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From Pythagoreans and Weierstrassians to True Infinitesimal Calculus

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Abstract

In teaching infinitesimal calculus we sought to present basic concepts like continuity and convergence by comparing and contrasting various definitions, rather than presenting "the definition" to the students as a monolithic absolute. We hope that our experiences could be useful to other instructors wishing to follow this method of instruction. A poll run at the conclusion of the course indicates that students tend to favor infinitesimal definitions over ϵ-δ ones.

1. Introduction

> *Lübsen defined the differential quotient first by means of the limit notion; but along side of this he placed (after the second edition) what he considered to be the* true infinitesimal calculus *– a mystical scheme of operating with infinitely small quantities.*
>
> –Felix Klein, [32, page 217].

Starting from the assumption that multiple approaches to the same concept can facilitate student learning, during the 2014-2015 academic year we taught *True Infinitesimal Calculus* (TIC) based on [31] to about 120 first-year college students. A similar course was taught during the 2015-2016 year to about 130 students, and is being taught during the 2016-2017 year to a similar number of students. Most of the students had already seen the basic techniques of the calculus in their high school courses.

Keisler's book was reviewed in [6] but Bishop's position is dictated by a broad opposition to all of classical mathematics, as spelled out four years earlier in his "Schizophrenia" text [5]; see [26] for details. A sympathetic historical account of infinitesimals can be found in [1].

In an effort to quantify student attitudes toward (1) the *Epsilontik* and (2) infinitesimals, we ran a poll at the end of the course. The goal was to compare student reactions to the two approaches, as well as to gauge the helpfulness of each approach in their eyes. A total of 84 students participated in the poll.

Two-thirds of the respondents to the poll felt that infinitesimal definitions of three key calculus concepts helped them understand the concept, while only one in seven felt that ϵ-δ definitions helped them understand the concept.

We refer to the approach used in the course as TIC to distinguish it from the traditional *Epsilon-Delta Calculus* (EDC). While EDC is often referred to as infinitesimal calculus, the use of the adjective *infinitesimal* in that term is something of a dead metaphor, since no infinitesimals are actually used in such courses except at best in some motivating discussions aimed to enhance student intuitions.

In our course, two types of definitions of three key mathematical concepts (continuity, uniform continuity, and convergence) were given:

(A) the usual ϵ-δ definition;
(B) the infinitesimal definition.

The course first presented the infinitesimal definition (B-track, for *Bernoullian*) and then the ϵ-δ definition (A-track, for *Archimedean*). We amplified the treatment in Keisler due to the demands of the second semester sequel taught the EDC way. In particular, we expanded Keisler's treatment of the ϵ-δ approach, and added a treatment of the concept of uniform continuity. The following points should be kept in mind.

- The first edition of Keisler's book [30] was the first ever calculus textbook using rigorous infinitesimals.

- Neither Archimedes nor Bernoulli envisioned anything like the set-theoretic ontology underpinning the construction of modern punctiform continua. Modern terms like *Archimedean continuum* and *Bernoullian continuum* refer not to *ontology* of mathematical entities but rather to the *procedures* typically used in the respective frameworks.[1]

- The procedures in Robinson's framework provide closer proxies for the procedures of historical infinitesimalists like Leibniz, Bernoulli, Euler, and Cauchy than do the procedures in the modern Weierstrassian framework, which similarly relies on a punctiform continuum.

- A popularisation of infinitesimals exploiting the field of rational functions was developed by D. Tall under the name *superreal number system* (similarly a punctiform continuum); see for instance [41]. However this system lacks a transfer principle (see Section 2.3) and cannot serve as a basis for a rigorous course in the calculus.

The ϵ-δ definitions were a triumph of formalisation mathematically speaking, but create pedagogical difficulties when introduced without preparation, according to most scholars who have studied the problem; see for example [12]. Our approach enables the teacher to prepare the students for ϵ-δ by explaining the concepts first using a rigorous infinitesimal approach. Studies of methodology involving modern infinitesimals include [16, 21, 35, 40, 42, 47].

We sought to impart the fundamental concepts of the calculus in a way that is the least painful to the students, while making sure that they have the necessary background in the ϵ-δ techniques to continue in the second semester course taught via EDC. Once the students understand the basic concepts via their intuitive B-track formulations, they have an easier time relating to the A-track paraphrases of the definitions.

