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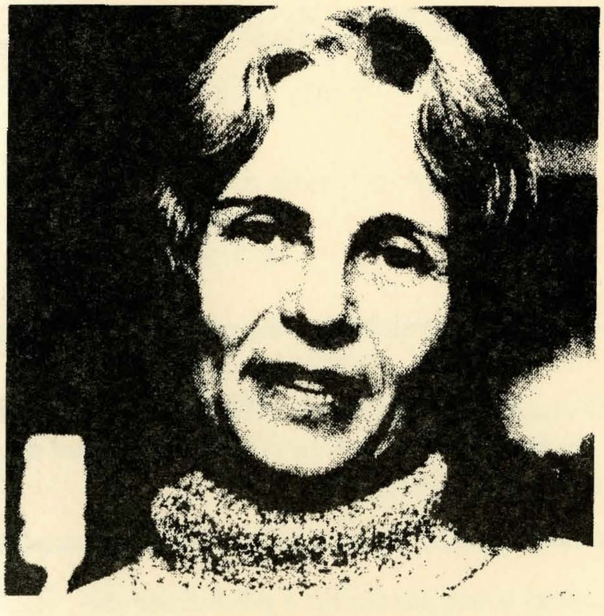

Anneli Lax

$(x-4)^2 - (x-4) - 6$

[1] [1]

$= x^2 - 4x - 4x + 16 - x + 4 - 6$

[1]

$= x^2 - 9x + 14$

[1]

$= (x-7)(x-2)$

1 point for simplifying $(x-4)^2$
1 point for simplifying $-(x-4)$
1 point for simplifying
1 point for factored expression

INVITATION TO AUTHORS

Essays, book reviews, syllabi, poetry, and letters are welcomed. Your essay should have a title, your name and address, e-mail address, and a brief summary of content. In addition, your telephone number (not for publication) would be helpful.

If possible, avoid footnotes; put references and bibliography at the end of the text, using a consistent style. Please put all figures on separate sheets of paper at the end of the text, with annotations as to where you would like them to fit within the text; these should be original photographs, or drawn in dark ink. These figures can later be returned to you if you so desire.

Two copies of your submission, double-spaced and preferably laser-printed, should be sent to:

Prof. Alvin White
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Claremont, CA 91711

Essays and other communications may also be transmitted by electronic mail to the editor at awhite@hmc.edu, or faxed to (909) 621-8366. The editor may be contacted at (909) 621-8867 if you have further questions.

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NOTE TO LIBRARIANS

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SUBSCRIPTIONS

The Humanistic Mathematics Network Journal is distributed free to all libraries and individuals who request copies and subscriptions. Please address subscription requests to Prof. Alvin White.

COVER

Return fire: In this issue Stephen Sproule, Ted Eisenberg and Margaret Schaffer respond to last issue's article by Jack Dancis. In other articles Elena Marchisotto looks back on the life of Anneli Lax, and Paul Alper and Yongzhi Yang use Pascal triangles to examine the mathematics of a lady tasting tea.

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Humanistic Mathematics Network Journal #21
December 1999

From Newsletter #1

Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other and how they might come to better understand mathematics as a meaningful rather than arbitrary discipline were among the ideas of the first theme.

The second theme focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, discovery to verification, mathematics to science, truth to utility, and in general, mathematics to the culture within which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."
b) An appreciation for the human dimensions that motivate discovery: competition, cooperation, the urge for holistic pictures.
c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for what is investigated, how it is investigated, and why it is investigated.
d) A need for new teaching/learning formats that will help discourage our students from a view of knowledge as certain or to-be-received.
e) The opportunity for students to think like mathematicians, including chances to work on tasks of low definition, generating new problems and participating in controversy over mathematical issues.
f) Opportunities for faculty to do research on issues relating to teaching and be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures. . . the network might formally support writing, team-teaching, exchanges, conferences. . . .

Alvin White
August 3, 1987

From the Editor

During the past two months issues of mathematics instruction in the schools of the United States roiled the national newspapers (*The NY Times, The Washington Post, The Los Angeles Times*) as well as email networks. In October 1999 the US Department of Education endorsed ten K-12 mathematics programs as "exemplary" or "promising."

The expert panel that made the final decisions did not include active research mathematicians. One of the expert panelists wrote in 1994 that it is time to abandon computational algorithms. "It's time to recognize that, for many students, real mathematical power...and facility with multidigit, pencil-and-paper algorithms...are mutually exclusive...[C]ontinuing to teach those skills is not only unnecessary but counterproductive and downright dangerous."

Two hundred mathematicians signed a letter asking that the endorsements be withdrawn. The letter, including the list of signers, was published in newspapers and on the internet. There ensued much discussion, pro and con, on the email networks.

Part of the discussion is an article in *The Washington Post* (10/10/99) by Alfie Kohn [http://search.washingtonpost.com/up-srv/WPlate/1999-10/10/1311-101099-idx.html]. "The best kind of teaching takes its cue from the understanding that people are active learners...It's simply not true that one must learn to read before being able to read for understanding; it makes a lot more sense to learn to read by reading for understanding...Wise educators don't teach addition and subtraction as prerequisites for pursuing interesting problems; they teach these skills through interesting problems."

But these discussions have no meaning for many students in Los Angeles and California. *The Los Angeles Times* reports (12/3/99) that the percentage of ill-trained teachers at schools serving minority students is on average six times the level at schools where most students are white. Schools serving third graders whose test scores are lowest have five times the proportion of untrained teachers as schools where children score the highest. Statewide, one in 10 teachers lacks a credential. But the ratio is higher in urban areas such as Los Angeles where one in four teachers is learning on the job. This year three out of every four teachers hired in Los Angeles had yet to obtain a credential.

Patrick Shields of SRI International said, "The state is completely committed to early literacy, and the students scoring the lowest—those who need the most help and are most at risk—are the most likely to have unqualified teachers. At nearly 1400 schools more than one in five teachers are underqualified. More than one million children attend these schools which are dysfunctional."

The question of which math programs the Department of Education recommends would seem to be irrelevant for many students.

Anneli Lax: In Memoriam

Anneli was a beloved friend and mentor. I first met her when I was studying for my Ph.D. at New York University in the 1980's. I was fortunate that she agreed to be my thesis advisor, and I benefited from her interest in my research—then and thereafter. I know I speak for many when I say that my mathematical life was deeply enriched by countless stimulating conversations with her. Her view of mathematics profoundly influenced my approach to the subject as well as how I teach it. She made me ever-conscious of connections with analysis in my investigations of geometry. She taught me how to capture the imagination of students, encouraging them to experience the rewards of doing mathematics. Her mathematical company for 20 years empowered me to make the transition from being her student to being her colleague.

