On Commensurability and Symmetry

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Abstract

Commensurability and symmetry have diverged from a common Greek origin. We review the history of this divergence. In mathematics, symmetry is now a kind of measure that is different from size, though analogous to it. Size being given by numbers, the concept of numbers and their equality comes into play. For Euclid, two magnitudes were symmetric when they had a common measure; also, numbers were magnitudes, commonly represented as bounded straight lines, for which equality was congruence. When Billingsley translated Euclid into English in the sixteenth century, he used the word "commensurable" for Euclid's symmetric magnitudes; but the word had been used differently before. Symmetry has always had also a vaguer sense, as a certain quality that contributes to the beauty of an object. Today we can precisely define the symmetry of a mathematical structure as the automorphism group of the structure, or as the isomorphism class of that group. However, when we consider symmetry philosophically as a component of beauty, we can have no foolproof algorithm for it.

1. Introduction

As a possible feature of two or more magnitudes, **commensurability** is the sharing of a common measure. Two feet and three feet are commensurable, each being a multiple of a foot; but the diagonal and side of a square are incommensurable.

As a possible feature of a shape, **symmetry** can be understood as the existence of a nontrivial measurement of the shape by itself. An isosceles triangle, flipped over, will still occupy its original position: the triangle is to that extent symmetric. An equilateral triangle is symmetric in five different ways.

In more technical language, **a symmetry** is an isometric permutation, trivial or not. A scalene triangle has only the trivial symmetry; an isosceles triangle, two symmetries; an equilateral triangle, six. More generally, every structure in the sense of model theory has *automorphisms;* even more generally, so does every object in a category. These automorphisms could be called symmetries.

Commensurability and symmetry are thus two distinct mathematical notions, though they can both be understood to spring from the notion of measurement. They also share a common linguistic origin. The adjective "commensurable" is the Anglicized form of the Latin COMMENSURABILIS, which is itself a loan-translation of the Greek *σύμμετρος*. The corresponding Greek abstract noun *συμμετρία* comes to us as "symmetry" via the Latin transliteration SYMMETRIA.

Thus "commensurability" and "symmetry" are cognate words, even doublets, in the sense of deriving from the same Greek source. "Analogy" and "proportion" are doublets in the same way, the latter word deriving from a Latin loan-translation of the Greek origin of the former. Indeed, the word "analogy" thus becomes one term in an analogy: as commensurability is to symmetry, so is proportion to analogy. However, this is an analogy of etymologies, not of concepts. The words "analogy" and "proportion" are nearly synonyms, though the former is perhaps looser or more abstract. The looseness of analogy is a theme of William M. Priestley in "Wandering About: Analogy, Ambiguity, and Humanistic Mathematics" [52] and will be taken up later in this Introduction.

In *Poetry and Mathematics* (which is one of Priestley's references), Scott Buchanan has a chapter called "Proportions"; but this is generally about analogy, which Buchanan defines as,

> the statement of the identity or similarity of at least two relations
> . . . Of course these relations may be of any degree of complexity,
> provided the identity or similarity is not violated. The complexity
> may be increased or diminished, apparently without limit. I shall
> call this property of analogies their expansiveness. [14]

Buchanan does not define proportions as such, though when he uses the term, it is with a mathematical meaning, as when he observes,

> Archimedes laid the foundation of a very permanent, at least a recurring, form of intellectual equilibrium by studying equilibrium in its physical forms. The problem is epitomized in the lever, and the principle of the lever is a proportion. [14]

For the Greeks, magnitudes *A*, *B*, *C*, and *D* are proportional when *A* has the same ratio to *B* that *C* has to *D*. Buchanan remarks on the ratio and its etymology:

> The symbolic key to many mathematical treasures is the ratio . . . At this point there is a fortunate linguistic bridge between poetry and mathematics . . . What we call reason was referred to by the Greeks as λόγος and by the Romans as *ratio.* We refresh our classical memory by associating "logical" and "rational" in English. Lying back of these words are distinct but related *Weltanshauungen* . . . *Logos* is still commemorated in the names of most of our sciences; *ratio* goes with our popular and practical argumentation. We rationalize. [14]

Thus we can add a third pair to our analogy or proportion:

  commensurability : symmetry :: proportion : analogy :: rationality : logic.

Only in the first pair can the two terms be given precise mathematical definitions that are distinct from one another. How has this come to be?

The original intention of this essay was just to record my research into the various senses in which "commensurability" and "symmetry" have been used in the last two thousand years or so. One conclusion of this research is that measurement is the common aspect of the two concepts. An isosceles triangle is symmetric because it is congruent to its mirror image. The triangle thus measures its image, and so the triangle is, so to speak, "commensurable" with its image. In Greek then, one would say that the triangle and its image were "symmetric." However, I have not found that anybody actually did this in ancient times.

Researching symmetry leads to a few suggestions or recommendations for mathematical practice. One suggestion is that textbooks mentioning symmetry ought to define it. I have been inspired by a slogan from the textbook *Groups and Symmetry* by M. A. Armstrong [10]:

Numbers measure size, *groups measure symmetry.*

This is how Armstrong begins his Preface. The slogan makes a good case for why the theory of groups is worthy of study. However, as far as I can tell, Armstrong never defines symmetry explicitly. The word does not appear in the index of his book. The adjective form "symmetric" does appear, as the first element of the phrase "symmetric group," and this has one reference.

Perhaps Armstrong's slogan is to be taken as an implicit definition of symmetry. Groups get an explicit axiomatic definition in Armstrong's Chapter 2, "Axioms." Symmetry then might be understood as whatever a group can be used to measure.

Intelligence has been defined as whatever an IQ test measures. However, this definition was made derisively, in 1923, by Edwin Boring [12], who said,

> Thus we see that there is no such thing as a test for pure intelligence. Intelligence is not demonstrable except in connection with some special ability. It would never have been thought of as a separate entity had it not seemed that very different mental abilities had something in common, a "common factor."

If we have no independent sense of this "common factor," then we have no way to judge the accuracy of intelligence tests: accuracy becomes a meaningless notion.

I encountered a reference to Boring in "Fifty psychological and psychiatric terms to avoid: A list of inaccurate, misleading, misused, ambiguous, and logically confused words and phrases" by Lilienfeld *et al.* [37]. One term that the authors recommend avoiding is "Operational definition":

> Operational definitions are strict definitions of concepts in terms of their measurement operations. As a consequence, they are presumed to be exact and exhaustive definitions of these concepts. Perhaps the best known example in psychology is Boring's (1923) definition of intelligence as whatever intelligence tests measure . . . an "operational definition" of aggression as the amount of hot sauce a participant places in an experimental confederate's drink is not an operational definition at all, because no researcher seriously believes that the amount of hot sauce placed in a drink is a perfect or precise definition of aggression that

> exhausts all of its potential manifestations . . . an operational definition of length would imply that length as measured by a wooden ruler cannot be compared with length as measured by a metal ruler . . .

What is not good enough for psychology may be good enough for mathematics. There would seem to be a decent operational definition of *same length*: two objects have the same length when they can be applied to one another with no overlap in the direction of interest. We may well wish to give an operational definition of having the same symmetry, or of symmetry itself. How would we do it though? Whether there is any value in it or not, at least it is clear how to administer an IQ test. How would we administer a "symmetry test"? For example, if the symmetry of an object lies in its automorphism group, should we worry about how this group might actually be extracted?

From early childhood, we know how to administer a "size test." We can measure the size of a set by counting. To measure the size of a set *is* to count it, as to measure the heaviness of a body is to weigh it. However, as Georg Cantor observes, we can explain size without counting. Two sets **have the same size,** or **are equipollent,** if there is a one-to-one correspondence between them. By one definition then, **the size** of a set is its equipollence class, namely the class of all sets that have the same size as the original set. A **number** would then be the size of *some* set. This definition does not require counting.

Cantor still seems to want a number to be a particular element of an equipollence class. He tries to achieve this goal by letting the number of a set be the set itself, after all qualities have been abstracted whereby the elements of the set can be distinguished from one another. After this abstraction though, what keeps the set from collapsing to a set with a single element?

John von Neumann solves this problem, and perhaps his solution should be better known than it is, if only as an example of fundamental progress that is fairly recent in history. Ancient mathematicians such as Euclid are sometimes criticized for not doing mathematics according to current standards of rigor. Neither did Cantor meet such standards; but, like Euclid, he was helping to *create* the mathematics to which our standards could be applied. Von Neumann's example shows this.

There is a precise way to select from each equipollence class a standard element, and we can call *this* the number of each element of the class. For example, the sets having five elements are precisely those sets that can be put in one-to-one correspondence with the words "one," "two," "three," "four," and "five," by the process called counting. We can now think of **the number five** itself in two ways:

(1) as what all five-element sets have in common, or
(2) as the set of the five words listed above, or as some other standard set of five elements.

Von Neumann's 5 is the set $\{0, 1, 2, 3, 4\}$. If one wants a five-element set, it is handy to have von Neumann's 5 ready to serve. One may prefer to use $\{1, 2, 3, 4, 5\}$ as one's five-element set; but then one needs a new symbol for this set, instead of 5 itself. Still, in ordinary language, a collection of objects is a *number* of them. Thus we may think of **a number five**

(3) as *any* five-element set.

In this sense, the set $\{1, 2, 3, 4, 5\}$ is a five.

Our way of thinking of numbers depends on whether we consider *equal* numbers to be the *same* number. Equality of numbers might be considered to correspond to isomorphism of groups; and isomorphism is usually not sameness, though in practice we may blur the distinction between isomorphic groups.

In von Neumann's definition, equal numbers *are* the same number; but this has not always been the understanding of equality. As invented by Robert Recorde in 1557, the sign of equality that we use today is an *icon* of two distinct, but equal (and parallel), straight lines. Here Recorde follows the understanding of Euclid, from eighteen centuries earlier: equality of bounded straight lines is congruence, not sameness. When a bounded straight line is *measured,* whether twice or thrice or many times, by some specified unit length, then the original line is a number: it is the number of those lengths within itself that are counted out by means of the given unit length.

Organization of this paper. We shall start Section 2 by looking at Cantor's theory of sets. This will immediately raise the question of equality, and so we shall go back a few centuries to Recorde's treatment of the notion.

Returning to Cantor will raise more questions, of a kind that seem to persist in a geometry textbook that I happened to use in high school. Again we shall go back, all the way to the first textbook of all, the *Elements* of Euclid. Euclid's distinction between equality and identity has traces in another modern textbook (which I happened to encounter as a reference in a *Wikipedia* article). Finally, we move on to von Neumann's clarification of Cantor's ideas of number.

In Section 3, we shall take up Armstrong's slogan in earnest, making an investigation of the modern mathematical treatment of symmetry. I suggested that an analogy was more loosely defined than a proportion. In this sense, Armstrong's slogan is an analogy, rather than a proportion. In the strict sense of Euclid, when four magnitudes are in proportion, any three of them determine the fourth, at least up to equality; but if one knows how to count, and if one knows the axiomatic definition of a group, this does not mean that one can figure out what symmetry is, even if one is told that groups measure symmetry as numbers measure size.

Section 4 is a broader historical investigation of what has happened to the Greek notion of $\sigma\upsilon\mu\mu\epsilon\tau\rho\acute{\iota}\alpha$, which has given us both symmetry and commensurability. In particular we explore the notion through a detailed investigation of three senses of the word. First comes the geometrical sense, from Euclid and others to Billingsley. Then there is the numerical sense of the term which leads us to Boethius and Recorde. Finally we approach somewhat more philosophical questions involving aesthetic values as we explore the philosophical sense of the term, reaching all the way back to Plato and Aristotle. While symmetry can be understood as an aspect or component of *beauty,* this is not exactly the symmetry defined in terms of automorphism groups. However, one can sometimes, if not always, understand a negative conclusion in positive terms. We shall be able to do so in the present case.