Recently Robinson's framework has become more visible, thanks to high-profile advocates like Terence Tao; see for instance [43, 44]. The field has also had its share of high-profile detractors, such as Errett Bishop and Alain Connes. Their critiques were critically analyzed in [25, 26, 28]. This conversation is clearly not over. Readers can find other relatively unfriendly

[1] Readers who would like to learn more about the specific terms used and the point made here on ontology vs procedures might find [8] helpful.

treatments of Robinson's framework, often in the context of other historical discussions of mathematics, in works by J. Earman [13], K. Easwaran [14], H. M. Edwards [15], G. Ferraro [17], J. Gray [18], P. Halmos [20], H. Ishiguro [23], G. Schubring [38], and Y. Sergeyev [39], and rebuttals and counterarguments against some of these in [2, 3, 4, 7, 8, 9, 10, 19, 24, 27, 29].

2. Review of definitions

In this section we will review both the A-track and the B-track definitions of the three calculus concepts that the students were polled on.

2.1. Procedure vs ontology

As a prefatory remark, we would like to respond to a common objection that to do calculus with infinitesimals you need to get a PhD in nonstandard analysis first, and that is obviously not a good way of teaching first-year college calculus.

One possible response is that a certain amount of foundational material needs to be taken for granted in either approach. Thus, the real number system is not constructed in the EDC approach. Instead, certain subtle properties, like the existence of limits, closely related to the completeness of the reals, are assumed on faith.

There is general agreement that in a calculus course we do not elaborate the exact details concerning the real numbers with respect to their ontological status in foundational theories such as the Zermelo-Fraenkel set theory (ZFC). This is because the *procedures* of the calculus do not depend on the *ontological* issues of set-theoretic axiomatisations. In the traditional approach to the calculus, we present all the procedures rigorously, including all the epsilon-delta definitions, while staying away from such ontological and foundational issues.

Similarly in developing TIC, we do not elaborate the set-theoretic issues of the precise ontological status of the hyperreals in a ZFC framework. Rather, we teach our students the procedures of the calculus exploiting infinitesimals in a fully rigorous way, including the more intuitive infinitesimal definitions of the key concepts.

2.2. A-track definitions of key concepts

A function f is said to be *continuous* at a point $c \in \mathbb{R}$ if the following condition is satisfied:

$$(\forall \epsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \mathbb{R})\big[|x-c| < \delta \;\Rightarrow\; |f(x)-f(c)| < \epsilon\big]. \tag{1}$$

A function f is said to be *uniformly continuous* in a domain $D \subseteq \mathbb{R}$ if the following condition is satisfied:

$$\begin{aligned}&(\forall \epsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in D)(\forall x' \in D)\\ &\qquad\qquad \big[|x'-x| < \delta \;\Rightarrow\; |f(x')-f(x)| < \epsilon\big].\end{aligned}$$

A sequence (u_n) is said to *converge* to $L \in \mathbb{R}$ if the following condition is satisfied:

$$(\forall \epsilon \in \mathbb{R}^+)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})\big[n > N \;\Rightarrow\; |u_n - L| < \epsilon\big].$$

The above A-track definitions are succinct summaries of the distilled mathematical content of these concepts; we did not present them in this fashion in first-year college calculus, but rather in a much slower pace.

2.3. B-track definitions of key concepts

We now review the corresponding B-track definitions in more detail, since they are less likely to be familiar to modern readers usually educated in EDC frameworks. What is involved is a hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$, where ${}^*\mathbb{R}$ is an ordered field including both infinitesimal (see below) and infinite numbers. A key tool in working with such an extension is the *transfer principle* (see below).

Such fields can be constructed from sets of sequences of real numbers, similar to the construction of the real numbers from the rational numbers.

An *infinitesimal* $\alpha \in {}^*\mathbb{R}$ is a number satisfying $|\alpha| < r$ for every positive real r. An *infinite number* H is a number satisfying $|H| > r$ for every real number r.

We teach the students to work with these new numbers, and to apply the basic rules of arithmetic to them (infinitesimal times infinitesimal is infinitesimal, infinitesimal times infinite can have any order of magnitude, etc.), and what "being infinitely close" means (see Section 2.4).