Here, however, I would like to share some (more personal) images of our friendship.

My memories of Anneli are vivid with pictures. When I think of her, the first thing I see is her smile. With that smile, Anneli invited you to talk, to share ideas, to reveal concerns. She was a great listener and a great problem solver.

My second image is of her strength—not only intellectually, but physically. I picture her carrying logs at Prince Camp, swimming in the ocean in Malibu during the winter, hiking in Colorado.

I have many memories of times shared with Anneli, for which I am very grateful. I'd like to share one recent experience with you that is very dear to me. I spent some time with Anneli and Peter in the Adirondacks the summer before last—Peter calls it "a little bit of heaven on earth." Anneli was already stricken with cancer, but she was (thankfully!) the picture of good health and cheer. She swam in the lake ever morning (too cold for me!), canoed, hiked, and baked bread (from scratch!). One evening, Anneli suggested we go to pick some berries. I am a city person. I had visions of strolling down a lane with a white basket daintly selecting the delicate fruit. We set out, and Anneli drove us deep into the woods. We then walked down bug-infested paths to the berry bushes. In order to retrieve the fruit, we had to be up front and personal with these prickly growths. Anneli dove right in and filled her bucket to the brim. I, however, spent my time engaged in fending off the pesty bugs and avoiding the thorns. Still, I was invited to reap the fruits of everyone's labor and was not denied a delicious dessert which Anneli prepared that evening. When I think of that day, I carry in my heart many of the things I loved about Anneli—her quiet determination, her openness and acceptance of the weaknesses of others, and her joy in simple pleasures.

Anneli was an extraordinary person, and it was a privilege to know her and to love her. I miss her very much.

Elena Marchisotto
California State University, Northridge

Teaching a Humanities Course: A Mathematician's View

Bill Marion
Valparaiso University

ABSTRACT
When asked by a number of my friends in academe about what I am doing these days, I tell them that I am now into my third academic career: first, a mathematician; second, a computer scientist; and, third, a budding humanities instructor. During the fall of 1998 I had the opportunity to teach a first-semester section of a new two-semester Core sequence in the humanities which has become part of the General Education Program at Valparaiso University. In what follows I intend to give you some idea of what it was like for me, an outsider to the humanities' disciplines, to teach such a course. I will describe some of my thoughts about what made the experience worthwhile and what were the difficulties. Also, I will offer some reflections on how this experience has helped me, both to think more carefully about the humanistic dimensions of mathematics and to address the issue of teaching mathematics humanistically.

INTRODUCTION
First, some background information. Valparaiso University (VU) is a comprehensive, national university in the Lutheran tradition with a total of around 3 800 students. Within the University there are four undergraduate colleges (Arts and Sciences, Business Administration, Engineering and Nursing), a Law School and a modest graduate program. For the past eighteen years students in their first year of study took specifically designated courses in English, History, Theology and a Freshman Seminar. Beginning with the 1998-9 academic year this twelve-credit Freshman Studies program has been replaced by two five-credit courses taken over both semesters.

In the Core, as it has become known, all new students read the same texts and study the same material at approximately the same time, all in sections of about 20. (See the Appendix for a list of the required texts and readings for the first semester.)

This ambitious project has rather ambitious goals:
1. to initiate the students into academic study and the VU academic community through a sustained conversation about important aspects of human experience;
2. to encourage careful reflection and better understanding of our lives together;
3. to develop the students' ability and encourage habits of skilled writing, critical thinking, careful analysis and interpretation of texts, and persuasive presentation of ideas.

Since the theme focuses on "common" human experiences, six sub-themes are examined: birth and creation, coming of age and education, citizenship and service, love, work and play, and loss and dying; the first three provide a framework for the material covered in the first semester. To help the students reflect on various aspects of each of these life experiences, they are required to read specific texts and to come to class prepared to participate in a conversation with their peers, with their instructor and with the text itself. In addition to the in-class dialogue, further reflection takes place through extensive writing assignments—in the first semester there were five papers to be written, a reading journal to be kept and a writing portfolio to be developed. Attendance at certain out-of-class activities, such as films, plays, lectures and musical events, is required to help broaden the students' own life experiences.

You might wonder how it happened that I, a mathematician/computer scientist, found myself teaching this first-time offering of the first semester of the Core. At times during the summer of preparation, when I was seized with a fit of anxiety, I asked myself this very question. It certainly wasn't because I'm a thrill-seeker nor because "it was there." It happened, as is usually the case, for a mix of reasons. Idealistically, I wanted to reintroduce myself in a systematic way to conversations which occur regularly in the study of the liberal arts. After all, eight years of Jesuit schooling in high school and college had awakened my curiosity to the possibility of such conversations. Practically, someone from my department "had" to do it since the call had gone out to all areas of the University to participate. Never one to shy away from try-

ing something new, I volunteered my services.

SUMMER OF PREPARATION

Once I had committed myself in early May 1998 to being one of the instructors for the start-up version of the Core, I quickly realized that I would need to devote much of the summer to study and preparation for what seemed to me to be a rather daunting task ahead. I began to think about some basic questions, such as how does one teach writing? How does one teach texts in the humanities? What adjustments to my teaching style will I need to make to facilitate classroom discussion about the texts? What type of background material do I need to learn and do my students need to know so that the discussions are meaningful and become more than just a sharing of ignorances?

Having the entire summer to get ready helped immensely. There was sufficient time to read all of the fall semester's required texts and readings in a leisurely manner. I even reread a couple of the novels while communing with nature and with friends on an island retreat in the northern part of Lake of the Woods in Ontario, Canada.

In addition, two workshops covering a total of nine days-four in the latter part of May and five in the middle of August—were organized by the Core Director to bring all of the Core faculty together to wrestle with some of the very same issues about which I had been concerned. (It was nice to find out that I wasn't the only one feeling like a fish out of water.) In the May sessions our conversations dealt principally with writing as a means to learning. There was also time set aside to organize ourselves into six cohort groups, each one of which would be responsible for preparing curricular materials, including developing a daily syllabus, related to one of the sub-themes. I was placed in the "Coming of Age and Education" cohort. The August sessions were devoted to pounding out a daily syllabus for the entire year and to providing guidance on many of the how and what questions.

MY EXPERIENCE

What can I say about the experience of teaching in the Core? The short answer is it was challenging, intellectually exciting, scary, quite difficult at times, and, yes, fun. However, in order for me to do justice to this question I feel obliged to elaborate—first, some perspective gained during the first few weeks of the course which helped carry me through the semester and, then, some specific examples which illustrate some of the pedagogic techniques I learned from my colleagues in the humanities to help me to teach a discussion-based course focused on works of literature.