2. Numbers and Size

2.1. Cantor's aggregates

Again, a number can be understood in either of two ways: as an equipollence class of sets, or else as a particular member of such a class, chosen once for all. In his *Contributions to the Founding of the Theory of Transfinite Numbers,*

Cantor initially takes something like the first approach. First he has to *define* sets, or what in translation from his German are called aggregates [15, §1, page 85]:

> By an **aggregate** (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects m of our intuition or our thought. These objects are called the **elements** of M. In signs we express this thus:
> $$M = \{m\}.$$

Here I transcribe Philip E.B. Jourdain's translation verbatim, down to his parenthetical inclusion of Cantor's German, although I do not know German myself. However, where Jourdain puts words between quotation marks, I put the words in boldface (if they are being defined) or in italics (if they are otherwise being emphasized). For the aggregate M, Jourdain uses an upright M, although its arbitrary element m is italic, as here.

After considering what we call *unions* of sets, and *subsets* of particular sets, Cantor continues [15, §1, page 86]:

> Every aggregate M has a definite *power,* which we will also call its *cardinal number.*
> We will call by the name **power** or **cardinal number** of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given.
> We denote the result of this double act of abstraction, the cardinal number or power of M, by
> $$\overline{\overline{M}}.$$

In writing the equation $M = \{m\}$, presumably Cantor asserts the *identity* of the whole M with the collection $\{m\}$ of objects. However, as we mentioned before, equality has not always meant sameness. Euclid *distinguishes* between equality and sameness. An isosceles triangle has two equal sides, but of course they are not the same side. By contrast, when four magnitudes are in proportion, this does not mean that, taken in pairs, they have *equal* ratios; they have the *same* ratio.

2.2. Recorde's equality

We may say that the two equal sides of an isosceles triangle have the same *length.* The sign "=" of equality is an *icon* of just this situation, in the precise sense of Charles Sanders Peirce [45, page 104]:

> A sign is either an *icon,* an *index,* or a *symbol.* An *icon* is a sign which would possess the character which renders it significant, even though its object had no existence; such as a lead-pencil streak as representing a geometrical line.

Robert Recorde had just this idea, when he introduced the equals sign in 1557. I want to pause here to consider Recorde's idea, in both its mathematical and its typographical context. Our ultimate concern is with the notions of symmetry and commensurability, as they have developed over time. Our notions are bound up with our ability to express them, and this ability itself has developed over time. Recorde's work is a reminder of this.

Recorde introduced the equals sign on the verso of folio **𝕱𝖋.𝖎.** (in roman font, Ff.i.) of *The Whetstone of Witte* [53]:

> Howbeit, for easie alteratiõ of equations. I will propounde a few exãples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of thele woordes : is equalle to : I will lette as I doe often in woorke ule, a paire of paralleles, or Gemowe lines of one lengthe, thus: ———— , bicause noe. 2. thynges, can be moare equalle. And now marke thele nombers.

> Howbeit, for easie alteration of *equations.* I will propounde a few examples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes : is equalle to : I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =, bicause noe. 2. thynges, can be moare equalle. And now marke these nombers.

The sign of equality consists of "gemowe lines." English has gathered many words to itself in the last thousand years, but not all of them have stuck. Recorde's "gemowe" is an obsolete word, found in the *Oxford English Dictionary* [42] under "gemew, gemow": it derives from the Old French plural

gemeaux, whose singular is *gemel.* The modern French singular for the same word is *jumeau,* meaning "twin," although the form *gémeau* was created in 1546, on the basis of the Latin GEMELLUS, the diminutive of GEMINUS, to indicate the sign of the Zodiac called in English "Gemini" [20, 54]. The older singular *gemel* also came into English, where, in the plural form "gemels," it is a heraldic term meaning "bars, or rather barrulets, placed together as a couple" [42]. Thus two gemels would seem to be like Recorde's sign of equality.

Recorde's passage above is reproduced in facsimile in the *Wikipedia* article "Equals sign." I have tried to reproduce the blackletter of Recorde's book by means of the 𝕲𝖔𝖙𝖍𝖎𝖈 font of the LATEX package called `yfont`. The package provides also 𝕾𝖈𝖍𝖜𝖆𝖇𝖆𝖈𝖍𝖊𝖗 and 𝕱𝖗𝖆𝖐𝖙𝖚𝖗 fonts. The Gothic font uses as many of Gutenberg's ligatures as possible [40, page 395]. Recorde's printer uses no obvious ligatures, except maybe between cee (𝖈) and tee (𝖙), albeit not with the loop of 𝖙. I have tried to maintain Recorde's spellings, including the tilde in place of a following en (as in 𝖔̃ for 𝖔𝖓). The `yfont` package does not provide the italic letters that Recorde's printer uses. In place of these, I have used Schwabacher, as for example to set the word 𝖊𝖖𝖚𝖆𝖙𝖎𝖔𝖓𝖘 (as opposed to 𝖊𝖖𝖚𝖆𝖙𝖎𝖔𝖓𝖘), which is italic in the original. Despite the evidence of Recorde's book, the use of Schwabacher (rather than italic) for emphasis within Gothic text is said to be "historical practice" [40, page 394].

Recorde's printer's numerals are not so heavy and stylized as in `yfont` Gothic. I try to follow the printer's use of periods, which come before and after most numerals, though not all.

Recorde's book is evidently a quarto. The sheets used in printing are numbered, and the four leaves that result from folding each sheet twice are numbered. On the recto of each of first three leaves is printed a letter for the number of the original sheet, followed by a Roman numeral for the number of the leaf. Thus what we should call pages 1, 3, 5, and 9 are designated respectively 𝕬.𝖎, 𝕬.𝖎𝖎, 𝕬.𝖎𝖎𝖎, and 𝕭.𝖎; the intervening pages are unmarked. The 23-letter Latin alphabet is used: 𝕬, 𝕭, 𝕮, 𝕯, 𝕰, 𝕱, 𝕲, 𝕳, 𝕵, 𝕶, 𝕷, 𝕸, 𝕹, 𝕺, 𝕻, 𝕼, 𝕽, 𝕾, 𝕿, 𝖀, 𝖃, 𝖄, and 𝖅. After this come the double-lettered sheets, 𝕬𝖆, 𝕭𝖇, 𝕮𝖈, and so on to 𝕽𝖗. The front matter consists of sheet 𝖆 for the title and 𝕿𝖍𝖊 𝕰𝖕𝖎𝖘𝖙𝖑𝖊 𝕯𝖊𝖉𝖎𝖈𝖆𝖙𝖔𝖗𝖎𝖊, and sheet 𝖇 for 𝕿𝖍𝖊 𝕻𝖗𝖊𝖋𝖆𝖈𝖊 𝖙𝖔 𝖙𝖍𝖊 𝖌𝖊𝖓𝖙𝖑𝖊 𝕽𝖊𝖆𝖉𝖊𝖗. Thus the book is made of 2+23+17 or 42 sheets, making 336 pages, except that there are oddities: the leaves 𝕽.𝖎. and 𝕯𝖔.𝖎𝖎𝖎.

are larger, with tables. Having the book only as a `pdf` image, I do not know how these larger leaves were made.

After defining his sign of equality, Recorde goes on to give several examples of equations, numbered in the left margin; with $\mathcal{AMS}$-LaTeX, I reproduce them as follows:

$$14.x. + .15.u = 71.u. \tag{1}$$
$$20.x. - .18.u = .102.u. \tag{2}$$
$$26.z + 10x = 9.z - 10x + 213.u. \tag{3}$$
$$19.x + 192.u = 10x + 108u - 19x \tag{4}$$
$$18.x + 24.u. = 8.z. + 2.x. \tag{5}$$
$$34z - 12x = 40x + 480u - 9.z \tag{6}$$

Periods are thus used freely, but inconsistently. I have approximated Recorde's peculiar indeterminates or "cossic signs" with Latin letters; see Figure 1 for a fascimile of the originals.

0.	1.	2.	3.	4.	5.	6.
u	x	z	y	zz	sz	zy

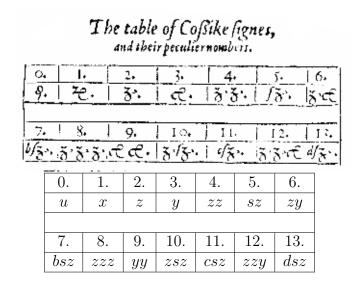| 7.	8.	9.	10.	11.	12.	13.
bsz	zzz	yy	zsz	csz	zzy	dsz

Figure 1: Recorde's cossic signs [53], with my transliteration.

One should understand as unity, or x^0, the symbol appearing as u above. Standing for another single symbol in Recorde, the z above has the meaning of our x^2. On the verso of folio **S.i.**, Recorde tells how to express each of what we should call the powers of x, from the zeroth to the twenty-fourth:

u	Betokeneth nomber abſolute aſ if it had no ſigne.
x	Signifieth the roote of any nomber.
z	Repreſenteth a ſquare nomber.
y	Expreſſeth a Cubike nomber.
zz	Is the ſigne of a ſquare of ſquares, or Zenzizenzike.
sz	Standeth for a Surſolide.
zy	Doeth ſignifie a Zenzicubike, or a ſquare of Cubes.
bsz	Doeth betoken a ſeconde Surſolide.
	. . . .
$zzzy$	Signifieth a ſquare of ſquares, of ſquared Cubes.

Recorde thus varies the word ("betokeneth, signifieth," *&c.*) used to say that the meaning of a sign is being given. Along with the zeroth and first, each prime power of what we call x is for Recorde a different new symbol. The fifth power, the *sursolid,* is obtained from the second power by prefixing an elongated ess, like our integral sign $\int$. The higher prime powers, from seventh to 23rd, are the second to sixth sursolids respectively; their symbols are obtained from that of the first sursolid by prefixing the letters from b to f. The symbols for composite powers are the appropriate composites of the symbols for prime powers. Six pages later (on the verso of the folio that would be numbered **S.iiii.**, if it were given a number), **The table of Coßike ſignes, and their peculier nombers** (Figure 1) gives what we should call the exponents for the first 14 signs, and it is explained that multiplying the signs corresponds to adding the exponents.

Recorde's peculiar indeterminates did not catch on. His sign of equality did, though not right away. Eighty years later, in 1637, as an example of a solution of a four line locus problem, Descartes wrote out an equation as

$$yy \infty 2y - xy + 5x - xx$$

[18, page 333], and this has the meaning that we have learned from him (though we have usually forgotten that x and y, while measured in different directions, need not be orthogonal). Descartes wrote yyy as y^3, and so forth, as we do, though instead of y^2 he still wrote yy (" *Cependent Descartes répète presque toujours les facteurs égaux lorsqu'ils ne sont qu'au nombre de deux*" [19, page 2, note 1]). I have written Descartes's sign of equality as ∞, and this is how it is written in the Hermann edition of *La Géométrie,* in the note that explains that the sign has been replaced with $=$ in the text [19, page 3, note 1].

The Dover facsimile edition shows that the left loop of ∞ is usually broken, though it is intact in one case that I could find [18, page 333], where the equation quoted above is being developed.

Without pursuing the matter further, I can only speculate that the equals sign of Recorde was ultimately found superior, precisely for its mnemonic value as an icon of two equal, but distinct, straight lines. Equality in origin is not sameness, though today we use the sign of equality to indicate that two different expressions denote the same thing. This is what Cantor will do explicitly.

2.3. Cantor's cardinal numbers

Cantor's cardinal number or power $\overline{\overline{M}}$ of the aggregate M is, as he says, a "general concept." This is as vague as "what all sets having the size of M have in common." However, Cantor has not yet defined having the same size. He immediately starts groping towards a second approach to number, where a number is a standard element of an equipollence class:

> Since every single element m, if we abstract from its nature, becomes a **unit,** $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M.
> We say that two aggregates M and N are **equivalent,** in signs
> $$M \sim N \text{ or } N \sim M,$$
> if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.

