

1-1-1957

Some Remarks on a Paper of Aronszajn and Panitchpakdi

Melvin Henriksen
Harvey Mudd College

Recommended Citation

Henriksen, Melvin. "Some remarks on a paper of Aronszajn and Panitchpakdi." *Pacific Journal of Mathematics* 7.4 (1957): 1619-1621.

This Article is brought to you for free and open access by the HMC Faculty Scholarship at Scholarship @ Claremont. It has been accepted for inclusion in All HMC Faculty Publications and Research by an authorized administrator of Scholarship @ Claremont. For more information, please contact scholarship@cuc.claremont.edu.

SOME REMARKS ON A PAPER OF ARONSZAJN
AND PANITCHPAKDI

MELVIN HENRIKSEN

In the paper of the title [1], a number of problems are posed. Negative solutions of two of them (Problems 2 and 3) are derived in a straightforward way from a paper of L. Gillman and the present author [2].

Motivation will not be supplied since it is given amply in [1], but enough definitions are given to keep the presentation reasonably self-contained.

1. A Hausdorff space X is said to satisfy (Q_m) , where m is an finite cardinal, if, whenever U and V are disjoint open subsets of X such that each is a union of the closures of less than m open subsets of X , then U and V have disjoint closures. In particular, a normal (Hausdorff) space X satisfies (Q_{\aleph_1}) if and only if disjoint open F_σ -subsets of X have disjoint closures. (For, an open set that is the union of less than \aleph_1 closed sets is a fortiori an F_σ . Conversely if U is the union of countably many closed subsets F_n , then since X is normal, for each n there is an open set U_n containing F_n whose closure is contained in U . Thus U is the union of the closures of the open sets U_n .) In Problem 3 of [1], it is asked if every compact (Hausdorff) space satisfying (Q_m) for some $m > \aleph_0$ is necessarily totally disconnected, and it is remarked that this is the case if the first axiom of countability is also assumed.

If X is a completely regular space, let $C(X)$ denote the ring of all continuous real-valued functions on X , and let $Z(f) = \{x \in X : f(x) = 0\}$, let $P(f) = \{x \in X : f(x) > 0\}$, and let $N(f) = P(-f)$. As usual, let βX denote the Stone-Čech compactification of X . If every finitely generated ideal of $C(X)$ is a principal ideal, then X is called an F -space. The following are equivalent.

- (i) X is an F -space.
- (ii) If $f \in C(X)$, then $P(f)$ and $N(f)$ are completely separated [2, Theorem 2.3].
- (iii) If $f \in C(X)$, then every bounded $g \in C(X - Z(f))$ has an extension $\bar{g} \in C(X)$ [2, Theorem 2.6].

A good supply of compact F -spaces is provided by the fact that if X is locally compact and σ -compact, then $\beta X - X$ is an F -space [2, Theorem 2.7].

Received April 1, 1957. In revised form April 29, 1957. This paper was written while the author was an Alfred P. Sloan fellow.

We remark first that a normal (Hausdorff) space X satisfies (Q_{\aleph_1}) if and only if it is an F -space.

For, suppose first that X is an F -space, and let U, V be disjoint open F_σ -subsets of X . Since $X - (U \cup V)$ is a closed G_δ in a normal space, there is a bounded $f \in C(X)$ such that $Z(f) = X - (U \cup V)$. Hence by (iii), there is a $\bar{g} \in C(X)$ such that $\bar{g}[U] = 0$ and $\bar{g}[V] = 1$. In particular, U and V have disjoint closures, so X satisfies (Q_{\aleph_1}) . Conversely let X satisfy (Q_{\aleph_1}) , and let $f \in C(X)$. Then $P(f)$ and $N(f)$ are disjoint open F_σ -subsets of X , which by (Q_{\aleph_1}) have disjoint closures. So, by Urysohn's lemma, $P(f)$ and $N(f)$ are completely separated. Thus X is an F -space by (ii).

Compact connected F -spaces exist. In particular it is known that if R^+ denotes the space of nonnegative real numbers, then $\beta R^+ - R^+$ is such a space [2, Example 2.8]. Hence Problem 3 of [1] has a negative solution.

We remark finally that if the first axiom of countability holds at a point of an F -space, then the point is isolated [2, Corollary 2.4]. In particular, every compact F -space satisfying the first axiom of countability is finite.

2. In Problem 2 of [1], it is asked (in different but equivalent language) if for every totally disconnected compact space X satisfying (Q_m) for some $m > \aleph_0$, the Boolean algebra $B(X)$ of open and closed subsets of X has the property that every subset of less than m elements has a least upper bound. A lattice is said to be (conditionally) σ -complete if every bounded countable subset has a least upper bound and a greatest lower bound. In view of the above (and since every subset of $B(X)$ is bounded), in case $m = \aleph_1$, the problem asks if for every compact totally disconnected F -space X , the Boolean algebra $B(X)$ is σ -complete.

In [3, Theorem 18], it is shown that if X is compact and totally disconnected, then $B(X)$ is σ -complete if and only if $C(X)$ is σ -complete (as a lattice). It is noted in [2, Theorem 8.3, f.f.] that for a completely regular space Y , the lattice $C(Y)$ is σ -complete if and only if $f \in C(Y)$ implies $\bar{P}(f)$ and $\bar{N}(f)$ are disjoint open and closed subsets of Y ($\bar{P}(f)$ denotes the closure of $P(f)$). It is easily seen that Y has this latter property if and only if βY has [2, Lemma 1.6].

In [2, Example 8.10], an example is given of a completely regular space X such that βX is a totally disconnected F -space, and such that $C(X)$ is not σ -complete. By the above, it follows that $B(\beta X)$ yields a negative solution to Problem 2.

We remark also (as was pointed out by J. R. Isbell) that if N denotes the countable discrete space, then $\beta N - N$ is also a totally disconnected compact F -space such that $B(\beta N - N)$ is not σ -complete. The

former assertion follows easily from the remarks in § 1, and the latter follows from the fact that $B(\beta N - N)$ is isomorphic to the Boolean algebra of all subsets of N modulo the ideal of finite subsets of N (under the correspondence induced by sending a subset of N to the intersection of its closure in βN with $\beta N - N$). It is easily verified that this latter Boolean algebra is not σ -complete.

REFERENCES

1. N. Aronszajn and P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405-439.
2. L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. **82** (1956), 366-391.
3. M. H. Stone, *Boundedness properties in function lattices*, Canadian J. Math. **1** (1949), 176-186.

THE INSTITUTE FOR ADVANCED STUDY