) $mI = \{a \in R : I + \mathcal{A}(a) = R\}$

(c) $mI \subset mK$

(d) $m(I + J(R)) = mI$.

Recall that the Jacobson radical J(R) of a commutative ring R with identity is the intersection of all the maximal ideals of R, and that $a \in J(R)$ if and only if $(1 - ax)$ is a unit for every $x \in R$ [11, Section 30].

Let $\mathcal{U}(R)$ denote the set of units of a ring R, let $\mathcal{M}(R)$ denote the set of maximal ideals of R, and let $S(R) = \Sigma\{mM : M \in \mathcal{M}(R)\}$. By Lemma 1, $S(R) = \Sigma\{mI : I \text{ a proper ideal of } R\} = \Sigma\{\mathcal{A}(1 - i) : i \in R \backslash \mathcal{U}(R)\}$ is the smallest ideal containing $\mathcal{A}(1 - i)$ for every non unit $i \in R$.

The next lemma indicates the importance of the ideals mI in the study of rings with pseudoprime Jacobson radical.

2. Lemma. *The Jacobson radical* J(R) *of a ring* R *is pseudoprime if and only if* $S(R) \subset J(R)$.

Proof. To prove the lemma, it suffices to show that J(R) is pseudoprime if and only if $mI \subset J(R)$ for every proper ideal I of R.

If J(R) is pseudoprime, I is a proper ideal of J(R), and $a \varepsilon m\mathrm{I}$, there is an $i \varepsilon \mathrm{I}$ such that $a(1 - i) = 0$. But $(1 - i) \notin \mathrm{J(R)}$, so $a \varepsilon \mathrm{J(R)}$.

Suppose, conversely, that $m\mathrm{I} \subset \mathrm{J(R)}$ for every proper ideal I of R, $ab = 0$, and $b \notin \mathrm{J(R)}$. Then there is a ı $x \varepsilon \mathrm{R}$ such that $1 - bx$ is not a unit. Thus $a(1 - bx) = a$, so $a \varepsilon m((1 - bx)\mathrm{R}) \subset \mathrm{J(R)}$.

Suppose I is a proper ideal of a ring R (which need not have an identity element). A proper prime ideal of R that fails to contain properly any other prime ideal of R is said to be a *minimal prime ideal* of R. Let $\mathcal{P}(\mathrm{R})$ denote the set of minimal prime ideals of R. It is well known that $\bigcap\{\mathrm{P}: \mathrm{P} \varepsilon \mathcal{P}(\mathrm{R})\}$ is the set N(R) of nilpotent elements of R [11, p. 100], and that a prime ideal P is minimal if for every $a \varepsilon \mathrm{P}$, there is a $b \notin \mathrm{P}$ such that $ab \varepsilon \mathrm{N(R)}$ [5, lemma 3.1].

If $\mathrm{N(R)} = \{0\}$, then R is called a *semiprime* ring.

For any ideal I of R, the *radical* $\sqrt{\mathrm{I}}$ of I is the intersection of all the prime ideals of R containing I. Equivalently, $\sqrt{\mathrm{I}} = \{a : a^n \varepsilon \mathrm{I}\}$ for some positive integer n. The next proposition describes $\sqrt{m\mathrm{I}}$ as an intersection of minimal prime ideals of R.

3. Proposition. *Suppose* I *is a proper ideal of* R *and* P *is a minimal prime ideal of* R

(a) $m\mathrm{I} \subset \mathrm{P}$ *if and only if* $\mathrm{I} + \mathrm{P} \neq \mathrm{R}$

(b) $\sqrt{m\mathrm{I}} = \bigcap\{\mathrm{P} \varepsilon \mathcal{P}(\mathrm{R}) : \mathrm{I} + \mathrm{P} \neq \mathrm{R}\}$

(c) *If* M *is a maximal ideal of* R, *then*

$$\sqrt{m\mathrm{M}} = \bigcap\{\mathrm{P} \varepsilon \mathcal{P}(\mathrm{R}) : \mathrm{P} \subset \mathrm{M}\}$$

(d) *If* R *is semiprime, then* $\sqrt{m\mathrm{I}} = m\mathrm{I}$.