(Cantor's symbol for equipollence, at least in Jourdain's translation, is curvier than the $\sim$ of TEX.) Cantor goes on to observe that "equivalence" (what we have called equipollence, or having the same size) is indeed what we now call an equivalence relation: it is symmetric (as above), reflexive, and transitive. Moreover,

Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if, they are equivalent: thus,

$$\text{from } M \sim N \text{ we get } \overline{\overline{M}} = \overline{\overline{N}},$$

and

$$\text{from } \overline{\overline{M}} = \overline{\overline{N}} \text{ we get } M \sim N.$$

Thus the equivalence of aggregates forms the necessary and sufficient condition for the equality of their cardinal numbers.

Here is where sameness and equality are explicitly confused. In any case, Cantor derives his latter implication from the general equivalence

$$M \sim \overline{\overline{M}}$$

and the transitivity of equivalence. The former implication might be said to follow similarly from the implication

$$\overline{\overline{M}} \sim \overline{\overline{N}} \implies \overline{\overline{M}} = \overline{\overline{N}};$$

but Cantor himself does not seem to suggest such an intermediate step. He argues:

In fact, according to the above definition of power, the cardinal number $\overline{\overline{M}}$ remains unaltered if in the place of each of one or many or even all elements m of M other things are substituted. If, now, $M \sim N$, there is a law of co-ordination by means of which M and N are uniquely and reciprocally referred to one another; and by it to the element m of M corresponds the element n of N. Then we can imagine, in the place of every element m of M, the corresponding element n of N substituted, and, in this way, M transforms into N without alteration of cardinal number. Consequently

$$\overline{\overline{M}} = \overline{\overline{N}}.$$

One may question the validity of this argument, just as one may question some of Euclid's arguments. We engage in such questioning in the next subsection.

2.4. Ambiguity of equality

In the fourth proposition of Book ɪ of the *Elements* [22], Euclid proves what today we call the "Side-Angle-Side" or SAS condition for congruence of triangles. We may say that the proposition is not a theorem, but a postulate, as it is for example in the Weeks–Adkins textbook that I used in high school [60, page 61]. Nonetheless, Euclid gives a proof; but here he "applies" one triangle to another, and this is not accounted for among his postulates. As Fitzpatrick says in a note to his own translation, "The application of one figure to another should be counted as an additional postulate" [26, page 11]. In that case though, Cantor would seem to need such a postulate; for in proving

$$M \sim N \implies \overline{\overline{M}} = \overline{\overline{N}},$$

he seems to use the assumption $M \sim N$ in order to apply N to M, element by element, so as to obtain $\overline{\overline{M}} = \overline{\overline{N}}$.

I believe I can understand Fitzpatrick's inclination to give Euclid more postulates than he himself makes explicit. On the originally blank last page of my copy of the Weeks–Adkins geometry text [60], I find a list, in my own hand, of "Statements unmentioned but neccessary [*sic*]":

- If $A = B$ at one place and time, then $A = B$ at any place and time, provided A and B always represent the same things.
- Line AB is the same as line BA, provided each A and each B represent the same points.
- If two people are to discuss geometry, they must have a common language.

Such were my concerns in high school. Though our proofs in geometry class were supposed to make every assumption explicit, I had evidently been troubled to realize that we were not achieving this goal.

I do not think my list of tacit conventions in our text was the direct result of a lecture by the teacher, though above the list I find something that I could have copied from the blackboard: a table showing the converse, inverse, and contrapositive of the statement "If A, then B."

Although I have not been able to confirm this memory with textual evidence, I seem to recall an exercise from geometry class that involved the "trisector" of a line segment or angle. I refused to perform the exercise, since the concept

of trisection had not been formally defined. I was not the only student troubled by this exercise. The teacher ridiculed us, observing that it was obvious what trisection meant. She was right, though I was incensed at the time. Had not the whole purpose of the geometry course been to establish that "obviousness" was not a sufficient criterion for mathematical truth?

It had; but I think our text itself had gone overboard with this idea. The book lists "Algebraic Properties of Equality and Inequality" [60, page 41]. I see that I crossed out "Properties" and wrote "Theorems" above. The first of the properties or theorems is

$$\text{If } a = b \text{ and } c = d, \text{ then } a + c = b + d.$$

This is the "Addition Property of Equality." If equality were identity, then this property would be logically immediate. Not so the "Addition Property of Inequality," which is of different logical status, though this is not said:

$$\text{If } a > b \text{ and } c > d, \text{ then } a + c > b + d.$$

Subtraction, multiplication, and division properties of equality and inequality are also given. As has been explained in the text, "The letters a, b, c, and d are symbols for positive numbers"; and before that,

> Statements of the form "a is equal to b" occur throughout algebra and geometry. The symbols a, b refer to elements of some set and the basic meaning of $a = b$ is that a and b are names for the same element . . . In our geometry, $AB = CD$ means that line segment AB and line segment CD have the same length, and $\angle X = \angle Y$ means that angle X and angle Y have the same measure. In each case the equality is a statement that the *same number* gives the measure of both geometric quantities involved.

If the "basic meaning" of equality is sameness, then the word "basic" is being used in its slang sense of "approximate," as in, "The proof is basically correct, but has some small errors." For as we have just seen, Weeks and Adkins go on to tell us that in geometry, equality is not actually sameness, but sameness of some *property.* Thus, with geometric objects, it does need to be made explicit somehow that equality is preserved under addition. Recognizing this, Euclid gives what is counted now as his second "common notion":

$$\text{If equals be added to equals, the wholes are equal.}$$

But in the Weeks–Adkins "Addition Property of Equality," the letters stand for numbers, and equality of these numbers *is* sameness. In this case, the "Addition Property" and the other properties of equality should go without saying. Indeed, my classmates and I were told this by a different teacher in the following year, in a precalculus class, when we started proving things from the axioms of $\mathbb{R}$ as an ordered field, and we asked the teacher why we were not proving the "Addition Property" as a theorem.

Since Euclid introduces no symbolism for the length of a line segment, as opposed to the segment itself, his notion of equality is unambiguous. It is congruence. This is made explicit in the common notion that is now numbered fifth, following J. L. Heiberg's bracketing of two earlier common notions in manuscripts:

> Things congruent to one another are equal to one another.

Thomas L. Heath uses "coincide" for "congruent" [25]; but Heiberg's Latin is,

QUAE INTER SE CONGRUUNT, AEQUALIA SUNT.

The Greek verb is ἐφαρμόζω, or ἐπί + ἁρμόζω, the root verb being the origin of our "harmony." To say that two line segments are equal is to say that one can be picked up and placed on the other so that they "harmonize," that is, coincide. In Euclid's Proposition ɪ.4, it is assumed about given triangles ABΓ and ΔEZ that sides AB and AΓ and their included angle are respectively equal to ΔE and ΔZ and their included angle. By definition of equality, this means AB can be placed on ΔE so that they coincide, and then the angles will coincide, and then AΓ and ΔZ will coincide, so that the remaining features of the triangle are respectively equal.

That is a proof. Or we can call it an "intuitive justification" for what is "really" a postulate. But Cantor's quoted argument for the implication

$$M \sim N \implies \overline{\overline{M}} = \overline{\overline{N}}$$

does not even rise to this level. I think the argument fails at the start for not observing more precisely that $\overline{\overline{M}}$ is unchanged if distinct elements of M are replaced with other *distinct* things. Despite the earlier description, $\overline{\overline{M}}$ cannot consist of "units" simply, without any way to distinguish between different units. Cantor does not provide a way to distinguish.

2.5. Euclid's numbers

Euclid does not have Cantor's problem in the *Elements,* even though the
definitions at the head of Book VII [23] may be vague:

<p align="center">Μονάς ἐστιν, καθ᾽ ἣν
ἕκαστον τῶν ὄντων ἓν λέγεται.
Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.</p>

<p align="center">Unity is that according to which
each entity is said to be one thing.
And a number is a multitude of unities.</p>

I translate Euclid's *μονάς* as "unity" here, although Heath uses "unit" [25].
In his "Mathematicall Preface" [17] to Billingsley's 1570 translation of the
Elements, John Dee notes explicitly in the margin that he has *created* the
word "unit" precisely to translate Euclid's *μονάς.* However, Billingsley uses
"unity" in his own translation [21]. The editors of the *Oxford English Dic-
tionary* [42] found the relevant passages of both Dee and Billingsley worth
quoting, in the articles "Unit" and "Unity" respectively.

An abstract noun does seem called for, in translating at least in the first in-
stance above of *μονάς.* I always thought it was strange for Heath to translate
Euclid as,

> An *unit* is that by virtue of which each of the things that exist
> is called one.

An alternative for "unit" or "unity" might be "oneness." Euclid's *ἕν* "one"
has neuter gender, but the feminine form of the adjective is *μία*, and both
forms (along with the masculine *εἷς*) have the root SEM. However, it is not
clear whether the M here relates these words to *μονάς* in the way that "one"
is related to "oneness." How thoroughgoing is the analogy

<p align="center">{εἷς, μία, ἕν} : μονάς :: one : oneness?</p>

Pierre Chantraine gives no indication of a connection between *μία* and *μονάς*
in the *Dictionaire étymologique de la langue grecque* [16]. On the other hand,
neither does he suggest a connection between *εἷς*, *μία*, *ἕν* and the prefix
συν-, which was originally *ξυν-*, and which appears as *συμ-* in *συμμετρία*.

The *American Heritage Dictionary* [41] alludes to a presumed connection. Here the entry **syn-** in the dictionary proper refers to **sem-¹** in the Appendix of Indo-European Roots. This may be an error, since in the Appendix itself, the modern "syn-" is found not under **sem-¹,** but under **ksun.** However, both **sem-¹** and **ksun** are referred to the same entry *sem-* in Pokorny's *Indogermanisches Etymologisches Wörterbuch.* Perhaps an editor of the *American Heritage Dictionary* came to think Pokorny too bold in tracing $\sigma\upsilon\nu$- and $\ \!\!\breve{\varepsilon}\nu$ unequivocally to a common root; but the editor failed to make all changes needed to reflect this change of heart.

So there could be an etymological connection between $\mu\iota\alpha$ and $\mu o\nu\acute{a}\varsigma$ that Chantraine failed to note. However, English does have the option of coining the word "monad" as a translation of $\mu o\nu\acute{a}\varsigma$, and English has in fact done this, as for example to render the philosophy of Leibniz. The *American Heritage Dictionary* traces "monad" to the Indo-European root **men-⁴**, meaning "small, isolated." This suggests that "oneness" is really not an etymologically justifiable translation of $\mu o\nu\acute{a}\varsigma$. Benjamin Jowett uses "monad" in translating $\mu o\nu\acute{a}\varsigma$ among the words of Socrates in Plato's *Phaedo* [48, 105B–C, page 245]:

> I mean that if any one asks you "what that is, of which the inherence makes the body hot," you will reply not heat (this is what I call the safe and stupid answer), but fire, a far superior answer . . . and instead of saying that oddness is the cause of odd numbers, you will say that the monad is the cause of them . . .

Thus

$$\text{hot : fire :: odd : monad.}$$

The example of Jowett is quoted in the *Oxford English Dictionary,* precisely to illustrate the English use of "monad." The Loeb translation by Fowler of the same passage [49, page 363] has "the number one" for Socrates's $\mu o\nu\acute{a}\varsigma$, but this is misleading, inasmuch as a monad is not a number of things, but one thing. One is not a number.

It is possible that the definitions found in the *Elements* were not put there by Euclid. As the diagrams of Euclid's *propositions* indicate, the unities or units or monads that make up Euclid's numbers are not so abstract as to be devoid of distinctions. Each of Euclid's numbers can be conceived of as a bounded straight line, each of its units being a different part of the whole.

