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Calculus and the Computer: A Conservative Approach

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Introduction

This paper describes a program for making the use of numerical methods an integral part of the freshman college course in single variable calculus. It began because Milton Lees of Case Western Reserve University, and the author felt that since the high speed digital computer has had such a profound effect on how mathematics is applied, it should also have an effect on how it is taught.

We realized also that any effective approach to integrating numerical methods into the calculus had to be a conservative one, since mathematics departments are under pressure from their colleagues in engineering and the physical sciences to present a large number of ideas and techniques in rapid order in this freshman course. Hence only a limited amount of new material would be tolerated, and it would have to blend naturally into the standard topics of the calculus. In particular, no large amount of the student's time could be occupied with programming, and one could not rely on the ability or willingness of the typical calculus instructor to teach programming or supervise programming activity. Also, the computing facilities of many colleges and universities are so over-taxed at present that they could not accommodate the large number of freshmen who study calculus and provide them with adequate programming assistance.

For these reasons, Milton Lees and the author began preparing text materials in the summer of 1966 which would not depend on access to automatic computing equipment. We felt it feasible at Case, beginning in the spring of 1967, to run an auxiliary computing facility, and, after providing a small amount of programming instruction, we could assign a small number of homework problems to be run on a digital computer. The text materials and computer laboratory materials were class-tested and revised in 1967, 1968 and 1969 with cooperation and support from the Worth Publishers, Inc., and Case Western Reserve University. The final version of the text is now available [HL] and will be accompanied by a computing manual written by Professor and Mrs. Judah Rosenblatt of Case Western Reserve University [R]. The text materials have also been class-tested at Briggs College of Michigan State University, the University of Pennsylvania, Tulane University and Baldwin-Wallace College.

The Numerical Methods

To orient the course naturally in the direction of numerical methods we introduce the limit process via sequences. That is, we say that a sequence $\langle x_n \rangle$ has limit ℓ if, for every $\varepsilon > 0$, there is a positive integer N such that

(1) $\left| x_n - \ell \right| < \varepsilon$ whenever $n \geq N$.

The standard rules for calculating derivatives and manipulating continuous functions are developed side by side with the rules for calculating limits of sums, products, and quotients of convergent sequences. Then the student is faced with the problem of approximating to within some prescribed tolerance ϵ, the limit of a sequence $\langle x_n \rangle$, where ℓ is not a rational number. The first such example is the Newton sequence $\langle x_n \rangle$ for approximating the square root of a positive (non-square) rational number. That is, we let x_0 be any positive number, and we let

(2) $$x_{n+1} = \frac{x_n^2 + a}{2x_n} \text{ for } n \geq 1.$$

Since $\langle x_n \rangle$ is a bounded decreasing sequence for $n \geq 2$, it converges, and, by (2), its limit must be $\sqrt{a}$. To approximate $\sqrt{a}$ to within ε, one calculates successively, x_1, $(x_1-\varepsilon)^2$, x_2, $(x_2-\varepsilon)^2$, ..., and stops when $(x_N-\varepsilon)^2 < a$.

This has the virtue of determining the term of least index such that $\left| x_N - \sqrt{a} \right| < \varepsilon$; it also gives a concrete meaning to N in the definition of limit given in (1).

Later we generalized this procedure to provide a stopping rule for Newton's method in general in the case of the Newton sequence $\langle x_n \rangle$ given by

(3) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n \geq 1.$$

As is shown in [H] or [HL, p.209]

(4) if f' is differentiatiable and is either strictly positive or strictly negative on the interval $[a;b]$, if $f(a)\,f(b) < 0$, if f'' fails to change sign on $[a;b]$, and if $|f(c)/f'(c)| \leq (b-a)$, where c is that end point of $[a;b]$ at which $|f''(x)|$ is smallest, then the Nedon sequence $\langle x_n \rangle$ given by (3) converges monotonely to the unique zero r of f in $[a;b]$. Moreover, x_n is increasing or decreasing according as $f'f'' \leq 0$ or $f'f'' \geq 0$ on $[a;b]$.

When the hypotheses of (4) hold, one may approximate r to within ϵ as follows, say in case $\langle x_n \rangle$ is increasing. Calculate successively x_1, $f(x_1)$, $f(x_1+\epsilon)$, x_2, $f(x_2)$, $f(x_2+\epsilon)$, and stop when $f(x_n)$ and $f(x_n+\epsilon)$ have opposite signs.