Proof of (a). If $\mathrm{I} + \mathrm{P} = \mathrm{R}$, there is an $i \varepsilon \mathrm{I}$ and a $p \varepsilon \mathrm{P}$ such that $i + p = 1$. Since $\mathrm{P} \varepsilon \mathcal{P}(\mathrm{R})$, there is a $q \notin \mathrm{P}$ such that $q(1 - i) = qp \varepsilon \mathrm{N(R)}$. Hence there is a positive integer n such that $q^n(1 - i)^n = 0$. By the binomial theorem $(1 - i)^n = (1 - i')$ for some $i' \varepsilon \mathrm{I}$, so $q^n \varepsilon m\mathrm{I} \backslash \mathrm{P}$. We have shown that $\mathrm{I} + \mathrm{P} = \mathrm{R}$ implies $m\mathrm{I} \not\subset \mathrm{P}$.

If, conversely, there is an $a \varepsilon m\mathrm{I} \backslash \mathrm{P}$, then $a(1 - i) = 0$ for some $i \varepsilon \mathrm{I}$. Since $a \notin \mathrm{P}$, $1 - i \varepsilon \mathrm{P}$ and $\mathrm{I} + \mathrm{P} = \mathrm{R}$. This completes the proof of (a).

To get (b) from (a), it suffices to show that $\sqrt{m\mathrm{I}}$ is the intersection of all the minimal prime ideals containing it. It follows from [7, Theorem 10], that $m\mathrm{I}$ is the intersection of all the prime ideals of R such that $\mathrm{P}/_{m\mathrm{I}}$ is a minimal prime ideal of $\mathrm{R}/_{m\mathrm{I}}$. Suppose a is an element of such a prime ideal P. Then there is a $b \notin \mathrm{P}$ and a positive integer n such that $(ab)^n \varepsilon m\mathrm{I}$. Hence $a^n b^n(1 - i) = 0$ for some $i \varepsilon \mathrm{I}$. Suppose $b^n(1 - i)\varepsilon\mathrm{P}$. Now $b \notin \mathrm{P}$, so $(1 - i)\varepsilon\mathrm{P}$ since P is a prime ideal. Thus $\mathrm{I} + \mathrm{P} = \mathrm{R}$, and by (a), $m\mathrm{I} \not\subset \mathrm{P}$. This contradiction shows that $a^n[b^n(1 - i)] = 0$ and $b^n(1 - i) \notin \mathrm{P}$. Hence $\mathrm{P}\varepsilon\mathcal{D}(\mathrm{R})$ and (b) holds.

Clearly (c) follows from (b).

If $a\varepsilon\sqrt{m\mathrm{I}}$, then $a^n \varepsilon m\mathrm{I}$ for some positive integer n. So there is an $i\varepsilon\mathrm{I}$ such that $a^n(1 - i) = 0 = [a(1 - i)]^n$. Since R is semiprime, $a(1 - i) = 0$ and $a\varepsilon m\mathrm{I}$. Thus (d) holds.

For any ring R, let G(R) denote the multiplicative semigroup generated by $\{(1 - i): i\varepsilon\mathrm{R}\backslash\mathcal{U}(\mathrm{R})\}$ and let $\mathrm{T}(\mathrm{R}) = \{a\varepsilon\mathrm{R}: ax = 0$ for some $x\varepsilon\mathrm{G}(\mathrm{R})\}$. Note that T(R) is an ideal of R which is proper if and only if $0 \notin \mathrm{G}(\mathrm{R})$. Also, $\mathrm{S}(\mathrm{R}) \subset \mathrm{T}(\mathrm{R})$. For, if $a\varepsilon\mathrm{S}(\mathrm{R})$, then there is a finite set $\{\mathrm{M}_1,\ldots,\mathrm{M}_n\}$ of maximal ideals, and elements $m_i\varepsilon\mathrm{M}_i$ for $i = 1,\ldots,n$ such that $a\varepsilon\Sigma_{i=1}^n \mathcal{A}(1 - m_i)$. Then $a\Pi_{i=1}^n(1 - m_i) = 0$, so $a\varepsilon\mathrm{T}(\mathrm{R})$.