The number itself is then the *set* of these parts. Two *different* numbers can be equal: Euclid makes this clear in Proposition VII.8, where he lays down one number that is equal to another, though different from it. He does this for the convenience of diagramming the argument, since the equal numbers are going to be divided differently into parts.

At least one modern textbook may allow different numbers to be equal, although this is not clear. Near the beginning of his *Fundamental Concepts of Algebra* [39, pages 2–3], Bruce Meserve writes:

> The numbers that primitive man first used in counting the elements of a set of objects are called *natural numbers* or *positive integers.* Technically, the positive integers are symbols. They may be written as $/$, $//$, $///$, . . . ; i, ii, iii, . . . ; 1, 2, 3, . . . ; or in many other ways . . .
>
> Comparisons between cardinal numbers must agree with the corresponding comparisons between the sets of elements represented by the cardinal numbers. Accordingly, the cardinal numbers a, b associated with the sets A, B are equal (written $a = b$) and the sets are said to be *equivalent* if there exists a one-to-one correspondence between the elements of the two sets . . .

On page 1 of Meserve's book, a footnote has explained that "new terms will be italicized when they are defined or first identified." However, the word "equal" is not italicized in the passage above. It is not clear whether Meserve would write such equations as

$$/// = 3, \qquad\qquad 3 = \text{iii}.$$

Still, $///$, 3, and iii would seem to be different as symbols, and Meserve has said that numbers are symbols. On the other hand, he does not say that $///$ and 3 are themselves numbers, but only that a certain number or numbers are written this way. Presently he does seem to treat equality as sameness [39, page 4]:

> Given any two finite sets A, B with cardinal numbers a, b, we may compare the cardinal numbers using the subsets 1, 2, . . . , a and 1, 2, . . . , b of the set of positive integers. Let C be the set 1, 2, . . . , c of positive integers that are in both these subsets.

> If $c = a$ and $c \neq b$, then $a < b$. If $c = a$ and $c = b$, then $a = b$.
> If $c = b$ and $c \neq a$, then $b < a$. Thus we have proved that for
> any two finite sets A, B with cardinal numbers a, b exactly one
> of the relations $a < b$, $a = b$, $a > b$ must hold.

It is not clear why a third letter c is needed here after a and b; but its
introduction is reminiscent of Euclid's introduction of a new number that is
different from but equal to an earlier number. That which is denoted by c
is said to be common to the two indicated sets, and so it must not only be
equal to a or b; it must *be a* or b.

Meserve goes on to treat equality as a typical or generic equivalence relation
[39, pages 7, 8]:

> Any relation having the three properties:
>   reflexive, $a = a$,
>   symmetric, $a = b$ implies $b = a$,
>   transitive, $a = b$ and $b = c$ imply $a = c$,
> is called an *equivalence relation.* The equivalence of sets and
> therefore the equality of cardinal numbers as defined [above] can
> be proved to be an equivalence relation as follows . . .
>   One can also prove under the usual definitions that "iden-
> tity" ($\equiv$), "congruence" ($\cong$) of geometric figures, and "similar-
> ity" ($\sim$) of geometric figures are equivalence relations. Thus each
> of the symbols $=$, $\equiv$, $\cong$, $\sim$ represents "equals" in a well-defined
> mathematical sense. We now use the equivalence relation $=$ in
> a characterization of the positive integers by means of Peano's
> postulates . . .

It is not clear what Meserve means by identity symbolized by $\equiv$. His book's
word index features identity only in the phrases "identity element under an
operation," "identity relation," and "identity transformation." Under "iden-
tity relation," the corresponding pages are only 102 and 134, where it is
established that an equation of polynomials is an identity if it holds for all
values of the indeterminates; otherwise the equation is conditional. Meserve's
index of symbols and notation features $\equiv$ only for congruence of integers
with respect to a modulus. Gauss establishes this use of the symbol at the
beginning of the *Disquisitiones Arithmeticae* [27, page 1] and remarks in a
footnote,

We have adopted this symbol because of the analogy between
equality and congruence. For the same reason Legendre . . . used
the same sign for equality and congruence. To avoid ambiguity
we have made a distinction.

Presumably the analogy between equality and congruence lies in their being
what we now call equivalence relations.

Meserve *is* sensitive to one foundational issue. Unlike what many people,
including Peano himself [44], seem to think, while induction establishes that
only one operation of addition can be defined recursively by the rules

$$a + 1 = a^+, \qquad\qquad a + b^+ = (a + b)^+,$$

induction does not *obviously* establish that such an operation exists at all.
Meserve knows this, at least through Landau [33], whom he cites after noting,

Peano's postulates are not sufficient to define addition and mul-
tiplication explicitly, but they may be used to prove that each
of these operations may be defined in exactly one way to satisfy
certain conditions [39, page 10].

To prove that addition and multiplication can be defined, one does not need
the postulates that the operation $x \mapsto x^+$ of succession is injective and 1 is
not a successor. This is why modular arithmetic is possible. However, there
is no "modular exponentiation," defined by

$$a^1 = 1, \qquad\qquad a^{b+1} = a^b \cdot a,$$

where equality is congruence with respect to, say, 3. This shows that def-
inition by recursion requires more than proof by induction. See my article
"Induction and Recursion" [46].

We have seen that Euclid's geometry provides a way to understand numbers
as sets of distinct units, which is something that Cantor and some of his
successors have failed to do.

2.6. *Von Neumann's ordinal numbers*

Today, unlike Euclid (and for that matter Descartes), we may prefer not to
rely on geometry as a foundation of our mathematics. For example, geometry
may not well accommodate a straight line consisting of uncountably many

units. In this case, for an alternative foundation, we can understand numbers as von Neumann does.

Before reviewing the definition, we should note that, in addition to cardinal numbers, Cantor defines *ordinal numbers* [15, §7, pages 111–2, & §12, page 137]:

> Every ordered aggregate M has a definite **ordinal type,** or more shortly a **type,** which we will denote by
>
> $$\overline{M}.$$
>
> By this we understand the general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them. Thus the ordinal type $\overline{M}$ is itself an ordered aggregate whose elements are units which have the same order of precedence amongst one another as the corresponding elements of M, from which they are derived by abstraction.
>
> . . . .
>
> Among simply ordered aggregates *well-ordered aggregates* deserve a special place; their ordinal types, which we call **ordinal numbers,** form the natural material for an exact definition of the higher transfinite cardinal numbers or powers,—a definition which is throughout conformable to that which was given us for the least transfinite cardinal number Aleph-zero by the system of all finite numbers ν (§6).

On the contrary, Cantor's definitions are not exact. Von Neumann points this out as follows [59].

> The aim of the present paper is to give unequivocal and concrete form to Cantor's notion of ordinal number.
>
> Ordinarily, following Cantor's procedure, we obtain this notion by "abstracting" a common property from certain classes of sets [15]. We wish to replace this somewhat vague procedure by one that rests upon unequivocal set operations. The procedure will be presented below in the language of naive set theory, but, unlike Cantor's procedure, it remains valid even in a "formalistic" axiomatized set theory . . .

What we really wish to do is to take as the basis of our considerations the proposition: "Every ordinal is the type of the set of all ordinals that precede it." But, in order to avoid the vague notion "type," we express it in the form: "Every ordinal is the set of ordinals that precede it." This is not a proposition proved about ordinals; rather, it would be a definition of them if transfinite induction had already been established. According to it, we have

$$0 = \varnothing,$$
$$1 = \{0\},$$
$$2 = \{0, 1\},$$
$$3 = \{0, 1, 2\},$$
$$\dots\dots\dots\dots,$$
$$\omega = \{0, 1, 2, \dots\},$$
$$\omega + 1 = \{0, 1, 2, \dots, \omega\},$$
$$\dots\dots\dots\dots\dots$$

I have simplified von Neumann's equations by allowing numbers already defined to be used in later definitions. Von Neumann writes out all of the definitions here in terms of the empty set, which he denotes by O; and he denotes sets by (...) rather than by { ... }. The number five becomes a certain set of five elements, written out in full as

$$(O, (O), (O, (O)), (O, (O), (O, (O))), (O, (O), (O, (O)), (O, (O), (O, (O))))),$$

or more simply as $(0, 1, 2, 3, 4)$, or in our terms $\{0, 1, 2, 3, 4\}$.

Many mathematicians seem not to think of numbers as sets. When we need a set with five elements, we use $\{1, 2, 3, 4, 5\}$. When we need a set with n elements, we use $\{1, \dots, n\}$. We may however prefer a simpler notation for this set. During the development of groups in his *Algebra,* Serge Lang writes in two different places [34, pages 13, 30]:

Let $J_n = \{1, \dots, n\}$. Let S_n be the group of permutations of J_n. We define a **transposition** to be a permutation τ such that there exist two elements $r \neq s$ in J_n for which $\tau(r) = s$, $\tau(s) = r$, and $\tau(k) = k$ for all $k \neq r, s$ ...

. . . .

> Let S_n be the group of permutations of a set with n elements. This set may be taken to be the set of integers $J_n = \{1, \ldots, n\}$. Given any $\sigma \in S_n$, and any integer i, $1 \leqq i \leqq n$, we may form the orbit of i under the cyclic group generated by σ. Such an orbit is called a **cycle** for σ . . .

This seems like a needless profusion of symbols. In another sense, Lang displays parsimony with symbols, or at least with words, allowing the expression

$$J_n = \{1, \ldots, n\}$$

to serve both for the clause "J_n be equal to $\{1, \ldots, n\}$" and for the noun phrase "J_n, which is equal to $\{1, \ldots, n\}$." The inequation

$$r \neq s$$

stands for the noun phrase "r and s, which are unequal"; strictly, one need not say that they are unequal, since they have already been described as "two"; one might say "two *distinct* elements" for emphasis. The equation

$$\tau(r) = s$$

stands not for a noun, but for the declarative sentence "$\tau(r)$ is equal to s." I have known students to be confused by the ambiguous use of equations, and Paul Halmos somewhere inveighs against it. Nonetheless, the prevalence of ambiguity does show that there is a difference between expressing mathematics well and just doing good mathematics. We shall return to Lang's mathematics in the next section.

If one uses von Neumann's definition, then n itself is an n-element set, and one has no need for notation like J_n. One may well blanch at the thought of saying "Let S_n be the group of permutations of n." One may then introduce notation like Lang's J_n; but in that case, why not define J_n to mean $\{0, \ldots, n-1\}$, namely von Neumann's n? Our theme is that numbers measure size; and the beginning of size in general is not 1 but 0. When we measure a line with a ruler, at one end of the line we place the point of the ruler that is marked 0. See Figure 2.
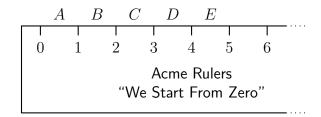
Figure 2: The measure of the set $\{A, B, C, D, E\}$ is 5.

3. Groups and Symmetries

3.1. Symmetry as a concept

We suggested in the Introduction (Section 1) that groups measure symmetry as numbers measure size. We can write out this slogan as a proportion:

$$\text{numbers : size :: groups : symmetry.}$$

However, for Euclid, the proportion of magnitudes that we may express as

$$A : B :: C : D$$

means A has the *same ratio* to B that C has to D. There are certain things that we can do with a pair of magnitudes having a ratio: we can multiply each magnitude individually and then compare the multiples, and we can subtract the less magnitude from the greater. When by means of such activities, we cannot find any difference between two pairs of magnitudes, this is what it means for the four magnitudes to be in proportion. We generalize this idea to allow proportions like

$$\text{hand : mitten :: foot : sock.}$$

Mittens and socks are knitted from yarn, and a hand fits into a mitten the way a foot fits into a sock, without separation of the digits. But extracting the group of symmetries of an object is somewhat different from counting a set. If you want to know how many candies are in a jar, you can just pull them out, one by one, saying the next number in the standard sequence as you go. However, in order to compute $\mathbb{Z} \times (\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})$ as the symmetry group of a certain pattern of tiles in Istanbul's Rüstem Pasha Mosque (see Figure 3), you should both know some theory and have some practice.