We then introduce the *standard part function* (sometimes called *shadow*). This is a function from the finite (i.e., not infinite) hyperreals to the reals, which rounds off each finite hyperreal to its nearest real number.

The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are "transfered") to an extended number system. Thus, the familiar extension $\mathbb{Q} \hookrightarrow \mathbb{R}$ preserves the properties of an ordered field. To give a negative example, the extension $\mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm\infty\}$ of the real numbers to the so-called *extended reals* does not preserve the properties of an ordered field. The hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ preserves *all* first-order properties. For example, the identity $\sin^2 x + \cos^2 x = 1$ remains valid for all hyperreal x, including infinitesimal and infinite values of $x \in {}^*\mathbb{R}$. Another example of a transferable statement is the property that

$$\text{for all positive reals } x, y, \text{ if } x < y, \text{ then } \frac{1}{y} < \frac{1}{x}.$$

Transfer applies to formulas like (1) that quantify over elements of $\mathbb{R}$, but not directly to statements that quantify over *sets* of elements. Thus, the completeness property of the reals, which involves quantification over sets, does not transfer directly. For a more detailed discussion, see the textbook *Elementary Calculus* [31].

2.4. Microcontinuity

Both continuity and uniform continuity can be defined in terms of the auxiliary concept of *microcontinuity* (this term is not used by Keisler but it is used in [11]). The definition of microcontinuity exploits the natural extensions ${}^*f$ of a real function f and ${}^*D$ of a real set D, available in a hyperreal setting.

Let $D = D_f \subseteq \mathbb{R}$ be the domain of a real function f. We say that ${}^*f$ is *microcontinuous* at x if

$$\text{whenever } x' \approx x, \text{ one also has } {}^*f(x') \approx {}^*f(x), \tag{2}$$

for all x' is in the domain ${}^*D_f \subseteq {}^*\mathbb{R}$ of ${}^*f$. Here the relation $\approx$ is the relation of infinite proximity, i.e., $x' \approx x$ if and only if the difference $x' - x$ is infinitesimal.

The condition of microcontinuity can be tested not only at a real point $x \in D_f$ but also at an arbitrary hyperreal point $x \in {}^*D_f$.

Thus, the squaring function

$$y = x^2 \tag{3}$$

fails to be microcontinuous at an infinite point $H \in {}^*\mathbb{R}$. Indeed, let $\alpha = \frac{1}{H}$ so that $H \approx H + \alpha$. Note that the corresponding y-increment is

$$\Delta y = (H + \alpha)^2 - H^2 = H^2 + 2H\alpha + \alpha^2 - H^2 = 2 + \alpha^2.$$

It follows that the y-increment is appreciable rather than infinitesimal, showing that the squaring function is not microcontinuous at H.

It turns out that a real function f is *continuous* at $c \in \mathbb{R}$ if and only if ${}^*f$ is microcontinuous at c, and *uniformly continuous* in its domain D_f if and only if the following condition is satisfied:

$$(\forall x \in {}^*D_f)(\forall x' \in {}^*D_f)\left[x \approx x' \;\Rightarrow\; {}^*f(x) \approx {}^*f(x')\right].$$

Equivalently, a real function f is uniformly continuous on D_f if ${}^*f$ is microcontinuous at x for each $x \in {}^*D_f$.

To continue with the example (3) presented above, one can now state that the failure of the squaring function to be uniformly continuous on $\mathbb{R}$ is due to its failure to be microcontinuous at a single infinite point. This proof of the failure of uniform continuity of a function is of reduced quantifier complexity when compared to A-track proofs of the same fact.

2.5. Convergence

The last of the three concepts we focused on for this study is *convergence*. A sequence (u_n) *converges to* L if and only if

$$\mathrm{st}(u_H) = L$$

for all infinite values H of the index, where "st" denotes the *standard part function*.

Consider for example the sequence $u_n = \frac{n+1}{n}$. To prove that the limit is 1, we write

$$\lim_{n\to\infty} u_n = \mathrm{st}(u_H) = \mathrm{st}\left(\frac{H+1}{H}\right) = \mathrm{st}\left(1 + \frac{1}{H}\right) = 1$$

by the additive property of the shadow, since $\frac{1}{H}$ is infinitesimal. Note that rather than having to deal with an *inverse* problem, as in the EDC framework, the proof is a *direct* calculation. Moreover, it is just as rigorous as the EDC proof, the difference being that the EDC proof would necessarily involve preliminary calculations or at least a guess for a limit value.