4. **Proposition.** *The following properties of a minimal prime* P *of a ring* R *are equivalent*

(a) $\mathrm{P} \subset \mathrm{J}(\mathrm{R})$.

(b) $\mathrm{P} \supset \mathrm{S}(\mathrm{R})$.

(c) $\mathrm{P} \supset \mathrm{T}(\mathrm{R})$.

Proof. If $\mathrm{P} \subset \mathrm{J}(\mathrm{R})$ and M is a maximal ideal of R, then $\mathrm{P} \subset \mathrm{M}$. Hence by Proposition 3, $m\mathrm{M} \subset \mathrm{P}$, so $\mathrm{S}(\mathrm{R}) = \Sigma\{m\mathrm{M}:\mathrm{M}\varepsilon\mathcal{M}(\mathrm{R})\} \subset \mathrm{P}$. Thus (a) implies (b).

Suppose next that there is an $a\varepsilon\mathrm{T}(\mathrm{R})\backslash\mathrm{P}$. Then there is an $x\varepsilon\mathrm{G}(\mathrm{R})$ such that $ax = 0\varepsilon\mathrm{P}$. Since $a \notin \mathrm{P}$, we have $x\varepsilon\mathrm{P}$. Since $x\varepsilon\mathrm{G}(\mathrm{R})$, there is finite set $\{\mathrm{M}_1,\ldots,\mathrm{M}_n\}$ of maximal ideals of R and elements $m_i\varepsilon\mathrm{M}_i$ for $i = 1,\ldots,n$ such that $x = (1 - m_1)\ldots(1 - m_n)\varepsilon\mathrm{P}$. Hence $(1 - m_i)\varepsilon\mathrm{P}$ for some i, so $\mathrm{P} + \mathrm{M}_i = \mathrm{R}$. By Proposition 3, $m\mathrm{M}_i \not\subset \mathrm{P}$ and therefore $\mathrm{P} \not\supset \mathrm{S}(\mathrm{R})$. Thus we have shown that (b) implies (c).

If $P \supset T(R)$, then $P \supset S(R) \supset mM$ for every maximal ideal M of R. So, by Proposition 3, $P \subset J(R)$. Thus (c) implies (a) and the proof of Proposition 4 is complete.

Since every proper ideal of R is contained in a prime ideal, the following corollary follows immediately from Proposition 4 and the remarks preceding it. It may also be derived easily from [2, Proposition 3.3].

5. COROLLARY. *The Jacobson radical of a ring* R *contains a prime ideal if for every positive integer* n, *whenever* $m_1,\ldots,m_n$ *is a finite set of non units of* R, *it follows that* $\Pi_{i=1}^{n}(1 - m_i) \neq 0$.

Another easy consequence of Proposition 4 follows.

6. COROLLLARY. *If* R *is a ring with pseudoprime Jacobson radical* J(R), *and* P *is a minimal prime ideal of* R *such that* $P \supset J(R)$, *then* $P = J(R)$.

PROOF. Since J(R) is pseudoprime and $P \supset J(R)$, then $P \supset J(R) \supset S(R)$. Hence by Proposition 4, $P \subset J(R)$, so $P = J(R)$.

The next theorem and its corollaries solves the problem posed by M. Larsen, W. Lewis, and R. Shores in [9, p. 233]. Recall that if I_1 and I_2 are proper ideals of a ring R and $I_1 + I_2 = R$, then I_1 and I_2 are said to be *co-maximal.*

7. THEOREM. *Suppose* R *is a ring with pseudoprime Jacobson radical.*

(a) *If* S(R) *contains a prime ideal* P, *then* $P = S(R)$ *is the unique minimal prime ideal of* R *contained in* J(R).