Figure 3: Rüstem Pasha Mosque interior detail. Photos taken in 2013 by the author.

The upper semi-lattice of Turing degrees was studied for decades before the late Barry Cooper found a nontrivial automorphism in the 1990s.

As in the examples just given, the object whose symmetry is being measured is usually not simply a set. It may best be considered, if only implicitly, as an object in a so-called *category.* From one object to another in a category, there may be *homomorphisms,* also called simply *morphisms.* These can be *composed,* as if they were functions—which they usually are, although category theory does not require them to be. Some morphisms may be *invertible,* in which case they are *isomorphisms.* An invertible homomorphism from an object to itself is an *automorphism.* The automorphisms of an object compose a *group,* the group operation being composition. Then by the most general definition, two objects, possibly in two different categories, have the **same symmetry** if their automorphism groups are isomorphic to one another as objects in the category of groups.

The objects of a *concrete* category have "underlying sets," and the objects themselves are "sets with structure"; a morphism from one object to another is then indeed a function from the one underlying set to the other that "preserves" this structure. Then two objects of (possibly different) concrete categories have the **same size** if their underlying sets are isomorphic to one another in the category of sets.

Is there now perhaps some lack of parallelism, some *asymmetry,* in the slogan, "Numbers measure size, groups measure symmetry"? In the "categorical" definition of sameness of symmetry, groups are mentioned; in the "categorical" definition of sameness of size, not numbers but *sets* are mentioned. One might say that it is sets that measure size; more precisely, the underlying set of an object of a concrete category is the measure of the size of the object itself.

In this case, one might ask whether extracting this underlying set is parallel to extracting the automorphism group of an arbitrary category. Symbolically, let an object A of a category have the automorphism group $\mathrm{Aut}(A)$; if the category is concrete, let A have the underlying set $\mathrm{Dom}(A)$, the "domain" of A.

Objects A and B have the same size if

$$\mathrm{Dom}(A) \cong \mathrm{Dom}(B);$$

A and B have the same symmetry, if

$$\mathrm{Aut}(A) \cong \mathrm{Aut}(B).$$

The operation $X \mapsto \mathrm{Dom}(X)$ somewhat corresponds to Cantor's operation $X \mapsto \overline{\overline{X}}$ defined above in §2.1 on page 97; but the new operation has the advantage of a clear meaning.

If the slogan "Numbers measure size, groups measure symmetry" is to express a thorough-going analogy, we should understand a number to be nothing other than a pure set, that is, an object in the category of sets. The number of an object in a concrete category would then be the underlying set of the object. This usage of "number" would be compatible with Euclid's usage, though not with ours, since equipollent sets are not necessarily equal for us.

Today, under the Axiom of Choice, every equipollence class of sets contains an ordinal number and therefore a least ordinal number, which can be defined to be the cardinal number of every set in the class. This may temporarily give us hope. However, there is no useful way to designate, within every isomorphism class of automorphism groups, a particular element that shall serve as *the* group of every object whose automorphism group belongs to the class. Gödel's universe of *constructible* sets is well-ordered, and so, if one works there, then one has *a* way to select a representative from each isomorphism class of groups; but this would seem not to be a useful way, for present purposes, to select a representative.

It is however worth noting that every group G is isomorphic to the automorphism group of at least one object. This object can be the labelled directed graph in which the vertices are the elements of G, and every ordered pair (a, b) of these elements determines a directed edge from a to b, the arrow being labelled by the element $a^{-1}b$. If σ is an automorphism of the labelled directed graph, then σ permutes the vertices in a way that respects the labels on the edges. Thus the edge (a^{σ}, b^{σ}) must also be labelled with $a^{-1}b$. Since it was already labelled with $(a^{\sigma})^{-1}b^{\sigma}$, we conclude

$$a^{\sigma}a^{-1}b = b^{\sigma}.$$

In particular $1^{\sigma}b = b^{\sigma}$, and so σ is $x \mapsto gx$, where $g = 1^{\sigma}$. Conversely, for any choice of g, the permutation $x \mapsto gx$ is an automorphism of the labelled directed graph.

Concerning the slogan that numbers and groups measure size and symmetry respectively, it would seem to be more accurate to say,

  numbers measure size, *isomorphism classes* of groups measure symmetry;

or

<div style="text-align: center;">sets measure size, groups measure symmetry;</div>

or even

<div style="text-align: center;">sets have size, groups have symmetry.</div>

3.2. Groups of symmetries

We looked at Lang's *Algebra* in §2.6 for its *ad hoc* approach to selecting a standard set of each finite size. We look again for what Lang may suggest about groups as measures of symmetry.

Lang hints at the understanding of groups as automorphism groups. Right after the abstract definition of a group as a monoid with inverses, he gives several examples, although they are abstract as well:

- If a group and a set are given, then the set of maps from the set into the group is itself a group.
- The set of permutations of a set is a group.
- The set of invertible linear maps of a vector space into itself is a group, as is the set of invertible $n \times n$ matrices over a field.

This is at [34, I, §2, page 8]. The next "example" is:

> **The group of automorphisms.** We recommend that the reader now refer to §11, where the notion of a category is defined, and where several examples are given. For any object A in a category, its automorphisms form a group denoted by $\mathrm{Aut}(A)$. Permutations of a set and the linear automorphisms of a vector space are merely examples of this more general structure.

We may understand $\mathrm{Aut}(A)$, or rather its isomorphism class, as the measure of the symmetry of A. Lang however does not speak of symmetry as such. Between the two instances quoted in our §2.6 where the notation J_n is used, Lang observes [34, I, §5, p. 28]:

> The symmetric group S_n operates transitively on $\{1, 2, \ldots, n\}$.

The term "symmetric group" here is not given any special typographical treatment, although it represents the first use of the term "symmetric" in the index (and the term "symmetry" is not in the index). Other terms are made bold when Lang defines them.

According to the index in his own *Algebra,* Hungerford uses the term "symmetry" once, to refer to any of the eight symmetries of the square, defined as an example [30, I.1, page 26]. In his philosophical book *Mathematics: Form and Function,* Mac Lane defines a symmetry this way, as a rigid motion of a figure ("a collection of points") onto itself [38, I.6, pages 17 & 19].

Armstrong uses the term "symmetry" in this way too, but also more abstractly. Again, he does not actually define the term: perhaps this would not be in keeping with his informal treatment. After his opening slogan, Armstrong says what he expects of his audience, which is basically that they have some experience of undergraduate mathematics:

> The first statement ["numbers measure size"] comes as no surprise; after all, that is what numbers "are for". The second ["groups measure symmetry"] will be exploited here in an attempt to introduce the vocabulary and some of the highlights of elementary group theory.
>
> A word about content and style seems appropriate. In this volume, the emphasis is on examples throughout, with a weighting towards the symmetry groups of solids and patterns. Almost all the topics have been chosen so as to show groups in their most natural role, acting on (or permuting) the members of a set, whether it be the diagonals of a cube, the edges of a tree, or even some collection of subgroups of the given group . . .
>
> . . . .
>
> As prerequisites I assume a first course in linear algebra (including matrix multiplication and the representation of linear maps between Euclidean spaces by matrices, though not the abstract theory of vector spaces) plus familiarity with the basic properties of the real and complex numbers. It would seem a pity to teach group theory without matrix groups available as a rich source of examples, especially since matrices are so heavily used in applications.

Armstrong goes on to use the word "symmetry" as if it were a word like "language": it denotes a concept, but also an instance of the concept. When we observe that we use *language* to express ourselves, we are referring to the general concept of language; but we may also observe that English is *one language* among many, such as Turkish or Russian or Greek.

The definitions in the *Elements* discussed in §2.5 use *μονάς* in this twofold way: it is the concept of unity, and it is anything that has unity. Thanks to John Dee, we now have two words for the two uses of *μονάς*: something with unity is a unit.

Armstrong's twofold use of "symmetry" is seen, even at the beginning of his Chapter 1, "Symmetries of the Tetrahedron":

> How much symmetry has a tetrahedron? Consider a regular tetrahedron T and, for simplicity, think only of rotational symmetry. Figure 1.1 [Figure 4] shows two axes. One, labelled L, passes through a vertex of the tetrahedron and through the centroid of the opposite face; the other, labelled M, is determined by the midpoints of a pair of opposite edges. There are four axes like L and two rotations about each of these, through $2\pi/3$ and $4\pi/3$, which send the tetrahedron to itself. The sense of the rotations is as shown [not in Figure 4]: looking along the axis from the
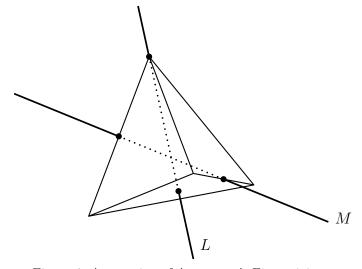

Figure 4: A recasting of Armstrong's Figure 1.1.

vertex in question the opposite face is rotated anticlockwise. Of course, rotating through $2\pi/3$ (or $4\pi/3$) in the opposite sense has the same effect on T as our rotation through $4\pi/3$ (respectively $2\pi/3$). As for axis M, all we can do is rotate through π, and there are three axes of this kind. So far we have $(4 \times 2) + 3 = 11$ symmetries. Throwing in the identity symmetry, which leaves T fixed and is equivalent to a full rotation through 2π about any of our axes, gives a total of twelve rotations.

Each of these twelve rotations is *a symmetry* of the tetrahedron. Presumably twelve of them together constitute a measure of *the symmetry* of the tetrahedron. However, Armstrong goes on to observe that this measure is not simply the number twelve:

> We seem to have answered our original question. There are precisely twelve rotations, counting the identity, which move the tetrahedron onto itself. But this is not the end of the story. A flat hexagonal plate with equal sides also has twelve rotational symmetries (Fig. 1.2), as does a right regular pyramid on a twelve sided base (Fig. 1.3) [both figures omitted].

The respective groups of rotational symmetries of the three objects have order twelve, but no two are isomorphic to one another, and therefore none embeds in another.

The collection of isomorphism-classes of symmetry groups is thus only partially ordered. It is not even a semi-lattice in either sense: finite subsets need not have suprema or infima. Armstrong's examples can show this; so can simpler ones, as in Figure 5.
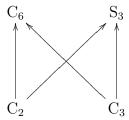

Figure 5: Isomorphism classes of groups are not a lattice.

The finite cyclic groups C_2 and C_3 embed both in C_6 and the symmetric group S_3, but neither embedding can be nontrivially factorized, and neither of the latter two groups embeds in the other: so $\{C_2, C_3\}$ has no supremum; $\{C_6, S_3\}$, no infimum. This lack of a lattice structure on the collection of isomorphism classes of groups can be contrasted with the existence of a lattice structure on the set of subgroups of a specific group. If we consider S_6 as the group of permutations of $\{0, 1, 2, 3, 4, 5\}$, we can define C_6 as the subgroup $\langle (0\ 1\ 2\ 3\ 4\ 5) \rangle$. If we now make the definitions

$$C_2 = \langle (0\ 1) \rangle, \qquad C_3 = \langle (0\ 1\ 2) \rangle, \qquad S_3 = \langle (0\ 1), (0\ 1\ 2) \rangle,$$

then

$$\sup\{C_2, C_3\} = S_3, \qquad\qquad \inf\{C_6, S_3\} = \langle\ \rangle.$$

However, alternative definitions are possible:

$$C_2 = \langle (0\ 3)(1\ 4)(2\ 5) \rangle,$$
$$C_3 = \langle (0\ 2\ 4)(1\ 3\ 5) \rangle,$$
$$S_3 = \langle (0\ 2)(1\ 3), (0\ 2\ 4)(1\ 3\ 5) \rangle,$$

and in this case

$$\sup\{C_2, C_3\} = C_6, \qquad\qquad \inf\{C_6, S_3\} = C_3.$$

3.3. Arithmetic as leading to set theory

In the form, "Every set can be well ordered," the Axiom of Choice evidently implies the comparability of any two sizes of sets. Hartogs showed the converse in 1915 [35, page 161]. Indeed, for any set A, the set of ordinals that do embed in A is itself an ordinal α, and this must not embed in A; if A must therefore embed in α, then a well-ordering of A is induced.