Our disucssion of numerical integration is fairly standard. We consider Euler's method, the trapezoidal rule and Simpson's rule, and we obtain the standard error bounds for them in terms of the absulute values of appropriate derivatives. But we take pains to indicate how, innately, these error bounds are conservative and refer the reader interested in learning more to the excellent treatment of numerical integration by Davis and Rabinovich [DR].

Infinite series arise naturally when the transcendental functions are introduced. After appropriate physical motivation we define the natural exponential function e^x as the solution of the initial value problem.

(5) $$\phi' = \phi \quad , \quad \phi(0) = 1$$

Its qualitative properties are derived directly from (5) and we show next that

(6) $$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This is an infinite series of positive terms if $x > 0$ and is an alternating series if $x < 0$. Indeed, almost all of the infinite series that arise in the calculus are of one of these two types.

The classical integral test tells us that

(7) if $\sum_{k=1}^{\infty} a_k$ converges and f is a decreasing positive continuous function such that $f(k) = a_k$ for every positive integer k, then $\int_{n+1}^{\infty} f(x)\,dx \leq \left(\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right) \leq \int_{n}^{\infty} f(x)\,dx.$

For infinite series $\sum_{k=1}^{\infty} a_k$ of positive terms for which $\lim_{n\to\infty} (a_{n+1}/a_n) < 1$, the following error bound is more efficient. See [HL, p. 379].

<u>if</u> $(a_{n+2}/a_{n+1}) \leq p < 1$ for $n \geq N$, <u>then</u>

(8) $$a_{n+1} \leq \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{\infty} a_k \right| \leq \frac{a_{n+1}}{1-p}$$

For example, using (8) on the series for e^x given in (6) yields

(9) $$\frac{x^{n+1}}{(n+1)!} < \left(e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right) < \frac{x^{n+1}}{(n+1)!} \frac{n+2}{n+2-x} \quad \text{if } 0 < x < n+2,$$

while applying (7) to the more slowly convergent series $\sum_{k=1}^{\infty} \frac{1}{k}$ yields

$$(10) \quad \frac{1}{(\alpha-1)(n+1)^{\alpha-1}} \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} - \sum_{k=1}^{n} \frac{1}{k^{\alpha}}\right) \leq \frac{1}{(\alpha-1)n^{\alpha-1}} \text{ if } \alpha > 1.$$

The standard alternating series test tells us that

(11) if $\langle a_n \rangle$ is a decreasing null sequence of positive numbers, then

$$\left| \sum_{k=1}^{n} (-1)^{k+1} a_k - \sum_{k=1}^{\infty} (-1)^{k+1} a_k \right| \leq a_{n+1}$$

If we apply it to the series for e^{-x} obtained by replacing x by $-x$ in (6), we get

$$(12) \quad \left| e^{-x} - \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} \right| < \frac{x^{n+1}}{(n+1)!} \text{ if } n \geq x > 0.$$

For alternating series that converge more slowly, we use instead; see [HL, p. 394]

(13) if $\langle a_n \rangle$ is a decreasing null sequence of positive numbers such that $a_{n+1} \leq (a_n + a_{n+2})/2$ for every positive integer n, then

$$\left| \left[\sum_{k=1}^{n-1} (-1)^{k+1} a_k + \frac{(-1)^n}{2} a_n \right] - \sum_{k=1}^{\infty} (-1)^{k+1} a_k \right| \leq \frac{1}{2} \left| a_n - a_{n+1} \right|$$

The hypothesis of (13) will be satisfied if there is a function f that decreases to 0 as $x \to \infty$, with $f(k) = a_k$ for $k \geq 1$, and $f''(x) \geq 0$ for $x \geq 1$. If we apply (13) to the alternating harmonic series approximation of ln 2, we get

$$(14) \quad \left| \left[\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} + \frac{(-1)^n}{n} \right] - \ln 2 \right| < \frac{1}{2n(n+1)}$$

The natural logarithm is developed as the inverse function of e^x and may be approximated efficiently by using the series

$$\ln\left(\frac{1+x}{1-x}\right) = 2 \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1} \text{ if } |x| < 1$$

For details, see [0] or [HL, p. 403 ff].