Insights

One, I realized that as a mathematician and a teacher/scholar, I am not without my own resources. I know how to learn. I know how to read carefully. I know how to carry on a conversation about what I have read. I know how to raise questions about that which I don't understand. Having written articles previously for a variety of audiences, including readers of VU's literary periodical, I do know something about the process of writing. And, since much has changed in the way we in the undergraduate community teach mathematics these days, I am not uncomfortable in a classroom situation where lecturing is not the main mode of aiding student learning. The knowledge that each day in class I had something to bring to the table increased my confidence in my ability to work with students in this new environment.

Two, most 18-year old college students who attend a place such as Valparaiso University have some experience with reading good literature in high school and expressing their thoughts in writing about what they have read, neither of which they seemed to dislike. This situation lends itself to a nice start to the course. However, their view of the world is quite different from those of us who came of age in the '60s, and their experiences of the world are limited as compared to our own. (I'm sure that this comes as no surprise to those who have taught first-year humanities courses in recent years.) Hence, although these new students were eager to learn more about their world, it began to dawn on me that one of my roles as their instructor was to help them see how and why the texts and readings we had chosen for them to read could contribute to their own understanding of their place in the world. It was one of those "aha moments" when I realized that teaching works of literature involves struggling with issues about life experiences.

Specifics

One, early on I struggled with focusing the students' attention at the beginning of class on what they had

read the night before. For the first couple of weeks I would ask students to respond to some of the guiding questions which were part of the daily syllabus. However, on some days all I got were those blank stares with which we are all familiar. I raised the issue with colleagues at one of our weekly cohort meetings and received the following suggestion: begin each class day with a short writing assignment in which the students would be asked to respond in their writing journals to a question I would pose about the current reading assignment; then, ask for some volunteers to read aloud what they had written. It was amazing to me to see how well this method helped to set a tone for discussions that followed.

Two, especially during the first few weeks of the course, finding a proper balance between my role as the instructor, providing sufficient context for the students to understand what they had read, and my role as a moderator, facilitating the discussion so that the students-all the students-could contribute their insights to the issues being raised was a real challenge. For example, to appreciate better Toni Morrison's first major novel, "The Bluest Eye," it is helpful for the reader to have some understanding not only of the historical, philosophical and cultural background out of which she wrote but also some contextual background of the time about which she wrote. For me one of the difficulties was how to provide my students with that information without telling them of Ms. Morrison's intent in writing the novel, nor even laying out the issues she was raising.

> **"**
> ***It was a real learning experience for me just to get a feel for where they were as writers and how far I could expect to take them—much less how to achieve the desired effect.***

One approach (which met with partial success) to resolving this dilemma was to use the first day (of the three to five set aside for reading and discussing a text) for background, and then, on the remaining days, to divide up the class into small groups either to respond to the guiding questions included in the syllabus or to dissect some particularly meaningful and rich passages in the text.

Three, it took me the entire semester and then some to develop a sense of how I could help my students become better writers. It was a real learning experience for me just to get a feel for where they were as writers and how far I could expect to take them—much less how to achieve the desired effect. Based on my own observations and on discussions with my colleagues in the Core, I now understand that for writing assignments outside of class I need to articulate more clearly my expectations, to have more in-class discussions about the process of writing, to provide models of good writing by first-year students, to be more consistent in evaluating what they have written and to offer more reflective criticism of their work.

EFFECTS ON THE TEACHING OF MATHEMATICS

In a letter [3] summarizing the conclusions reached by the participants in the initial 1986 Humanistic Mathematics Conference, Alvin White describes "two related themes that emerged from the conference, one, teaching mathematics humanistically and, two, teaching humanistic mathematics." While on sabbatical this Spring semester (1999), I have had the luxury of time to think about these issues more carefully, especially just after having the experience of teaching a course in the humanities. Having taught only one humanities course one time, I will not be so bold as to give a definitive answer to the more philosophical question of whether mathematics is a humanistic discipline. (A very tentative answer would be, no, if by humanistic discipline we mean the system of study in which human interests, values and dignity are taken to be of primary importance [2]. The answer might be more affirming if by humanistic mathematics we are referring to humanizing this systematic study by paying more attention to its historical growth, philosophical underpinnings and regional applications.) However, fortuitously the time was right for me to give some thought to how some of the experiences I have had might inform my own teaching of mathematics.

It turns out that my teaching in the Core coincides with a realization on my part that some of the pedagogic tools I have used no longer seem to be as successful as they once were in helping students become better learners of mathematics. For example, one of the techniques I have employed for the past ten

years in almost all mathematics courses is something I call same-day quizzes. In order to focus students' attention the night before on material they have been assigned to read and problems they have been asked to work out for the next class period-for the most part, new material and problems whose solutions or solution-types have not yet been introduced in class-I make it known from day one that they should be prepared daily to take a quiz in the last ten minutes of class. The quiz consists of one or two of the homework problems. In theory, having the quiz at the end of the class period gives the students sufficient time to ask questions either before class or during class about that which they don't understand. In practice, one of the difficulties with this approach has been that, especially in lower-division math classes, there are too many students who struggle with the material the night before class to such an extent that they either give up out of frustration or come to class completely confused. For these students the class time before the quiz is unproductive, and they are no better prepared to do well on the quiz than they were before they came to class. To say the least, this situation is not conducive to providing a healthy learning environment.

So, what have I learned from teaching a discussion-based, intensive writing course about literary works which might carry over to the teaching of mathematics? First of all, carefully constructed writing assignments both inside and outside the classroom can be valuable aids to learning in that the process of writing can help clarify in the writer's mind ideas and concepts which at first appear to be rather fuzzy. Precision about the meaning of what one has read is important both in literature and in mathematics. In addition, articulating on paper what one understands about a problem and its solution can lead one to making connections with what has come before. And, as we all have experienced, mathematics is much about observation, patterns and making connections.

Second, group work assignments, if carefully crafted and monitored, can add to the process of learning. (This just confirms what I have found from assigning certain types of software design projects as team projects to my students in computer science courses.) Some of the same benefits as with writing can accrue to the learner since struggling to articulate to others in the group what one does and does not understand can lead to clarity. An additional benefit can come from the dynamics inherent in a group, namely, through the process of discussion different approaches can be brought to bear on a problem and its solution. Hence, insights that one might never have had when working alone can emerge.

For a number of you the observations in the previous two paragraphs are not new. Some of you have been using writing and group work in the mathematics classroom for some time. (See [1] for a sample of the many innovative ideas that currently are being implemented around the country.) Yet, there is much we can learn from discussions with our colleagues in the humanities about teaching humanistically.