Gödel proved the consistency of the Axiom of Choice with the Zermelo–Fraenkel axioms of set theory. There is apparently no such consistency result for the Axiom of Determinacy: that for all choices of subsets A of the interval $[0, 1]$, there is a winning strategy for one of the two players of the game in which the players alternately select digits e_k from the set $\{0, 1\}$, and the first player wins if the sum $\sum_{k=1}^{\infty} e_k/2^k$ is in A. One may find this axiom plausible;

and yet it contradicts the Axiom of Choice. This is one reason not to accept the Axiom of Choice as blithely as the (other) Zermelo–Fraenkel axioms.

Without Choice, but with the Axiom of Foundation, one can assign to each set an ordinal rank. The class of all sets of a given rank is a set. One may then define the size of a set A to be the set of sets of minimal rank that are equipollent with A. I do not know anything about the partial ordering of sizes in this sense: whether for example it can be required to be a lattice without imposing the full Axiom of Choice.

When I have the opportunity to teach undergraduate set theory, I try to emphasize several points about ordinal arithmetic:

1) that it is the natural generalization of the arithmetic of the finite ordinals;
2) that *Cantor normal forms* are ordinals written in base ω, just as our ordinary numbers are finite ordinals written in base ten;
3) that *normal* operations on the class of ordinals, such as

$$\xi \mapsto \alpha + \xi, \qquad\qquad \xi \mapsto \beta \cdot \xi, \qquad\qquad \xi \mapsto \gamma^\xi$$

(where $\beta > 0$ and $\gamma > 1$) are analogous to the continuous functions studied in calculus.

It does seem harder to make the pedagogical case that group theory is a natural generalization of school arithmetic. If one is going to make the case, one may forget about symmetries and instead talk about modular arithmetic, and Fermat's theorem, and Euler's phi-function. Matrix multiplication can serve as an example of a noncommutative operation. Sooner or later though one will have to talk about symmetries; and then one is off in a new world. Perhaps this is what is signified by the analogy,

$$\text{numbers : size :: groups : symmetry.}$$

Groups are as different from numbers as the concept of symmetry is different from size.

4. *Symmetria*

Symmetry then is a way of understanding a mathematical structure that is more subtle than simply counting the number of its underlying individuals.

Why is it called symmetry? Let us review the history of the relevant terms once again.

Having been born to a Roman patrician family just after the 476 extinction of the Western Roman Empire, Boethius coined the Latin adjective COM-MENSURABILIS for either of two numbers that are *not* relatively prime. In the same book that gave us our sign of equality, Robert Recorde used (and perhaps created) the English term "commensurable," which had the meaning given by Boethius to COMMENSURABILIS. For Recorde then, commensurable numbers had a common measure that was a *number* of units, and not just unity itself. Thirteen years later, in translating Euclid, Billingsley used the term "commensurable" with Euclid's meaning of *σύμμετρος*, namely, having *any* common measure, even unity in the case of numbers.

The abstract noun "symmetry" also came into English in the sixteenth century, but not with a technical mathematical sense. Like its Greek source, *συμμετρία*, it referred to an interrelation of parts, and to their *proportions,* as in architecture. The adjective "symmetric" seems to have taken two more centuries to come into use, as does the crystallographic or more generally geometric notion of symmetry with respect to a straight line, a point, or a plane.

The Greek abstract noun *συμμετρία* is evidently the source of the English noun. There would appear to be three historical senses of symmetry, which I would term (1) geometric, (2) numerical, and (3) philosophical.

The geometrical sense of *συμμετρία* is Euclid's, though it appears earlier in the work called *De Lineis Insecabilibus,* which is attributed (with some doubt) to Aristotle. In his translation of Euclid, Billingsley used "commensurable" in the same geometric sense, which is the sense that the word continues to have.

The numerical sense of *συμμετρία* is the negation of being relatively prime. I do not find this sense attested in Greek; but Boethius used the loan-translation *commensurabilis* with this sense. Writing before Billingsley, Recorde used "incommensurable" with the sense of Boethius. Boethius interpreted Nicomachus, though Nicomachus does not seem to have used *συμμετρία* with a clear technical sense; at any rate, he did not give the word Euclid's meaning.

By the philosophical sense of *συμμετρία*, I mean the sense of the word as found in Plato and Aristotle. One could just as well call it the vulgar or

popular sense. If any sense of the word gave us the modern mathematical sense of symmetry, it is this one.

4.1. Geometrical symmetry

4.1.1. Euclid

The citations in the *Greek–English Lexicon* of Liddell and Scott [36] of the adjective *σύμμετρος*, *-ον* do not strictly include the first of the definitions at the head of Book x of Euclid's *Elements* [24]:

> Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα,
> ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.
> Magnitudes measured by the same measure
> are called **commensurable;**
> those that admit no common measure, **incommensurable.**

As John Dee coined "unit" in order to translate Euclid's *μονάς*, so, using the Latin CON- for the Greek *συν-*, and the Latin MENSURA for the Greek *μέτρον*, Dee or somebody else could have composed the English word "commensurable," precisely to translate Euclid's *σύμμετρος*. The actual history will turn out to be more complicated.

4.1.2. De Lineis Insecabilibus

The *Lexicon* gives Euclid's *meaning* for the word *σύμμετρος*. It also quotes the *words* of Euclid given above; but it does so in an earlier expression, attributed to Aristotle, with the feminine gender of *γραμμή* "line," instead of the neuter gender of *μέγεθος* "magnitude." (The masculine and feminine of *σύμμετρος* are identical.) The lexicon entry reads:

> *commensurate with, of like measure* or *size with* . . . : esp. of Time, *commensurate with, keeping even with* . . . 2. in Mathematics, *having a common measure, σύμμετροι αἱ τῷ αὐτῷ μέτρῳ μετρούμεναι* (sc. *γραμμαί*) Arist. *LI*968ᵇ6; freq. denied of the relation between the diagonal of a square and its side . . . *μήκει οὐ σύμμετροι τῇ ποδιαίᾳ* not lineally *commensurate* with the one-foot side, Pl. *Tht.* 147d, cf. 148b . . . II. *in measure with, proportionable, exactly suitable* . . .

Here "Arist. *LI*" is *De Lineis Insecabilibus,* an obscure work attributed to Aristotle, but not with certainty, as Harold H. Joachim says in his Introductory Note [2]. His comments serve as a reminder of the difficulty of making sense of ancient mathematics: it needs the knowledge, skills, and experience of both the classicist and the mathematician:

> THE treatise Περὶ ἀτόμων γραμμῶν, as it is printed in Bekker's Text of Aristotle, is to a large extent unintelligible. But . . . Otto Apelt, profiting by Hayduck's labours and by a fresh collation of the manuscripts, published a more satisfactory text . . .
>
> In the following paraphrase, I have endeavoured to make a full use of the work of Hayduck and Apelt, with a view to reproducing the subtle and somewhat intricate thought of the author, whoever he might have been . . . there are grounds for ascribing [the treatise] to Theophrastus: whilst, for all we can tell, it may have been . . . by Strato, or possibly some one otherwise unknown. But the work . . . is interesting . . . Its value for the student of the History of Mathematics is no doubt considerable: but my own ignorance of this subject makes me hesitate to express an opinion.

In Bekker's edition, *De Lineis Insecabilibus* is five pages [1, pages 968–72]. The quotation in the LSJ *Lexicon* is drawn from the following account of a specious argument:

> Again, the being of 'indivisible lines' (it is maintained) follows from the Mathematicians' own statements. For if we accept their definition of 'commensurate' lines as those which are measured by the same unit of measurement, and if we suppose that all commensurate lines actually are being measured, there will be some actual length, by which all of them will be measured. And this length must be indivisible. For if it is divisible, its parts— since they are commensurate with the whole—will involve some unit of measurement measuring both them and their whole. And thus the original unit of measurement would turn out to be twice one of its parts, viz. twice its half. But since this is impossible, there must be an indivisible unit of measurement.

The argument may be the following, which is more or less what Joachim suggests in his notes:

1. Every line is commensurable, in the sense of having a common measure with some other line.
2. Thus all lines are commensurable with one another.
3. In particular, all lines have a common measure.
4. A common measure of all lines must be indivisible.
5. Therefore there is an indivisible line.

The first step might then be symbolized as

$$\forall x \; \exists y \; \exists z \; (z \; \mathrm{m} \; x \wedge z \; \mathrm{m} \; y), \tag{7}$$

where $a \; \mathrm{m} \; b$ means a measures b. However, the first step may be even simpler: every line is commensurable in the sense of being *mensurable,* that is, measurable. This could then be an allusion to the "Archimedean" assumption in Book v of Euclid's *Elements*: of any two lines, some multiple of the shorter exceeds the longer, so that the shorter "measures" the longer, at least approximately. If we would make an approximate measurement of the longer by the shorter, it might be said, this can only be out of conviction that an exact measure is possible, in the sense that, when we apply to the two lines the so-called Euclidean algorithm, found in Propositions vii.1 and 2 and x.2 and 3 of the *Elements,* the process terminates. This would give step 2 of the proposed analysis:

$$\forall x \; \forall y \; \exists z \; (z \; \mathrm{m} \; x \wedge z \; \mathrm{m} \; y). \tag{8}$$

But then the third step is

$$\exists z \; \forall x \; \forall y \; (z \; \mathrm{m} \; x \wedge z \; \mathrm{m} \; y), \tag{9}$$

and this follows from neither (7) nor (8).

The confusion of the argument may be reflected in the superficial similarity of sentences having different logical form, such as "These two angles are acute" and "These two angles are equal." The first abbreviates "These two angles are *each* acute"; the second, "These two angles are equal *to one another.*" Perhaps having recognized the potential ambiguity, Euclid often (though not always) uses the qualification, "to one another," when it fits. (See the example of *Elements* v.9 in §4.3.1 below.)

Again at the head of Book x, Euclid does provide a way to to call an individual magnitude commensurable, once some line of reference has been fixed. This reference line is to be called ῥητός, as is any other straight line

on which the square is commensurable with the square on the reference line. Each of these squares is also to be called ῥητός. Heath translates the adjective as "rational." Etymologically speaking, the rational is what is capable of speech; ῥητός refers originally to something spoken, as in our "rhetoric." In the present context, the irrational is ἄλογος, something without speech or reason or, in Latin, RATIO.

In *De Lineis Insecabilibus,* the refutation of the argument above is at 969$^{\mathrm{b}}$6; but perhaps it is not very illuminating. Joachim renders it thus:

> As to what they say about 'commensurate lines'—that all lines, because commensurate, are measured by one and the same actual unit of measurement—this is sheer sophistry; nor is it in the least in accordance with the mathematical assumption as to commensurability. For the mathematicians do not make the assumption in this form, nor is it of any use to them.
> Moreover, it is actually inconsistent to postulate both <u>that every line becomes commensurate</u>, and that there is a common measure of all commensurate lines.