3. The poll

The questionnaire, which mostly followed a multiple-choice format, also contained a control question asking students to prove that

$$\lim_{x\to 2}(x+5)=7$$

in two different ways: ϵ-δ (A-track) and infinitesimal (B-track); see Section 1.

Almost all the students (98%) attempted to solve the control problem via B-track, while 71% attempted to give an A-track solution. Of those who attempted a B-track solution, 85% succeeded; of those who attempted an A-track solution, 20% succeeded.

It should be noted that our students had substantial practice specifically in using the ϵ-δ methods. Our TAs spent two entire sessions on this, and the students also had to submit homework assignments where they were required to use the ϵ-δ techniques. Also, they did many exercises similar to the above using ϵs and δs.

The students were asked to comment on the helpfulness of A-track and B-track definitions of three key concepts: continuity, uniform continuity, and convergence of a sequence. More specifically, they were presented with the statement "the definition helped me understand the concept," and were given the following five options for a possible answer: (1) agree strongly; (2) agree; (3) undecided; (4) disagree; (5) disagree strongly.

With respect to the B-track definition of continuity, 69% of the students felt that the definition helped them understand the concept ("agree" or "agree strongly"). Meanwhile, 10% of the students felt that the A-track definition of continuity helped them understand the concept. Among students who were able to define continuity correctly, 75% felt the B-track definition helped them understand the concept, while 9% felt the A-track definition helped them understand it.

With regard to uniform continuity, 74% felt that the B-track definition helped them understand the concept, whereas 21% felt that the A-track definition helped them understand the concept. Among students who were able to define uniform continuity correctly, 80% felt that the B-track definition helped them understand it, whereas 24% felt that the A-track definition helped them understand it.

With regard to the definition of a convergent sequence, 62% felt that the B-track definition helped them understand the concept, whereas 10% felt that the A-track definition helped them understand it. Among students who were able to define convergence correctly, 70% felt that the B-track definition helped them understand it, whereas 13% felt that the A-track definition helped them understand it.

4. Divide-and-conquer vs paraphrase

In our poll, the percentage of students who felt that the B-track definition helped them understand the concept increases by about 7% (of the respondents) when one calculates the percentage on the basis of those students who were able to give a correct definition of the appropriate concept. A similar phenomenon occurs among students who felt that the A-track definition helped them understand the concept in the case of the concepts of uniform continuity and convergent sequences.

On average among the three concepts, over two-thirds (68%) of the students felt that the B-track definition is helpful, while only about one in seven students (14%) felt that the A-track definition is helpful.

To summarize, what we tried to do in the course is to impart to the students the fundamental concepts of the calculus in a way that is the least painful to the students, while making sure that they have the necessary background in the ϵ-δ techniques to continue in the second semester course taught via EDC. The results of the poll suggest that starting with the intuitive B-track definitions succeeds in this sense. Once the students understand the basic concepts via their intuitive B-track formulations, they are able to relate more easily to the A-track paraphrases of the definitions.

4.1. From Pythagorians to Weierstrassians and beyond

To comment on the idea of paraphrase in more detail, suppose one is limited to working with the rational numbers (for fear of getting thrown overboard by enraged Pythagorians).

Yet in one's mathematical investigations there may arise a need to express the predicate that an unknown rational number x is greater than the diagonal of a unit square. However, one is only allowed to use inequalities of the form

$$x > q$$

where q is rational. Since one is forbidden to talk about irrationals, one says instead that x is greater than every rational q such that $q^2 < 2$, or in formulas

$$\forall q \in \mathbb{Q} \left[q^2 < 2 \implies q < x \right]. \tag{4}$$

This quantified formula looks more complicated than the intended inequality, but since one already knows what it means, one can readily understand it. In other words, the complicated quantified formula (4) is merely a long-winded paraphrase for the familiar inequality $x > \sqrt{2}$.

Similarly, someone interested in property (2) that an infinitesimal change in input should always produce an infinitesimal change in output may be led to exploit the ϵ-δ formula (1) with its notorious alternating quantifiers (to avoid Hippasus' fate at the hands of enraged Weierstrassians).