(b) If $\sqrt{mP}$ *is a prime ideal, then* $P = N(R)$.

PROOF. The prime ideal P contains a minimal prime ideal P_0, and by Lemma 2, $P_0 \subset P \subset S(R) \subset J(R)$. By Proposition 4, $S(R) \subset P_0$, so $P_0 = P = S(R)$. Using Proposition 4 again yields that S(R) is the unique minimal prime ideal contained in J(R), and (a) holds.

If $\sqrt{mP} = Q$ is a prime ideal, then by Proposition 4, $Q \varepsilon \mathcal{P}(R)$. But $\sqrt{mP} \subset P$, so $\sqrt{mP} = P \subset J(R)$. By Lemma 1, $mP = m\sqrt{mP} = \{0\}$. Hence $P = \sqrt{\{0\}} = N(R)$, and (b) holds.

8. Corollary. *The following properties of a ring* R *are equivalent.*

(a) J(R) *is pseudoprime and there is a minimal prime ideal* P of R *co-maximal with every other minimal prime ideal of* R

(b) N(R) *is a prime ideal.*

Proof. If (a) holds, then $\sqrt{mP} = P \subset J(R)$ by Proposition 3 and Lemma 2. Hence S(R) contains a prime ideal, so (b) holds by Theorem 7.

If (a) holds, then $N(R) \subsetneq J(R)$ and N(R) is the unique element of $\mathcal{D}(R)$. So (a) holds and Corollary 8 follows.

A ring is called *arithmetical* if its lattice of ideals is distributive. In [6, Corollary 2] C. Jensen has shown that incomparable prime ideals of an arithmetical ring are co-maximal. Hence we have:

9. Corollary. *If the Jacobson radical* J(R) *of an arithmetical ring* R *is pseudoprime, then* N(R) *is a prime ideal contained in* J(R).

3. Other classes of rings whose Jacobson radicals are pseudoprime

In [1], A. Bouvier calls a ring R *presimplifiable* if whenever $x,y \in R$ and $xy = x$, then $x = 0$ or y is a unit, and the studies factorization properties of such rings. By Lemma 1, R is presimplifiable if and only if $S(R) = \{0\}$. These rings are characterized in the next theorem.

10. Theorem. *The following properties of a ring* R *are equivalent.*

(a) R *is presimplifiable*

(b) $mM \subset N(R)$ *for every maximal ideal* M *of* R.

(c) *Every minimal prime ideal of* R *is contained in* J(R).

(d) *Every proper divisor of* 0 *in* R *is contained in* J(R).

Proof. If R is presimplifiable and M is a maximal ideal of R, then $mM = \{0\} \subset N(R)$, so (a) implies (b).

If (b) holds, then $mM \subset N(R) \subset P$ for every $P \in \mathcal{D}(R)$ and maximal ideal M of R. So, by Proposition 3, if $P \in \mathcal{D}(R)$, then P is contained in every maximal ideal of R. That is, $P \subset J(R)$, so (c) holds.

Every proper divisor of 0 is contained in some minimal prime ideal of R [4, Section 2], so (c) implies (d).

If (d) holds, then $\mathcal{A}(a) \subset M \subset J(R)$ for every maximal ideal M of R. Hence by Lemma 1, $mM = \{a \varepsilon R : M + \mathcal{A}(a) = R\} = \{0\}$, so R is presimplifiable. This completes the proof of Theorem 10.

In the remainder of the paper, rings satisfying various chain conditions, and which have a pseudoprime Jacobson radical are studied.