Joachim describes his work as a paraphrase, but he seems here to follow Bekker's Greek reasonably:

> τὸ δ' ἐπὶ τῶν συμμέτρων γραμμῶν, ὡς ὅτι αἱ πᾶσαι τῷ αὐτῷ τινὶ καὶ ἑνὶ μετροῦνται, κομιδῇ σοφιστικὸν καὶ ἥκιστα κατὰ τὴν ὑπόθεσιν τὴν ἐν τοῖς μαθήμασιν· οὔτε γὰρ ὑποτίθενται οὕτως, οὔτε χρήσιμον αὐτοῖς ἐστίν. ἅμα δὲ καὶ ἐναντίον <u>πᾶσαν μὲν γραμμὴν σύμμετρον γίνεσθαι</u>, πασῶν δὲ τῶν συμμέτρων κοινὸν μέτρον εἶναι ἀξιοῦν.

In particular, the clause "that every line becomes commensurate" is indeed singular in the Greek. However, we might try reading the whole last sentence to mean that, even if any two lines are commensurate, it does not follow that all lines have a common measure. At any rate, the proposed content would seem to be true. We might understand magnitudes of a given kind (lines, areas, solids) to compose an ordered commutative semigroup in which a less magnitude can always be subtracted from a greater. Then two magnitudes will be **commensurate** if the Euclidean algorithm can be applied effectively to produce a common measure. What we call the positive rational numbers compose such a structure, and any two of them are commensurate, but there is no least positive rational number.

4.1.3. Billingsley

The second oldest quotation in the *Oxford English Dictionary* [42] for "commensurable" gives this word the meaning that it continues to have, which is that of *σύμμετρος* in Euclid. The quotation is from Billingsley's version of the *Elements,* already mentioned above. The citation is:

> **1570** Billingsley *Euclid* x Def. i. 229 All numbers are commensurable one to another.

The quotation is actually on the verso of folio 228—facing the recto of 229—of Billingsley's book [21], and it is part of a commentary, possibly by John Dee, on the first definition in Book x, the definition itself having been translated,

> Magnitudes commensurable are such, which one and the selfe same measure doth measure.

As examples of *σύμμετρος*, in the Index of Greek Terms for Thomas's two volumes, *Selections Illustrating the History of Greek Mathematics,* in the Loeb series [56, 57], there are cited instances of what, following Billingsley or Dee, we should call commensurability or its negation:

1) Plato's *Theaetetus,* on Theodorus's theorem that the square roots of nonsquare numbers of square feet from two to seventeen are incommensurable with the foot;
2) Euclid's formal definition of commensurability, as above; and
3) Archimedes's theorem that commensurable magnitudes (*τὰ σύμμετρα μεγέθεα*) balance at distances inversely proportional to their weights. (By the Method of Exhaustion, the same is true for incommensurable magnitudes.)

In Heath's *History of Greek Mathematics* [28, 29], the Index of Greek Words does not show *συμμετρία* or *σύμμετρος* at all. Neither does Heath's English index show "symmetry" or "commensurability." In order to look up in Heath the topics listed from Thomas's index, one should check under the word "irrational."

4.2. Numerical symmetry

4.2.1. Nicomachus

According to the *Oxford English Dictionary,* "commensurable" derives from the Latin word COMMENSURABILIS, which Boethius coined or at least used; the English word may also be derived from Nicole Oresme's fourteenth-century French version of Boethius's word. The *Larousse dictionnaire d'étymologie* recognizes Oresme's 1361 derivation of the French *commensurable* from the sixth-century Latin of "Boèce" [20, page 168].

Boethius's *Arithmetic* is considered [13, page 212] an abridgment of Nicomachus's *Introductio,* and it was "the source of all arithmetic taught in the schools for a thousand years" [31, page 201]. D'Ooge's edition of Nicomachus does not provide the Greek, except implicitly through an index of Greek terms. There is one instance of *συμμετρία* and one of *σύμμετρος*. The instance of the former is translated as follows [43, I.14.3, page 208]:

> if when all the factors of a number are examined and added together in one sum, it proves upon investigation that the number's own factors exceed the number itself, this is called a superabundant number, for it oversteps the symmetry which exists between the perfect and its own parts.

Here "symmetry" seems to be a synonym for equality. In modern notation, a number n is superabundant ($\dot{v}\pi\epsilon\rho\tau\epsilon\lambda\grave{\eta}s$), perfect ($\tau\acute{\epsilon}\lambda\epsilon\iota os$), or deficient ($\dot{\epsilon}\lambda\lambda\iota\pi\acute{\eta}s$), according as

$$\sum_{d|n} d > 2n, \qquad \sum_{d|n} d = 2n, \qquad \sum_{d|n} d < 2n.$$

The number 28 is perfect because

$$\{d \colon d \mid 28\} = \{1, 2, 4, 7, 14, 28\},$$
$$28 = 14 + 7 + 4 + 2 + 1;$$

and this situation is one of "symmetry." By contrast, 12 is superabundant since $6 + 4 + 3 + 2 + 1 = 16 > 12$.

The one indexed instance of *σύμμετρος* in Nicomachus [43, II.3.2, page 232] could likewise be replaced with "equal." First Nicomachus sets up the general situation:

> Every multiple will stand at the head of as many superparticular ratios corresponding in name with itself as it itself chances to be removed from unity, and no more nor less under any circumstances.

What this means is that, for any number k, if for some n we take the nth power k^n, starting from there we obtain a continued proportion

$$k^n : k^{n-1}\ell : k^{n-2}\ell^2 : \cdots : k\ell^{n-1} : \ell^n,$$

where $\ell = k + 1$. In the proportion, there are n terms after the first, and the ratio of each of these terms to the preceding is that of ℓ to k; this ratio is superparticular because the excess of ℓ over k (namely unity) is a part of k (that is, it measures k). The way n appears in two senses is apparently considered "symmetric." Nicomachus himself explains with an example, and here, apparently, the adjective σύμμετρος is used:

> The doubles, then, will produce sesquialters, the first one, the second two, the third three, the fourth four, the fifth five, the sixth six, and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number agrees with the multiples that have generated them, at that point by a divine device, as it were, there is found the number which terminates them all because it naturally is not divisible by that factor whereby the progression of the superparticular ratios went on.

An illustration is provided as in Figure 6, where each column shows a continued proportion as above.

1	2	4	8	16	32	64
	3	6	12	24	48	96
		9	18	36	72	144
			27	54	108	216
				81	162	324
					243	486
						729

Figure 6: Superparticular ratios in Nicomachus.

It does not appear that Nicomachus uses συμμετρία as a technical term.

4.2.2. Boethius

Boethius, however, in *De Institutione Arithmetica* [11, I.18, page 39, l. 14], does use "commensurable" as a technical term for numbers that are *not* prime to one another. In his example, by applying what we know as the Euclidean Algorithm, he shows that viiii and xxviiii are prime to one another (contra se primos); but xxi and viiii have the common measure iii, and therefore Boethius calls them commensurabiles.

4.2.3. Recorde

Robert Recorde carried the usage of Boethius into English. He provides the *oldest* quotation for "commensurable" in the *Oxford English Dictionary*:

> **1557** Recorde *Whetst.* Bj, .20. and .36. be commensurable, seyng .4. is a common diuisor for theim bothe.

This from Recorde's *Whetstone of Witte* [53], cited earlier as the origin of our sign of equality. The book is formally a dialogue between the Scholar and the Master. It starts with an account of numbers that seems based on Euclid, though Recorde first mentions Euclid only to have the Scholar say,

> Yet one thyng more I must demaunde of you, why Euclide, and the other learned men, refuse to accompte fractions emongest nombers.

The Master responds as follows, alluding to the definition of number quoted above from the *Elements*:

> Bicause all nombers doe consiste of a multitude of unities : and every proper fraction is lesse then an unitie, and therefore can not fractions exactly be called nombers : but maie bee called rather fractions of nombers.

My quotations extend from the verso of 𝕬.ii. to 𝕭.i., which is the folio number cited in the *Oxford English Dictionary.* Presently the Master introduces the term *commensurable* to mean *not relatively prime,* that is, *having a common measure other than unity*; this is the meaning of Boethius. Billingsley will use the term differently, thirteen years later, to mean *having any common measure at all,* as noted above; however, the *OED* takes no note of the difference. Recorde writes as follows; the *OED* quotation is here.

Scholar . . . What saie you now of nõbers relatiue?

Master. Some tymes their relation hath regarde to their partes, namely, whether these. 2. that bee so compared, haue any common parte, that will diuide theim both. For if thei haue so, then are thei called nombers commensurable. As. 12. and. 21. bee nombers commensurable: for. 3. will diuide eche of theim.

Likewaies. 20. and. 36. be commensurable, seyng 4. is a commõ diuisor for them both. But if thei haue no suche common diuisor, then are thei called incommensurable. As 18 and 25. For 25 can bee diuided by no nomber more than by. 5. And. 18. can not be diuided by it.

In like maner. 36. and. 49. are incommensurable: For 49. hath no diuisor but. 7. And 7. can not diuide. 36.

Scholar. Do you meane then, that incommensurable nombers, haue no cõparison nor proportion together?

Master. Naie, nothyng lesse. For any. 2. nombers maie haue comparison et proportion together, although thei be incommensurable. As. 3. and. 4. are incommensurable, and yet are thei in a propor‐ tion together: as shall appeare anon.

(In the Master's last lines, I have used **et** where the original shows an obscure symbol; this symbol does not seem to be an ampersand, though it could be the "Tironian et.") Thus a number prime to another still has a ratio to the other; or in Recorde's terms, incommensurable numbers are still in propor- tion. One might here want to guard against the confusion that might have been seen in *De Lineis Insecabilibus* above: just because any two numbers are in proportion, it does not follow that they are in the *same* proportion as any other two numbers!

It might be convenient to have, as Recorde does, a single term for a pair of numbers that are not prime to one another; but it would seem that "com- mensurable" has not been used as such a term, at least not since Billingsley's rendition of Euclid.

4.3. Philosophical symmetry

4.3.1. Plato

In the Liddell–Scott *Lexicon,* the word $\sigma\upsilon\mu\mu\epsilon\tau\rho\iota\alpha$ is given two general mean- ings:

commensurability, opp. ἀσυμμετρία . . . **II.** *symmetry, due proportion,* one of the characteristics of beauty and goodness . . .

We have considered the first meaning. The second seems not to be specifically mathematical. A key citation is to Plato's *Philebus,* here in Fowler's translation in the Loeb edition [50, 64D–65A], with some parenthetical elaborations by me:

> SOCRATES. And it is quite easy to see the cause (αἰτία) which makes any mixture (μῖξις), whatsoever either of the highest value or none at all.
>
> PROTARCHUS. What do you mean?
>
> SOC. Why, everybody knows that.
>
> PRO. Knows what?
>
> SOC. That any compound (σύγκρασις), however made, which lacks measure and proportion (μέτρου καὶ τῆς συμμέτρου φύσεως μὴ τυχοῦσα, more literally, "which does not happen to have measure and a commensurate nature") must necessarily destroy its components, and first of all itself; for it is in truth no compound (κρᾶσις), but an uncompounded (ἄκρατος "unmixed, pure, perfect") jumble (συμπεφορημένη), and is always a misfortune to those who possess it.
>
> PRO. Perfectly true.
>
> SOC. So now the power of the good has taken refuge in the nature of the beautiful; for measure and proportion (μετριότης καὶ συμμετρία) are everywhere identified with beauty and virtue.
>
> PRO. Certainly.
>
> SOC. We said that truth also was mingled with them in the compound.
>
> PRO. Certainly.
>
> SOC. Then if we cannot catch the good with the aid of one idea, let us run it down with three—beauty, proportion, and truth, and let us say that these, considered as one, may more properly than all other components of the mixture be regarded as the cause, and that through the goodness of these the mixture itself has been made good.
>
> PRO. Quite right.

Thus Fowler translates συμμετρία as "proportion." Jowett uses "symmetry" [48, pages 637–8].