These quantified formulas look complicated, but they are merely long-winded paraphrases for simpler definitions exploiting infinitesimals that were used by Cauchy but have been suppressed since 1870 when Weierstrass and his followers broke with the infinitesimal mathematics of Leibniz, Euler, and Cauchy.

Our approach could therefore be described as an application of the divide-and-conquer algorithm. One first separates the inherent difficulty of the subject of the calculus into two parts:

Part I: the intrinsic difficulty of the concepts themselves;

Part II: the technical complications of the A-track paraphrases with their notorious quantifier alternations.

The good news is that the concepts are accessible without the A-track paraphrases, thanks to [22, 33, 36]. The idea is to start with Part I, contrary to the EDC approach that starts with Part II. Our approach is more consistent with Toeplitz's thinking, discussed in the next section.

5. All the stops out

Otto Toeplitz had the following to say in 1927 about teaching infinitesimal calculus:

> I consider it an inviolable axiom that by the end of a two-semester course, a beginner should have obtained a complete understanding and complete mastery of the technique of 'epsilontic' operations, and that he did not bring such techniques with him from high school. [45, page 303]

We heartily agree with the latter point, and empathize with the former. Toeplitz continues:

> The way this has been formulated already suggests the solution. Instead of launching the 'epsilontic' methodology right away at the beginning with all the stops out, as one says of the organ, one should lead the student gradually up a gentle ascent to the peak of this technique, just as the organist uses one register after the other in a well-composed piece of organ music - and in this way not one of the 45 percent spoken of above will be left out. (ibid.)

The explanation of Toeplitz's 45% figure is as follows. Toeplitz considers that about 5% of the students are the "natural" mathematicians that will grasp the *Epsilontik* immediately and do not even need to go to the lectures.

Toeplitz also considers that about 50% of the student body present in the mathematics courses is too weak, making it difficult to structure the course based on them. Dismissing half the class in this fashion is unacceptable, and arguably is a consequence of an obligatory adherence to the EDC approach. The TIC approach makes it possible to reach close to 100% of the students, by postponing the introduction of ϵ-δ definitions as we explained in Section 4.

Toeplitz is talking about the remaining 45% who are strong students. He is arguing that the top 5% should not be taught at the expense of the 45%.

Note that Toeplitz advocated using infinitesimals and differentials at a time (1927) when it was still considered that they are irremediably lost in a hazy fog of meaninglessness, as Courant colorfully put it in his textbook.[2] Thus, in discussing Kepler's Second Law, Toeplitz does not hesitate to exploit both (infinitesimal) differentials and the notion of *utter smallness* as a pedagogical device:

> In Figure 128, dF is the area of a narrow sector of area formed by two closely neighboring radii; it can be approximated by the right triangle SPQ (PQ perpendicular to r at P). The little chip by which it exceeds dF is *utterly small* in relation to dF, and, as dF itself becomes smaller, this chip diminishes even more rapidly and therefore can be neglected. [46, page 151] [emphasis added]

Toeplitz was hardly the only one to exploit the explanatory power of such terms. A majority of mathematics educators involved in teaching calculus routinely exploit such expressions. They would say things like "the function f has limit L at a point a if we can make $f(x)$ as close to L as we wish for all x sufficiently close to a." They would say "given epsilon positive and as small as we wish…" These expressions are variations on Toeplitz's "utterly small."

This is how a majority of educators explain things to students, and this is the language they use, because this is the way we think and the way we perceive these notions. It is not merely a pedagogical device, but this is how we understand these ideas. TIC offers us a possibility of making our intuitions precise with terms like "infinitesimal" (in place of "as small as we wish") and "infinitely close" (instead of "as close as we want"). Many mathematicians think in terms of Toeplitz's "utter smallness" and related ideas. The TIC approach makes effective use not only of the students' intuitions but also of the mathematicians' intuitions about infinitesimals.

[2] Courant described infinitesimals on page 81 of *Differential and Integral Calculus*, Vol I, as "devoid of any clear meaning" and "naive befogging." Similarly on page 101, Courant described them as "incompatible with the clarity of ideas demanded in mathematics," "entirely meaningless," "fog which hung round the foundations," and a "hazy idea." Cantor, Russell, and the mathematicians of Courant's generation were convinced that infinitesimals are self-contradictory. Following [36] we know this not to be the case.

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