Suppose R is a ring (which does not necessarily have an identity element). If $A \subset R$, let $h(A) = \{P \varepsilon \mathcal{P}(R) : A \subset P\}$, and if $\mathcal{S} \subset \mathcal{P}(R)$, let $k(\mathcal{S}) = \bigcap \{P \varepsilon \mathcal{P}(R) : P \varepsilon \mathcal{S}\}$. If we call a subset $\mathcal{S}$ of $\mathcal{P}$ *closed* if $\mathcal{S} = hk(\mathcal{S})$, then it is known that $\mathcal{P}(R)$ becomes a Hausdorff topological space with $B = \{h\mathcal{A}(a) : a \varepsilon R\}$ as a base for its open sets. Moreover, for any $a \varepsilon R, h\mathcal{A}(a) \bigcap h(a) = \varnothing$ and $h\mathcal{A}(a) \bigcup h(a) = \mathcal{P}(R)$, so the hull of each element of R is both closed and open. Moreover, $\mathcal{P}(R)$ and $\mathcal{P}(^{R}/N(R))$ are homeomorphic. [4, Section 2].

If R is semiprime and for every $x, y \varepsilon R$, there is a $z \varepsilon R$ such that $\mathcal{A}(x) \bigcap \mathcal{A}(y) = \mathcal{A}(z)$, R is said to satisfy the *annhilator condition*, or to be an *a.c.-ring*. The following assertions are proved in [4, Theorem 3,4]. Recall that if $a \varepsilon R$ is not a proper divisor of 0, then a is called a *regular element* of R, and an ideal containing a regular element is called a *regular ideal* of R.

11. LEMMA. (*Henriksen and Jerison*). *The following properties of a semiprime ring* (*not necessarily with an identity element*) *are equivalent.*

(a) $\mathcal{P}(R)$ *is compact and satisfies the annhilator condition.*

(b) $[h(a)\colon a \varepsilon R\}$ *is a base for the open subsets of* $\mathcal{P}(R)$.

If (a) *holds, then*

(c) R *has a regular element, and*

(d) *a proper ideal of* R *is contained in a minimal prime ideal of* R *if* (*and only if*) *it is not regular.*

The next lemma is probably known, but does not seem to appear in the literature.

12. Lemma. *If* R *is a semiprime ring* (*not necessarily with an identity element*) *and* $\mathcal{P}$(R) *is finite, then* R *satisfies the annhilator condition.*

Proof. By Lemma 11 (a,b,c), if $S \subset \mathcal{P}(R)$, there is an $a \varepsilon R$ such that $h(a) = S$. Hence if $x,y \varepsilon R$, there is a $z \varepsilon R$ such that $h(z) = h(x) \cap h(y)$. By [4, Lemma 3.1], since R is semiprime, $\mathcal{A}(z) = \mathcal{A}(x) \cap \mathcal{A}(y)$ and R an a.c.-ring.

13. Proposition. *The following properties of an a. c.-ring* R *such that* $\mathcal{P}$(R) *is compact are equivalent*

(a) J(R) *contains a prime ideal of* R.

(b) S(R) *is not a regular ideal.*

Proof. If (a) holds, then J(R) contains a $P \varepsilon \mathcal{P}(R)$, by Proposition 4, $S(R) \subset P$. But no element of a minimal prime ideal is regular, so (b) holds.

If (b) holds, then by Lemma 11 (c), S(R) is contained in some $P \varepsilon \mathcal{P}(R)$. So by Proposition 4, (b) holds.

The following corollary is an immediate consequence of Lemma 12 and Proposition 13.

14. Corollary. *If* R *is a semiprime ring such that* $\mathcal{P}$(R) *is finite, then* J(R) *contains a prime ideal if and only if* S(R) *is not a regular ideal.*

15. Remarks. (a) The hypothesis of Corollary 14 is satisfied if R is a semiprime ring that satisfies the ascending chain condition on annhilator ideals [7, Theorem 88], or if R has few zero divisors in the sense of [10. p. 152].

(b) Since $N(R) \subset J(R)$ and $N(R) \subset P$ for every $P \varepsilon \mathcal{P}(R)$, it follows easily that J(R) is pseudoprime (resp. J(R) contains a prime ideal of R) if and only if $J\left({}^{R}/N(R)\right)$ is pseudoprime (resp. $J\left({}^{R}/N(R)\right)$ contains a prime ideal of N(R)).