But, what effect will all of this have on my teaching of mathematics in the immediate future? This coming fall semester one of the courses I will be teaching is a section of Finite Mathematics which we offer mainly to business and social science majors. It is a typical Finite course in which the problems to be solved involve discrete quantitative data; the solutions of which usually require some understanding of mathematical modeling and the mathematical tools appropriate to solve the model. It has been my intention for the past year that come fall I want to explore how spreadsheets might enhance students' understanding of the material in this course, possibly via outside of class lab assignments. I now have a better idea as to how I might incorporate both writing and group work into the lab environment. In addition, I plan to introduce short writing assignments in the first few minutes of class to get a feel for what the students have understood from the nightly homework assignment. That way, on any given day there will be either a writing assignment at the beginning of the class or a quiz at the end of class.

CONCLUSION

My experience in teaching the first semester of our new two-semester Core sequence was definitely worthwhile. As a matter of fact, I am looking forward to teaching the entire sequence this coming academic year (1999-2000). The benefits of working with colleagues from across campus in a shared experience which is centered on reading great literature with an eye toward helping students reflect on what they have read and articulate what they have experienced far outweigh the costs in time and energy for an "outsider" to get up to speed. A bonus is that it provides

me with an opportunity to reflect on my role as a teacher of mathematics. An added bonus is that I now have a larger group of colleagues to whom I can turn for assistance with my own writing. (Two of my Core colleagues in the English Department have reviewed various drafts of this paper.)

I have had to work harder than I have in years to provide the students with an enriching experience, but in the process I have been intellectually stimulated by taking on such a challenge.

REFERENCES

1. Gold, B., Keith, S. & Marion, B. (Co-editors), "Assessment Practices in Undergraduate Mathematics," MAA Notes Series #49, Mathematical Association of America, 1999.
2. *Random House Unabridged Dictionary* (2nd ed.), Random House Inc., 1993.
3. White, A., "From Newsletter #1," *Humanistic Mathematics Network Journal*, Issue #18, Nov. 1998.

APPENDIX: FALL SEMESTER TEXTS AND READINGS

Required Texts

My Antonia by Willa Cather
Gilgamesh translated by David Ferry
The Bible, New Revised Standard Version
Frankenstein by Mary Shelley
Little Women by Louisa May Alcott
The Bluest Eye by Toni Morrison
Confessions (Books I-IX) by St. Augustine
Hunger of Memory by Richard Rodriguez
Antigone by Sophocles
Lest Innocent Blood Be Shed by Philip Hallie
The Bedford Handbook by Marilyn Hacker

Required Readings

Selected Poems by Anne Bradstreet
On Liberty (Chapter 4) by John Stuart Mill
Mencius (Selections)
The Book of Lord Shang
"Letter from Birmingham City Jail" by Martin Luther King Jr.
"The Courage To Stand Alone" by Wei Jingshen

Word Problems

Don Pfaff
Math Department, University of Nevada, Reno

(may be sung to the tune of Billy Joel's "We Didn't Start the Fire")

Words in problems bother me,
I'd like to drown them in the sea.
My brain is numb, I feel so dumb
I can't tell x from z.
Numbers swim within my brain, those methods used seem so arcane
How can I remember d is equal to rt?
Seems my work is never done,
Teach assigned another one,
This weirdo claims they're lots of fun,
How I wish I had a gun,
I'd have that joker on the run.
I can't solve word problems.
They're a pain to me
The answers I can't see.
If I have to to work one now,
My mood will fluctuate
From love to hate to hate to hate...

A Response to Dancis

Margaret Schaffer
Hale Middle School
Los Angeles, CA

In response to Jerome Dancis' article "Middle School Math Teaching and How It Harms Our Children," I would like to ask the author when he last taught in middle school. I've taught in an urban district for 31 years, grades 2-9.

Over the years the State Department of Education has grabbed every fad that has come down the line. Usually these fads come from colleges and universities, and they are someone's mandatory research project or doctorate. Really learning the basics has been "out of style" for quite a few years. The results are that we have generations of young adults unable to read or write complete thoughts. The same is true with math. Too many of the "average" students can't do simple multiplication facts in middle school. Yet, from sixth grade on, a large percentage of the work is based on multiplication. I'm sorry if Dr. Dancis feels that 7 or 8 year olds should not learn the basics. Doing manipulative projects are fun, but if Dr. Dancis wants his college students to do college level mathematics, a strong foundation must be built. A poor foundation in the earlier years of education will not withstand more advanced work.

As students advance from elementary school to middle school, elementary facts (the foundation) should have been mastered. It is very frustrating to have sixth and seventh graders working on a second and third grade level. They feel it, too. This is where the disruptions in class usually come from. We need parent responsibility. I don't want to hear, as I did today from a young man, "I have football practice everyday so I can't do homework."

Most teachers do not allow calling out in class simply for their own sanity. Most middle school teachers have learned that there are many modalities of learning and employ many of them to reach the greatest number of children. As for "cookbook" instruction, any good teacher will explain the what and why of any subject to make a point and again to reach as many children as possible.

We do not work with college age students but 11-14 year olds. We correct and discuss homework. Children learn from these class discussions. Also, at this age children need a lot of repetition. It is not a waste of time to help youngsters individually. Their attention span is known to be short (30-35 min.). After 45 minutes most of them are ready to pack up and leave. It's not insulting; this is a fact of life. Therefore, presenting lessons in a concise method with a variety of short activities helps them. Seat work is not glorified babysitting if the students want or need help. Most educators do not waste time. We are required to have one grade per week. These graded assignments can be a variety of assignments. Personally, I normally will have a quiz or a chapter test a week. But, I also assign projects with each chapter, do journal writing involving a concept we've learned that week and some extra credit.

The only time I have found homework not collected was in college math classes. In my classes we, meaning the class, correct, discuss and collect the work because not only the teachers but other students' methods and reasoning quite often help those who may be having difficulty.

Maybe college instructors do all or nothing grading but, in my experience, most middle school math teachers give partial credit for work when the process was correct if the students show their work.

As for Dr. Dancis' suggestion of having school principals wave a magic wand and have teachers jump, think again, it won't work. Dr. Dancis is entitled to his academic freedom. Most principals understand what academic freedom is and respect it. If any college administrator tried to take away Dr. Dancis' academic freedom there would be a hue and cry from him. The K-12 teachers have the same professional rights as their college colleagues.

In California secondary teachers, starting with grade

8, must have a valid California Teaching Credential in their subject. Therefore, a teacher with a General Credential lacking a math major or minor can't teach algebra or higher. A biology major who doesn't have chemistry as a minor cannot teach chemistry. If administrators allow this there are costly financial consequences for the school district.