We express e^x, $\sin x$, $\cos x$, Arcsin x, Arctan x, and $(1+x)^{\alpha}$ by their usual power series and apply (7), (8), (12), or (13) to approximate them. We conclude by noting that it can sometimes be more efficient to use an approximation technique than a closed form solution. For example, by using partial fractions, one may show that

$$(15) \quad \int_0^{1/3} \frac{dx}{1+x^3} = \frac{1}{4} \ln \frac{4}{3} - \frac{1}{6} \ln \frac{7}{9} - \frac{\sqrt{3}}{3} \text{Arctan} \frac{\sqrt{3}}{9} + \frac{\sqrt{3}\pi}{18}$$

But using (15) to obtain a decimal approximation of this integral is a gruesome task at best. On the other hand, if one writes $1/(1+x^3)$ as a geometric series, integrates the result term by term, and applies the alternating series test to the result, one obtains

$$\left| \int_0^{1/3} \frac{dx}{1+x^3} - \left(\frac{1}{3} - \frac{1}{4 \cdot 3^4} \right) \right| < \frac{1}{7 \cdot 3^7} < 7 \times 10^{-5}$$

From this, one obtains with a little arithmetic that

$$\left| \int_0^{1/3} \frac{dx}{1+x^3} - 0.3302 \right| < 1.2 \times 10^{-4}$$

See [HL, p. 530 ff] or [DR].

The Computer Laboratory

After some initial experiments involving about 90 students in the spring of 1967, we began at Case Western Reserve University to run a computer laboratory which would accommodate about 500 students studying from our test materials. Because of the simple nature of the numerical methods used, it was possible to teach all the programming needed in 4 evening lectures (given by Fred Way of the Computing Center). The language used was ALGOL 60. The cards were fed into a UNIVAC 1004 which communicated by telephone line to our main computer, a UNIVAC 1107 (replaced in 1968 by a UNIVAC 1108). During three 20-minute periods of the day, our programs received priority and turn-around time was quite rapid. All of the programs were short, averaging about 5 seconds of 1107 time (or 1.25 seconds of 1108), but many runs were needed because of programming errors. Students worked in teams of two and did 5 computer problems per semester.

Mrs. Ruth Lees was in charge of the computing laboratory from its inception in March, 1967 through May, 1968. She had two to three undergraduate assistants available at any time and a tab operator to help her. The undergraduate assistants helped students with programming difficulties, but referred students to their instructors if they had difficulties with the calculus.

The problems themselves were formulated by Milton Lees and me, but were run in advance by Mrs. Lees. Her advance runs sometimes uncovered difficulties (e.g., with round-off error) which forced modifications. Due dates of the assignments were staggered to avoid queing problems.

In the fall of 1968 Professor Judah Rosenblatt took charge of the laboratory with the assistance of Ruth Lees and Lisa Rosenblatt. The Rosenblatts wrote a brief ALGOL manual (about 18 pages) improving on an earlier one written by Ruth Lees, which gave instructions in programming adequate for problems in the calculus and augmented considerably our earlier list of computer problems. Since then they have also written an equally brief Fortran manual, and these two manuals, together with the list of computer problems, will appear shortly [R].

The latest modification of the program was a decision made in the spring of 1969 to convert the computing laboratory part of the calculus course into a separate one credit course to allow the students more time and to avoid grading difficulties. (What do you do with a student who learns calculus well, but not programming, or vice versa?) The costs of running the laboratory were defrayed (in part) by the National Science Foundation.

Other Programs

There are a number of other programs involving calculus and the computer being tried here and there, but, unfortunately, communication about such programs within the mathematical community is poor. The status of a number of such programs as of 1967 is reported on in a newsletter issued by the Committee on the Undergraduate Program in Mathematics in 1969 (see [C]).

The most avante garde program along these lines is the one issued by the Center for Research in College Instruction in Science and Mathematics (CRICISAM) with the support of the National Science Foundation. A group of mathematicians, including Warren Stenberg of the University of Minnesota and R. J. Walker of Cornell University, prepared text materials for computer oriented calculus while visiting Florida State University (see [CR]). These materials integrate the teaching of programming as well as numerical methods into single variable calculus. Until there is an adequate supply of instructors qualified to teach programming as well as calculus, the adoption of this program will be limited. With this problem in mind, a summer institute for training college teachers to use these materials will be held in the summer of 1970 at Florida State University.

Summary

Running programs of this sort takes considerable effort, and, since they call for closer cooperation between calculus instructors than is normally forthcoming, results in considerable trial and tribulation. Nevertheless, it is the opinion of the author that the calculus cannot survive in the college curriculum for much longer without somehow taking into account the impact of the digital computer on science and technology. Certainly the day is not far off when a knowledge of the elements of numerical analysis and programming languages will be regarded as equally essential to students of science and engineering as knowing how to differentiate and integrate.

References

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