Next, I examine consequences of the assumption that mI is finitely generated. For any ideal I of R let $\mathfrak{J}$(I) denote the set of finitely generated ideals F of I such that FI = F. It is shown in [7, Theorem 76] that:

(1) *If* $F \varepsilon \mathfrak{F}(I)$, *there is an* $i \varepsilon I$ *such that* $a(1 - i) = 0$ *for all* $a \varepsilon F$. *That is*, $F \subset mI$.

Suppose I is an ideal of a ring R. If $ab \varepsilon I$ and $a \notin I$ imply $b \varepsilon \sqrt{I}$, the I is called a *primary ideal* The radical of a primary ideal is a prime ideal [13, p. 152]. If whenever A and B are ideals of R, $AB \subset I$, and $A \not\subset I$ imply $B^n \subset I$ for some positive integer n, then I is called a *strongly primary* ideal. It is known that a pirmary ideal with finitely generated radical is strongly primary [13, p. 200, proof of 2)].

Let $I^{\omega} = \bigcap_{n=1}^{\infty} I^n$, and note that if $a \varepsilon mI$, there is an $i \varepsilon I$ such that $a = ai = ai^2 = \ldots = ai^n$ for every positive integer n. Thus $mI \subset I^{\omega}$.

16. Proposition. *Suppose* I *is an ideal of a ring* R.

(a) *If* mI *is finitely generated, then* mI *is the largest element of* $\mathfrak{F}(I)$ *and* $mI = \mathcal{A}(1 - i)$ *for some* $i \varepsilon I$.

(b) *If* I^{ω} *is finitely generated, then* $mI = I^{\omega}$ *if and only if* $I^{\omega}I = I^{\omega}$.

(c) *If* I^{ω} *is finitely generated and* $I^{\omega}I$ *is an intersection of strongly primary ideals, then* $mI = I^{\omega}$.

(d) *If* R *is Noetherian, then* $mI = I^{\omega}$.

Proof. Since $(mI)I = mI$, (a) follows from (1), and (b) follows from (a) and the fact that $mI \subset I^{\omega}$.

Suppose $I^{\omega}I$ is contained in a strongly primary ideal Q. If $I \not\subset \sqrt{Q}$, then $I^{\omega} \subset Q$ since Q is primary. If $I \subset \sqrt{Q}$, then there is a positive integer n such that $I^{\omega} \subset I^n \subset Q$ since Q is strongly primary. Hence $I^{\omega}I = I^{\omega}$ and (c) follows from (b).

Finally (d) follows from (c) since every ideal of a Noetherian ring is an intersection of (strongly) primary ideals [11, p. 199].

Proposition 16 (d) is also proved in [12, p. 49].

The next two examples show that some of the assumptions made in Proposition 16 (c) are necessary.

17. EXAMPLE. *An integral domain* D_1 *such that if* M *is a maximal ideal of* D_1 *then* $M^\omega M = J(D_1)$ *is a prime ideal, but* $mM \neq M^\omega$.

Let D_1 denote the ring of formal power series $a(x) = \Sigma_{n=0}^{\infty} a_n x^n$ with rational coefficients such that $a(0) = a_0$ is an integer. As is noted in [3, p. 162], M is a maximal ideal of D_1 if and only if there is a prime integer p such that $M_1 = pD_1$. Moreover $(pD_1)^\omega = \{a(x) \varepsilon D_1 : a(0) = 0\} = J(D_1)$, and, clearly $(pD_1)^\omega (pD_1) = (pP_1)^\omega$. Since D_1 is an integral domain $m(pD_1) = \{0\} \neq (pD_1)^\omega$. Note that $(pD_1)^\omega$ is not finitely generated since for $n = 0,1,2,\ldots$, $\left(\frac{1}{2^n} x\right) D_1$ is a strictly ascending chain of ideals contained in $(pD_1)^\omega$.