I'm sorry Dr. Dancis' experiences with the public school of his choice has not been up to his standards. Before crying wolf, I suggest he spend a year of so actually teaching middle school children. It is an eye-opener.

What Else Do We Forget to Tell Our Teachers?: A Response to Dancis

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In his article "Middle School Math Teaching and How It Harms Our Children," Jerome Dancis (*HMNJ* 20, 1999) raised a number of pertinent issues related to classroom practice. In particular he identified a number of fundamental teaching practices that were not described in his local school system's teaching guides. I would like to elaborate on one of the issues, that of assessment in mathematics teaching. Dancis describes a disconcerting "all-or-nothing" scoring procedure used by a teacher to score an algebraic simplification question out of 25 points. Although we have no idea how many years this teacher had been teaching, there are clearly aspects of assessment practice that s/he needs to learn. I pose the following question: What do we as mathematics teacher educators forget to tell our preservice teachers about assessment?

To ensure that future mathematics teachers employ a diversity of assessment strategies, we expose our preservice teachers to journal writing, mathematics project work, portfolios and other alternate assessment strategies. These strategies have their place in the teaching of mathematics and should be encouraged because of their educative role. Also, many of our preservice teachers did not encounter these forms of assessment at school, hence the need to introduce them to the teachers. However, when practicing teachers (in the USA) are expected to have in excess of 20 grades per student in a 6 week reporting period, we understand why teachers resort to assessing homework, testing 2-3 times a week and collecting grades at every opportunity. Ensuring that the assessment procedures are reasonable, that tests are well constructed and scored fairly, would go a long way to alleviating some of the difficulties expressed by Dancis. Far too frequently we forget to inform our preservice mathematics teachers of the basics of sound test construction, implementation and grading. In the teachers' "real world" they will be required to test, test with traditional pencil-and-paper quizzes and tests, and unfortunately test frequently.

AN EXAMPLE

In figure one I depict one question from a 50 minute test to illustrate the characteristics of test construction discussed below. I have specifically illustrated my argument with a very traditional algebra test because it relates directly to the experiences of practicing teachers. Each question tests some aspect of factoring. Figure 2 gives, as an example, the scoring rubric for question 2.4. Two possible solutions are given to guide the teacher's assessment of possible student solutions. Points are allocated for specific steps in the anticipated solutions.

I turn my attention to addressing some basic issues of test design we neglect to tell our preservice teachers. What every teacher should know before constructing his or her first class test:

a. Not all test questions should be of equal weighting. Dancis describes how the teacher had four problems, each worth 25 points. Did each of the questions require the same amount of work to

Question 2: Factor fully

2.1) $12x^2 - 27$	[3]
2.2) $-2x^2 - 4x + 6$	[4]
2.3) $4x^3 - 2x^2 - 6x$	[3]
2.4) $(x - 4)^2 - (x - 4) - 6$	[4]
2.5) $4(a - b)^2 - a^2(b - a)^2$	[6]
2.6) $z^3 - 3z^2y + 3zy^2 - y^3$	[7]

(* Question 2.6 can be solved in one step by students who recognize this as the expanded form of $(z - y)^3$. The students who sat this test were not familiar with this expression and grouped terms to find a common factor.)

Figure 1
Question two from a 50 minute algebra test on factoring

solve? We want a test with a range of complexity. We can not then assign equal grades to each question. Figure one illustrates a range of questions with varying levels of complexity. For example, question 2.5 is more complex and requires more work than question 2.1.

b. Assign points to the amount of work and complexity of the problem (also Cangelosi, 1996). How do you assign meaningful partial credit when scoring a question out of 25? Rather, each test item should be scored according to the number of logical steps required to complete the question. This tells the student that the process is important. The solution of question 2.4 (figure two) illustrates the allocation of each of the 4 points to various steps in the solution process.

c. The points awarded to each question should be made clear to the student on the test. In figure one the points allocated to each question are indicated for students to see.

d. Test items should be developmental (also Nitko, 1996). Starting with simple problems and moving to more complex questions with at least 10% of the test requiring the students to apply their knowledge to new contexts.

e. Teachers must create a scoring rubric (memorandum, blueprint or answer key (Nitko, 1996)) for the test before they administer the test. This gives an opportunity to identify errors in the questions and assign appropriate points to each question. The rubric should anticipate various student solution strategies (see Figure 2). It is then used (flexibly) as a guide to assign points to students' work. This process improves the fairness and hence validity of a test.

f. Teachers should not only be grading for their grade book. Unfortunately the educative role of assessment has been neglected in most mathematics classrooms, leaving assessment simply a means to audit learning. Teachers should review the scored test with the students with the hope that some students may learn from their mistakes.

$(x - 4)^2 - (x - 4) - 6$
[1] [1] [1]
$= ((x - 4) - 3)((x - 4) + 2)$
[1]
$= (x - 7)(x - 2)$

1 point for seeing (x-4) as a common factor
1 point for each parenthesis
1 point for simplifying the two parentheses

$(x - 4)^2 - (x - 4) - 6$
[1] [1]
$= x^2 - 4x - 4x + 16 - x + 4 - 6$
[1]
$= x^2 - 9x + 14$
[1]
$= (x - 7)(x - 2)$

1 point for simplifying $(x - 4)^2$
1 point for simplifying $-(x - 4)$
1 point for simplifying
1 point for factored expression

Figure 2
Scoring blueprint for question 2.4.

I hope this discussion serves as a reminder of some of the attributes of test construction that we take for granted and assume preservice teachers know because they have been through 12 years of schooling. Dancis showed that in many cases this might be a flawed assumption. If the teacher depicted in Dancis's article taught one of your students, then you could be sure that the teacher does not know how to construct a test. Although reform efforts in mathematics education introduce assessment techniques that may be more suitable and more motivating than traditional tests, let us not neglect to pass on the simple principles of sound test construction.

REFERENCES

Cangelosi, J. S. (1996). *Teaching Mathematics in Secondary and Middle School: An Interactive Approach*. Merrill: Englewood Cliff, N.J.

Dancis, J. (1999). "Middle School Math Teaching and How It Harms Our Children." *Humanistic Mathematics Network Journal* 20. 3-7.

Nitko, A.J. (1996). *Educational Assessment of Students* (2nd edition). Merrill: Englewood Cliff, N.J.

Another Response to Dancis

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The following is taken from an email message to Jerome Dancis.

I just read your paper "Middle School Math Teaching and How It Harms Our Children" in the *Humanistic Mathematics Network Journal*; I particularly liked your "sidebar" on the factoring problem. But you seem to think that the question would have been graded differently in some other school—who are you kidding?