Is there any connection to mathematics here? Presumably Plato knows the technical meaning of *συμμετρία* as commensurability. Thus the words that he puts in the mouth of Socrates suggest an architectural theory whereby the sides of rectangles used in beautiful buildings ought to be in the ratios of small whole numbers, just as musical harmonies are played on strings whose lengths are in such ratios (assuming uniform density and tension).

It has been argued in modern times that the Greeks in fact used a different design principle, based on what we call the golden ratio, but Euclid calls extreme and mean ratio (*ἄκρος καὶ μέσος λόγος*) in Book VI of the *Elements*: two magnitudes A and B are in this ratio, A being the greater, if they satisfy the proportion

$$A + B : A :: A : B, \tag{10}$$

where the one extreme, $A + B$, is the sum of the other extreme, B, and the mean, A. In this case, A and B are incommensurable. One proof of this theorem is that the Euclidean algorithm, applied to A and B, does not terminate, since by "separation" of the ratios in (10) as in Book V of the *Elements,*

$$B : A :: A - B : B.$$

Knorr argues [32, ch. II] that the first discovered instance of incommensurability was that of the diagonal and side of a square; even to *define* the extreme and mean ratio takes too much mathematical sophistication. However, using the theory of incommensurability alluded to in Plato's dialogue the *Theaetetus* [47, 147D–E, page 25], Theodorus could well have derived the incommensurability of two magnitudes in extreme and mean ratio—in our terms, the ratio of $\sqrt{5} + 1$ to 2—from that of the legs of the right triangle with sides that are, in our terms, 2, $\sqrt{5}$, and 3 [32, ch. VI]. In particular, Plato would likely have known that the extreme and mean ratio is, in our terms, "irrational." He might then have questioned its use in architecture, if it had been in use.

In any case, since we have seen that *συμμετρία* may be translated as "proportion," let us note that the word for a mathematical proportion is, for Euclid at least (as in Book V of the *Elements*), *ἀναλογία*, while to be proportional is to be *ἀνάλογος*, that is, "according to a [common] ratio." In particular, a proportion such as (10) is not an *equation* of ratios, but a "sameness" or *identification* of ratios. Knorr (for example) overlooks the distinction when he writes [32, page 15],

(c) A 'ratio' (λόγος) is a comparison of homogeneous quantities (i.e., numbers or magnitudes) in respect of size. A 'proportion' (ἀναλογία) is an equality of two ratios. Four magnitudes are 'in proportion' (ἀνάλογον) when the first and second have the same ratio to each other that the third and fourth have to each other . . .

We observed earlier that Euclid's equality is congruence, which can be detected by superposition. Equality is a possible property of two magnitudes. The presence of a *proportion* among *four* magnitudes is more subtle to detect. The magnitudes have ratios in pairs, but these ratios themselves are not magnitudes, and they cannot be placed alongside or atop one another. One does have such results as Proposition 9 of Book V of the *Elements*:

Τὰ πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν.

Those having to the same the same ratio are equal to one another; also, those to which the same has the same ratio, they are equal.

Symbolically,

$$A : C :: B : C \implies A = B,$$
$$C : A :: C : B \implies A = B.$$

This can be used to establish the equality of figures, such as pyramids, that are not congruent to one another, even part by part.

It is valuable to recognize the distinction between equality and sameness, if only because it can help prevent an error in interpreting Euclid's vague definition of proportions of numbers in Book VII of the *Elements.* The error has led modern mathematicians to think that the definition leads *Euclid* to error. The modern error is to think that, according to Euclid, we can establish a proportion

$$A : B :: C : D \tag{11}$$

of numbers simply by observing that for *some* numbers E and F and multipliers k and ℓ,

$$A = kE, \qquad B = \ell E, \qquad C = kF, \qquad D = \ell F.$$

Here the pair (k, ℓ) is not uniquely determined by the "ratio" (whatever that means) of A to B or of C to D. Since we are trying to establish *sameness* of those two ratios, and sameness *obviously* has the property that we call transitivity, while the proposed test for proportionality does not by itself establish transitivity, the test must not be Euclid's. We must first require E to be the *greatest* common measure of A and B; and F, of C and D. In other words, the proportion (11) means the Euclidean algorithm has the same steps, whether applied to A and B or C and D. I spell this out in another essay (in preparation).

4.3.2. Aristotle

In the *Metaphysics* [5, XIII.ɪɪɪ.10, 1078ᵃ35], Aristotle makes a general statement about *συμμετρία* that is more or less in agreement with Plato's *Philebus*:

> τοῦ δὲ καλοῦ μέγιστα εἴδη τάξις καὶ συμμετρία καὶ τὸ ὡρισμένον, ἃ μάλιστα δεικνύουσιν αἱ μαθηματικαὶ ἐπιστῆμαι.
>
> The main species of beauty are orderly arrangement, proportion, and definiteness; and these are especially manifested by the mathematical sciences.

It is not clear here whether mathematics *is* symmetric, or only concerns symmetrical (and orderly, well-defined) things. Aristotle's comment is preceeded by:

> And since goodness is distinct from beauty (for it is always in actions that goodness is present, whereas beauty is also in immovable things), they are in error who assert that the mathematical sciences tell us nothing about beauty or goodness . . .

The passage does not suggest what symmetry is. Earlier in the *Metaphysics* [6, IV.ɪɪ.18, 1004ᵇ11], Aristotle says:

> ἐπεὶ ὥσπερ ἔστι καὶ ἀριθμοῦ ᾗ ἀριθμὸς ἴδια πάθη, οἷον περιττότης ἀρτιότης, συμμετρία ἰσότης, ὑπεροχὴ ἔλλειψις, καὶ ταῦτα καὶ καθ᾽ αὐτοὺς καὶ πρὸς ἀλλήλους ὑπάρχει τοῖς ἀριθμοῖς . . . οὕτω καὶ τῷ ὄντι ᾗ ὂν ἔστι τινὰ ἴδια, καὶ ταῦτ᾽ ἐστὶ περὶ ὧν τοῦ φιλοσόφου ἐπισκέψασθαι τὸ ἀληθές.

For just as number *qua* number has its peculiar modifications, *e.g.* oddness and evenness, commensurability and equality, excess and defect, and these things are inherent in numbers both considered independently and in relation to other numbers . . . so Being *qua* Being has certain peculiar modifications, and it is about these that it is the philosopher's function to discover the truth.

Thus properties of numbers are given as examples, and they come in correlative pairs:

περιττότης	ἀρτιότης	oddness	evenness
συμμετρία	ἰσότης	symmetry	equality
ὑπεροχὴ	ἔλλειψις	excess	defect

Every number is even or odd, but not both. Excess and defect could be a number's superabundance and deficiency of factors, as discussed by Nicomachus. This leaves out perfection, unless this is implied by equality; but in that case, what is symmetry? Possibly for Aristotle every *pair* of numbers is either equal or, if not equal, then at least symmetric in the sense of having a common measure (be this unity or a number of units).

Aristotle does recognize the possibility of "asymmetric" or incommensurable pairs of mathematical objects [5, XI.iii.7 (1061ᵃ28)]:

And just as the mathematician makes a study of abstractions (for in his investigations he first abstracts everything that is sensible, such as weight and lightness, hardness and its contrary, and also heat and cold and all other sensible contrarieties, leaving only quantity and continuity—sometimes in one, sometimes in two and sometimes in three dimensions—and their affections *qua* quantitative and continuous, and does not study them with respect to any other thing; and in some cases investigates the relative positions of things and the properties of these, and in others their commensurability or incommensurability [τὰς συμμετρίας καὶ ἀσυμμετρίας], and in others their ratios; yet nevertheless we hold that there is one and the same science of all these things, viz. geometry), so it is the same with regard to Being.

Symmetry or commensurability in a more practical context arises in the *Nichomachean Ethics* [8, V.5, 1133ᵇ16, pages 100–1]:

> *τὸ δὴ νόμισμα ὥσπερ μέτρον σύμμετρα ποιῆσαν ἰσάζει· οὔτε γὰρ ἂν μὴ οὔσης ἀλλαγῆς κοινωνία ἦν, οὔτ' ἀλλαγὴ ἰσότητος μὴ οὔσης, οὔτ' ἰσότης μὴ οὔσης συμμετρίας. τῇ μὲν οὖν ἀληθείᾳ ἀδύνατον τὰ τοσοῦτον διαφέροντα σύμμετρα γενέσθαι, πρὸς δὲ τὴν χρείαν ἐνδέχεται ἱκανῶς. ἕν δή τι δεῖ εἶναι, τοῦτο δ' ἐξ ὑποθέσεως· διὸ νόμισμα καλεῖται· τοῦτο γὰρ πάντα ποιεῖ σύμμετρα· μετρεῖται γὰρ πάντα νομίσματι.*

Crisp translates thus [9, page 91]:

> So money makes things commensurable as a measure does, and equates them; for without exchange there would be no association between people, without equality no exchange, and without commensurability no equality. It is impossible that things differing to such a degree should become truly commensurable, but in relation to demand they can become commensurable enough. So there must be some one standard, and it must be on an agreed basis—which is why money is called *nomisma.* Money makes all things commensurable, since everything is measured by money.

The earlier Ross translation [3, page 1101–2] of the first part is,

> Money, then, acting as a measure, makes goods commensurate and equates them; for neither would there have been association if there were not exchange, nor exchange if there were not equality, nor equality if there were not commensurability.

The following might be more literal:

> Money equalizes, as measure makes commensurable. For, there being no exchange, there would be no association;—no exchange, there being no equality; no equality, there being no commensurability.

In particular, it seems to me that "measure" can be understood as the subject of "make commensurable," while "money" is only the subject of "equalize." Evidently equating or equalizing is not making things the *same*. One might translate the verb ἰσάζω here also as "balance." Money makes it possible to balance dissimilar goods, though as Aristotle says, the balance is never perfect.

Symmetry in the sense of balance is mentioned in the *Physics* [4, VII.iii, 246ᵇ3]:

> ἔτι δὲ καί φαμεν ἁπάσας εἶναι τὰς ἀρετὰς ἐν τῷ πρός τι πὼς ἔχειν. τὰς μὲν γὰρ τοῦ σώματος, οἷον ὑγίειαν καὶ εὐεξίαν, ἐν κράσει καὶ συμμετρίᾳ θερμῶν καὶ ψυχρῶν τίθεμεν, ἢ αὐτῶν πρὸς αὑτὰ τῶν ἐντὸς ἢ πρὸς τὸ περιέχον.

Apostle [7, pages 139–40] renders this:

> Further, we also speak of virtues as coming under things which are such that they are somehow related to something. For we take the virtues of the body, such as health and good physical condition, to be mixtures and right proportions of hot and cold, in relation either to one another or to the surroundings.

Apostol's "right proportion"—what I would understand as balance—is just Aristotle's συμμετρία.

If a holy temple or a human face exhibits what we call bilateral symmetry, it is balanced. This would seem to be the connection between the ancient *symmetria* and modern mathematical symmetry. The connection is tenuous, as we should expect, since there can be no strict rule, no practical formula, for determining unambiguously what is beautiful or balanced or symmetrical in life. Such a rule or formula might be proposed; but then one will be able to follow its letter, while ignoring its spirit.

There is likewise no strict rule for what is good mathematics. This can be understood as an implication of Gödel's Incompleteness Theorem. Mathematics cannot be the cranking out of all logical consequences of a given set of axioms. Negative in form, this conclusion is positive in content: "mathematical thinking is, and must remain, essentially creative," as Post said in 1944 [51, page 295], in a passage quoted by Soare in his 1987 recursion theory text [55, page x]. There *are* complete axiomatizations of some interesting theories,

such as the first-order theory of the ordered field of real numbers. A goal of model theory is to identify axiomatizable complete theories. However, one must still decide for oneself, and one must convince others, that this or that theory is worth studying. This obligation is also liberation. Likewise must one decide for oneself what is beautiful.

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