18. EXAMPLE. *An integral domain with a prime ideal* P *such that* P^ω *is both prime and principal, but* $mP \neq P^\omega$.

If D_1 is the ring of Example 17, let $D_2 = D_1 [[y]]$ denote the the ring of formal power series with coefficients in D_1. Let

$$P = \{a(y) = \Sigma_{n=0}^{\infty} a_n(x) y^n : a_n(x) \varepsilon D_1 \text{ for } n \geqslant 0 \text{ and } a_0(x) \varepsilon J(D_1)\}.$$

Thus $a(y) \varepsilon P$ if and only if when we write $a_0(x) = \Sigma_{n=0}^{\infty} a_{0n} x^n$, we have $a_{0n} = 0$. It is easily verified that P is a prime ideal, and $P^\omega = \{a(y) \varepsilon D_2 : a(0) = 0\} = yD_2$ is also a prime ideal. Since D_2 is an integral domain, $mP = \{0\} \neq yD_2 = P^\omega$. Note finally that $\sqrt{PP^\omega} = \sqrt{P \cap P^\omega} = \sqrt{P^\omega} = P^\omega$ is a prime ideal, but, by Proposition 16, PP^ω is not an intersection of strongly primary ideals.

The next proposition provides another sufficient condition for J(R) to contain a prime ideal.

19. PROPOSITION. *Suppose* P *is a minimal prime ideal of a ring* R *such that*

(i) P *is finitely generated, and*

(ii) *there is a maximal ideal* $M \supset P$ *and an ideal* B *of* R *for which* $P = MB$.

Then:

(a) $\sqrt{mP} = P$ *if* $P = M$ *and* $mM = P$ *if* $P \neq M$.

(b) *If* J(R) *is pseudoprime, then it contains a unique minimal prime ideal of* R.

PROOF. If $P = M = MR$, then (a) holds by Proposition 3. If $P \neq M$, then $B \subset P$ since P is prime, and $P = MB \subset MP \subset P$. Thus

P = MP, so $P \subset mM$ by (1) and $mM \subset P$ by Proposition 3. Hence $P = mP$ and (a) holds in this case as well.

Part (b) follows from (a) and Theorem 7.

An ideal B of a ring R is called a *multiplication ideal* if whenever A is an ideal of R such that $A \subset B$, there is an ideal C of R such that A = BC. If every ideal of R is a multiplication ideal, then R is called a *multiplication ring*. The ring R is called an *almost multiplication ring* if every ideal with a prime radical is a power of its radical. The following facts are known.

(2) *Every multiplication ring is an almost multiplication ring and every Noetherian almost multiplication ring is a multiplication ring* [10, p. 216 and p. 213, Theorem 9.21].

(3) *If* P *is a prime ideal and* M *is a maximal ideal of an almost multiplication ring such that* $P \subset M$ *and* $P \neq M$, *then*

P = MP. [10, p. 224, Ex. 9]

With the aid of (2) and (3) the following consequences of Proposition 19 follow.

20. Theorem. *If the Jacobson radical* J(R) *of a ring* R *is a pseudoprime multiplication ideal and if every radical ideal of* R *contained in* J(R) *is finitely generated, then* R *is an integral domain or* J(R) *is a minimal prime ideal of* R. *In particular, the Jacobson radical of a Noetherian* (*almost*) *multiplication ring contains a unique minimal prime.*

Proof. By Proposition 19 and (3), J(R) contains a unique minimal prime ideal P. Since J(R) is a multiplication ideal, if $P \neq J(R)$ there is an ideal B of R such that P = J(R)B. Since P is prime, $B \subset P$, so $P = J(R)B \subset J(R)P \subset P$, and hence P = J(R)P. Hence by (1) and Lemma 1, $P \subset mJ(R) = \{0\}$. Thus R is an integral domain. This completes the proof of the theorem.

The next example shows that a Noetherian ring may have a pseudoprime Jacobson radical which contains no prime ideal.