I once taught in a school where the principal was considered to be a good principal if he had no riots in the school. When that happened, the principal was moved up to an even higher position. (That was at Crane in Chicago in the middle 60's). And how did the principal ensure that there were no riots? He **never** (not once) came out of his office when the kids were in the building. All doors were locked after 8:00, and there were uniformed cops in the halls. No one worried about the kids learning anything—teachers were only interested in getting through the day. Compared to Crane, your "magnet school" sounds like a haven for learning!

"No amount of experimentation can ever prove me right; a single experiment can prove me wrong."
--Albert Einstein

Evaluation in the Mathematics Classroom

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INTRODUCTION

Evaluation is indispensable to the teaching and learning of mathematics. The National Council of Teachers of Mathematics [NCTM] (1989, 1991, 1995) recommends that evaluation should be an integral aspect of the teaching of mathematics and should be used to inform instructional procedures. Hoosain and Naraine (1995) listed evaluation as a component of effective mathematics teaching. In this paper we illustrate how the various types of evaluation can be used advantageously in classroom practice. We use the term evaluation as defined by Webb (1992) and NCTM (1995); that is, as a broader and more inclusive concept than assessment.

Evaluation may be defined as a systematic process of obtaining information for the purpose of making decisions. Thus the ultimate purpose of evaluation is decision-making. In education we conduct evaluation to make decisions about students, teachers, curricula and teaching methods and strategies. There is consensus that good teachers should be thoughtful decision makers (Carpenter & Fennema, 1988; Clark & Peterson, 1986). For example, we decide what grade to give to a student, whether to reteach a lesson or not, and so on. Good decisions depend on valid evidence (obtained from evaluation). Therefore, it is important to conduct evaluation objectively.

TYPES OF EVALUATION

There are three broad types of evaluation: (a) diagnostic, (b) formative, and (c) summative. These are not independent types because aspects of one type of evaluation may be found and used in another. However, there are differences among them, and these differences relate to the purpose for which each is conducted.

Diagnostic evaluation is usually done at the beginning of a course of study or a series of lessons to ascertain students' entry behaviors. This can be achieved by written or oral tests, teacher-made or otherwise. This is to facilitate a close fit between new material to be taught and students' cognitive level of development and current achievement. The assumption is that the closer the fit, the more likely learning will take place. This type of evaluation may also be useful during a program of instruction to identify the specific difficulties that students may be experiencing and to determine why they are having these difficulties. The information obtained can be used to design appropriate remediation, differentiated, and follow-up programs. By accurately diagnosing students' problems, teachers are in a better position to help the students. Diagnosis may be done through written and oral tests, written work, and interviews or one-on-one conferences involving the teacher and students. A combination of different sources of information about students (written work, interviews, observations, etc.) is likely to result in a more accurate diagnosis.

An example of diagnostic evaluation in mathematics is the use of a written test to determine students' readiness for a formal course in geometry. Respondents to the Priorities in School Mathematics survey (NCTM, 1981) believed that geometry is taught primarily to develop logical thinking abilities. These include the ability to understand and construct formal proofs. However, Usiskin (1982) reported that of all U.S. high school students, 60 percent did not study proof, and of those who did, only 13 percent were successful with proof. One reason for this poor performance is an apparent mismatch between teacher instruction and student readiness: the teacher is presenting information at one level while the student is functioning at a different (and usually lower) level. Research supports the existence of five levels of learning (the van Hiele levels) in geometry (Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1988). It is believed that a student can be helped to progress through this hierarchical system of levels if appropriate level-specific instruction is provided.

If a teacher is to provide instruction at a level that matches the student's current van Hiele level, that teacher must first determine the student's current van Hiele level. Usiskin (1982) found that as many as 90 percent of a sample of almost 3000 high school stu-

dents could be assigned a van Hiele level based on a multiple-choice geometry test. Using such a test to provide information on students' entry behaviors is an example of diagnostic evaluation.

Formative evaluation may be done during a program of work: for example, during a lesson. Its primary objective is to provide feedback to students and teachers who will then decide in what direction to proceed. For example, if during a lesson evaluation indicates that students are grasping what is being taught, the teacher may decide to continue as planned. On the other hand, if the evaluation indicates that students are not following the lesson, the teacher wisely deviates from his/her plan. The teacher may do additional and different examples, reteach part(s) of the lesson, use a different strategy or method, and so on. In other words, formative evaluation is intended to help the teacher to improve his/her instructional practices so as to promote better learning by the students. Some element of diagnosis is involved here, too.

The teacher can also use formative evaluation to keep students informed about their individual progress toward a goal so that the students can take the necessary measures to improve their performance. In a wider sense, formative evaluation may direct a reexamination of the appropriateness of objectives, materials, content, teaching methods and evaluation procedures related to a program of work.

Formative evaluation requires that the teacher monitor the students' progress closely. An examination of students' written work could reveal whether they are following the lesson or not. Therefore, it is necessary for the teacher to assign written activities during a lesson and to move around the class to spot check students' work. However, written work alone is inadequate (NCTM, 1991). Oral work could also be helpful (Buschman, 1995; NCTM, 1989; 1991). Asking students How? What? and Why? questions could help to identify students' specific misconceptions. For example, the teacher could ask students to explain how an answer was obtained or why a particular method was used. In this way the teacher can ascertain how students are thinking.

Observations by the teacher and self and peer evaluations are relevant in this context. Self and peer evaluations provide students with opportunities to identify their own mistakes and those of their peers. These forms of evaluations help students to develop their 'self-correcting' abilities, something we should aim at in mathematics teaching because in this way students become more independent learners.

As part of formative evaluation, instead of asking students 'Do you understand?' (a common practice among teachers), students could be asked specific questions relating to what was taught. The answers to these questions would be more helpful to the teacher and students because in many cases students claim they understand when in fact they do not.

One of the authors used the strategy of returning college students' assignments with comments, suggestions, and directions for improvement. These, coupled with the opportunity for further research, enabled the students to produce better work. There are at least two problems associated with this practice. First, although it worked very well with a small class, it would be more difficult to implement with much larger classes which are common in high and middle schools. Second, students may be reluctant to do an assignment twice.

Summative evaluation, the most common and most frequently used of the three types of evaluation, is usually done at the end of a program of work or a series of lessons. For example, it is done at the end of a month, quarter, semester, grading period or academic year. Its main intention is to obtain and report information about students. Based on this information a final grade (or certificate/diploma in some cases) is awarded. This final grade could be a combination of several grades. Generally, summative evaluation is not used often to provide feedback information. However, we do not see why it cannot be used for this purpose. The practice of providing students with the opportunity to redo their assignments with the objective of improving them (as well as their grades) as explained earlier in this paper is also relevant to summative evaluation. Summative evaluation usually takes the form of quizzes, tests, examinations, portfolios, individual and group projects, and presentations, or any combination of these. Based on the recommendation of NCTM (1989), an objective evaluation of the student is likely if several sources of information are used.

CONCLUSION

A critical question for the teacher is: What relative emphasis is to be placed on these three types of evaluation? This is difficult to answer. The answer may be related to the type of school: elementary, middle, or high. Much emphasis is usually placed on summative evaluation. This is understandable because parents, administrators, colleges, employers, and so forth, want to see what grades a student has. Unfortunately, the same emphasis has not been placed on diagnostic and formative evaluations. Since final grades depend to some extent on diagnostic and formative evaluation, greater emphasis on these types of evaluation is likely to result in improved as well as more accurate final grades. We, therefore, recommend that teachers emphasize diagnostic and formative evaluations more than they currently do. Another important concern of the teacher is the evaluation of aspects of the affective domain. Admittedly this is difficult for various reasons, but one cannot help observing that traditionally school evaluations have been exclusively concerned with the cognitive domain. We think that the time has come for teachers to begin to evaluate students' interests in and their attitudes toward mathematics, as part of diagnostic and formative evaluations.

REFERENCES

Burger, W. F., & Shaughnessy, J. M. "Characterizing the van Hiele levels of development in geometry." *Journal for Research in Mathematics Education, 17*, 31-34, 1986.

Buschman, L. "Communicating in the language of mathematics." *Teaching Children Mathematics, 1(6)*, 324-329, 1995.

Carpenter, T. P., & Fennema, E. *Research and cognitively guided instruction*. Madison, WI: National Center for Research in Mathematical Sciences Education, 1988.

Clark, C. M., & Peterson, P. L. "Teachers' thought processes." In M. C. Wittrock (Ed.). *Handbook of research on teaching* (3rd ed.), pp. 255-296). New York: Macmillan, 1986.

Fuys, D., Geddes, D., & Tischler, R. *The van Hiele model of thinking in geometry among adolescents*. Reston, VA: The National Council of Teachers of Mathematics, 1988.

Hoosain, E. & Naraine, B. "Effective Mathematics Teaching." *Minnesota Mathematics Magazine*, Winter 1995.

National Council of Teachers of Mathematics. (1981). *Priorities in school mathematics: Executive summary of the PRISM project*. Reston, VA: The Council.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: The Council, 1989.

National Council of Teachers of Mathematics. *Professional Standards for Teaching Mathematics*. Reston, VA: The Council, 1991.

National Council of Teachers of Mathematics. *Assessment Standards for School Mathematics*. Reston, VA: The Council, 1995.

Usiskin, Z. *Van Hiele levels and achievement in secondary school geometry*. Final Report of the Cognitive Development and Achievement in Secondary School Geometry Project. Chicago: University of Chicago. ERI Document Reproduction Service, 1982. (No. ED 220 288)

Webb, Norman L. "Assessment of students' knowledge of mathematics: Steps toward a theory." In D. B. Grouws (Ed.), *Handbook of Research in Mathematics Teaching and Learning* (pp. 661-683). NY: Macmillan Publishing Company, 1992.

"No amount of experimentation can ever prove me right; a single experiment can prove me wrong."

--Albert Einstein

What Is the Thing Called "Humanistic Mathematics?"

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My purpose in this article is to deal with two different interpretations given to Humanistic Mathematics in educational systems and to present the systematic implementation of a program based on both of them in Israeli elementary schools.

"What is **the thing** called 'Humanistic Mathematics?'" The question was raised by Bill Rosenthal in the 1993 *Humanistic Mathematics Network Journal*, echoing the voices of many of my colleagues in mathematics teaching who persistently have posed it expecting that the answer will lead to a debate that will end with the conclusion that there is no such thing as "Humanistic Mathematics." The arguments presented have been: "all teaching is humanistic" or "there is no need for separatism" and "there is no humanistic mathematics." Bill Rosenthal presents the question not to throw us into a chaotic situation where confusion reigns, but to give us the feeling that dealing with mathematics humanistically is more than finding a definition. It means starting with clarifying "the meaning" and continues with delineation of actions for forthcoming directions of involvement in a new pedagogy. It is like participating, in what I call—a protest movement in pedagogy—devoted to sociocultural synthesis of knowledge; a paradigm based on the accumulation of differential experience and of humanization of education; a world view whose images are formed by inversion of the usual lenses.

The examination of the difference between "teaching mathematics" as it is commonly understood and "teaching mathematics from an humanistic viewpoint" gives us the possibility of exploring the fields of mathematics and humanism in a meaningful way for both. This possibility was elucidated by Marchisotto in 1992 when she stated two pedagogical goals taken from the conference at Harvey Mudd College held in 1986:

1) "**teaching mathematics humanistically** as altering the nature of the teaching and learning environment in a way that makes the learning situation meaningful and relevant to the people involved in it."

2) "**teaching humanistic mathematics** as reconstructing the curriculum and the discipline of mathematics itself."

The first goal refers to changes of the pedagogical focus and the culture of the mathematics lesson so that the personal characteristics of the pupil as well as the teacher's are taken into account in the teaching plan. It means that there should be clarified goals and methodology involving what is accessible, valuable and interesting to human beings in mathematics. The "whom are we teaching" is not less important than the "what are we teaching."

The second means that a serious curricular operation has to be undertaken in order to expose the substantial relations between mathematics, science, arts, geography, language, etc. This operation will prepare and organize topics linking the mysterious and the evident in the relation between mathematics and other disciplines. It doesn't mean that all of the mathematics will be taught connected to other subjects, it means that the relevant connections will be introduced as a regular part of the formal curriculum and not as an anomaly in it.

On behalf of the tendency to choose one of the interpretations and focus on it as a main axis of action, we decided to take a synergetic approach and question the possibilities of delineating a version of "humanistic mathematics" that embodies both definitions.

We attempted a non-Euclidean task, two parallel lines that have to meet in one point in order to answer, "What is **the thing** called 'humanistic mathematics?'" We could have used various approaches with intent of reaching convincing conclusions about the nature of teaching, mathematics and the environment. Yet, we preferred a more esoteric and risky approach, an

approach related to one of the basics of humanism: the faculty of exercising doubt in those situations where everything seems clear and certain. Questioning not for the sake of questioning, but for the purpose of using questions as keys to new, exciting and vast realms of knowledge that can be used to improve the quality of human life.

This point of view and the respect we have for the human being, stimulated us to get engaged in an "against the stream" educational venture which promised interesting and worthy challenges. We expected, in return, to get more acquainted with "**the thing** called humanistic mathematics."