21. Example. *A semiprime Noetherian ring* R *with pseudoprime Jacobson radical* J(R) *which has exactly three minimal prime ideals, none of which are in* J(R). If F is any field, let $T = F[x_1,x_2,x_3]$ denote

the ring of polynomials in three indeterminates x_1,x_2,x_3. Let $I = x_1x_2T + x_1x_3T + x_2x_3T$, and let $T^* = \left\{\frac{a}{1-i} : a\varepsilon T, i\varepsilon I\right\}$ denote the quotient ring of R with respect to the multiplicative system $\{1-i : i\varepsilon I\}$. Finally, let $R = T^*/(x_1x_2x_3)T^*$, and let $\bar{b} = b + x_1x_2x_3T^*$ for any $b\varepsilon T^*$.

Since T is a Noetherian unique factorization domain, R is Noetherian, and each of its proper divisors of 0 is a multiple of $\bar{x}_1,\bar{x}_2$, or $\bar{x}_3$. Clearly, also, $\bar{I} = \bar{x}_1\bar{x}_2R + \bar{x}_1\bar{x}_3R + \bar{x}_2\bar{x}_3R \subset J(R)$, and it follows that J(R) is pseudoprime. Since every element of a minimal prime ideal is a proper divisor of 0, the minimal prime ideals of R are $P_i = \bar{x}_i\bar{R}$ for $i = 1,2,3$, none of which are contained in J(R) since $\bar{1} - \bar{x}_i$ is not a unit of R. Finally, R is sempirime because $\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3 = \{\bar{0}\}$.

In view of Example 22, the following proposition may not seem so special.

22. Proposition. *If* R *is a ring with no more than two minimal prime ideals and* J(R) *is pseudoprime, then* J(R) *contains a prime ideal.*

Proof. If R has exactly one minimal prime ideal, it must be $N(R) \subset J(R)$. Suppose R two minimal prime iedals P_1,P_2. By Remark 15(b), we may assume that R is semiprime. By Proposition 3, if $M\varepsilon\mathfrak{M}(R)$, then mM is P_1,P_2, or $P_1 \cap P_2 = \{0\}$. Hence $S(R) = \{0\}$. or S(R) contains a prime ideal. In the first case, the conclusion follows from Theorem 10, and in the second case it follows from Theorem 7.

I conclude with an example that shows that the hypothesis of Proposition 22 can be satisfied for a ring R without R being presimplifiable.

23. Example. *A semiprime Noetherian ring* R *with two minimal prime ideals such that* $J(R)\varepsilon\mathfrak{P}(R)$ *and* R *is not presimplifiable.* Let S denote the ring of formal power series with 0 constant term with coefficients from the ring of integers mod 2. clearly S is Noetherian and $J(S) = S$. If Z denotes the ring of integers, let $R = S^*Z = \{(a,n) : a\varepsilon S, n\varepsilon Z\}$ where for $a_1,a_2\varepsilon R, n_1n_2\varepsilon Z, (a_1,n_2) + (a_2,n_2) = (a_1,n_1) + (a_2,n_2)$ and $(a_1,n_1)(a_2,n_2) = (a_1a_2 + n_2a_1 + n_1a_2, n_1n_2)$. It is well known that R is a Noetherian ring with identity and the mapping

$a \to (a,0)$ is an injection of S onto a prime ideal $\bar{S}$ of R. It is easily verified that $\bar{S} = J(R)$. Also since $(a,0)(0,2) = (0,0)$ for every $a \varepsilon S$, $J(R) = \bar{S}$ is a minimal prime ideal of R. By the same reasoning $P = \{(0,2n) : n \varepsilon Z\} \varepsilon \mathcal{P}(R)$, and any other prime ideal of R contains a regular element. So $\mathcal{P}(R) = \{J(R),P\}$, and R is not presimplifiable since $P \not\subset J(R)$. Finally, R is semiprime since $P \cap J(R) \subset = \{0\}$.

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