On this basis we started to work in the year 1987 on the development of the "Teaching Mathematics with an Humanistic Approach" program in Israel. The program developed in Israel combines the two pedagogical definitions stated before in asymmetrical proportions. The main part of the program is conveyed to create a different culture in the classroom; the rest deals with the links between mathematics and other disciplines. Creating different mathematical activities in the classroom in correlation with a pupil's personal learning characteristics and developing interesting and supporting environmental situations lead to a refreshing attitude of teachers and pupils in respect to mathematics education.

The delineation of the program required the exploration of issues that dealt with five major categories of the questions which could take us one step forward in the route.

The questions can be organized in five categories: the Learner, the Mathematics, the Educational Environment, the Curriculum and the Teacher. The answers to the questions crystallized in the "Teaching Mathematics with a Humanistic Approach" program whose characteristics and dynamics mark a new focus in educational goals and their implementation. The following list presents some of the questions related to each of the categories:

THE LEARNER

The characteristics of the learner:
What is his learning style?
What is his cognitive style?
What is his motivational style?
What is his social style?
What are his communications patterns?
What are his mathematical abilities?
What is his performance level?
What are his physical abilities?
What are his beliefs?

THE MATHEMATICS

The Mathematics that is being taught:
Which mathematical ideas, skills, concepts have been chosen?
What substantive part of the body of knowledge is being dealt with?
What number systems, categories, lattices are being used?
What claims, arguments and symbols are involved?
What relation is there between the mathematical topic and other disciplines?
Which mathematical structures and models are being dealt with?

THE EDUCATIONAL ENVIRONMENT

The organizational quality of the environment:
How linear, systematic and organized is the learning setting?
How much diversification of activities are present in the surroundings?
How is the availability of activities securely established?
Which kind of activities and tasks are offered to the different learners?
Who is introducing the mathematics?
How is time organized for different learners, activities and tasks?
How is assessment incorporated with learning in a non-interfering way?
How flexible and adaptable is the environment for learner's and teacher's needs?
How is the contact with mathematics established?
How much mathematical "doing" is possible?
How do we assure that the mathematical aspects in arts, science, etc. are recognized by the learner?

THE CURRICULUM

The organization of a balanced curriculum:
How much authentic and relevant problem solving is included?
Which skills, principles and foundation elements are being included?
Which type of analytical tasks are bound to be rel-

evant for different learners?
How are linking topics between disciplines included?
How is a mathematics approach supported?

THE TEACHER

The personal and professional characteristics of the teacher:

Who is the teacher?
What is the teacher's teaching style? Learning style?
What are the teacher's communication patterns?
What are the teacher's preferences in mathematical topics?
How does the teacher use analytical expertise in the classroom?
What is the teacher's planning style? (Linear, systematic, global, intuitive)
What is the teacher's working incentive? Motivational force?
How does the teacher relate to argument and logical debate?
What are the teacher's possibilities of developing empathy to learners?
How does the teacher assure personal growth in mathematical and pedagogical knowledge?
What are the teacher's educational beliefs?

Deliberately we used the term learner and not pupil in order to clarify that learning is taking place in more than one capacity, and that the pupil and the teacher are both involved in the same activity, each from different and similar angles and positions. The pupil is learning mathematics, learning about his learning processes and also about all that is going on; the teacher is learning how to use pedagogical skills in order to develop his pupil's learning.

Though the list was meant to be exhaustive, there is a big probability that more items can be added. Moreover, I have a fair suspicion that the location of some of the questions could be changed. The reader is invited to choose one of the questions and find other questions to which it is related. A thorough study will reveal that there is no question that stands by itself, in isolation; moreover, few are the questions that are not linked to **all** of the rest.

> “
> ***...learning is taking place in more than one capacity...the pupil and the teacher are both involved in the same activity***

Using the questions as a dynamic pivot, experimenting and learning from the results of the hands-on work, we arrived at an integrated conception of a developing methodology which takes in account not only the two interpretations of Humanistic Mathematics stated above but also its implications for education in general.

Models for the development of differential planning and diagnostic sensitivity of teachers were created by the start-up staff. Courses for the preparation of teachers and instructors have been held continuously for the last six years, and tasks and activities for pupils are printed and distributed in order to support the ideas and ideals of the program.

With the support of the Ministry of Education, at the elementary and middle grades, the program expanded to kindergarten groups, in an outspreading, secure, step by step pace. Nowadays tens of schools and kindergartens in Israel are involved in different degrees of accomplishment of the program. A small, but growing, “Center for the Instruction of Mathematics with a Humanistic Approach” gives support in three different fields of action: Training of Teachers, Development of Instructional Materials, and Evaluation Tools based on Personal Interaction.

Evaluation tools compatible to the ideological foundations of the program were required so as to avoid pedagogical dissonance. The LTM—Let's Talk Math—dynamic assessment for identification of mathematics' performance and abilities is being used as an inquiry evaluation tool. Based on a structured interview, the LTM provides information on the learning processes used by the pupil and a blueprint of his mathematical concept world. This information can be used in order to supply the proper environment in the mathematics class.

The classical triangular model for the training of teachers relating subject matter to the pupil and the teacher proved till now to be a very static and rigid one. We have added another vertex to the triangle: the human-

istic approach. This has made it a dynamic quadrilateral, able to move in different directions and still maintain its basic essence: the essence of education oriented to the search of beauty and knowledge in Mathematics and to the development of confidence and goodness in human beings. This, as far as I'm concerned, is **the thing** called "Humanistic Mathematics."

REFERENCES

Apel, Tamar (1993) Monitored Teacher Pupil Interaction as a Strategy in Mathematics Teacher Training. *Paper presented at the First International Conference: "Teacher Education: From Theory to Practice"* Mofet Institute, Israel.

Burns, M. Susan (1991) Comparison of Two Types of Dynamic Assessment and Static Assessment with Young Children, *The International Journal of Dynamic Assessment and Instruction, 2*, (1) , 29-38

Carpenter, T.P. (1985) Research on the Role of Structure in Thinking, *Arithmetic Teacher*, February 1985.

Cohen, Audrey. (1993) A New Educational Paradigm, PHI DELTA KAPPAN, June (791-795)

Fullan, Michael. (1997) Emotion and Hope: Constructive Concepts for Complex Times, *Rethinking Educational Change with Heart and Mind, 1997 ASCD Yearbook*, 216-237

Gamlin, Peter J. (1989) Issues in Dynamic Assessment/Instruction. *The International Journal of Dynamic Assessment and Instruction*, 1, (1) , 13-25

Grouws, Douglas A. (1988) Teaching Tomorrow's Skills Responsibly, *Arithmetic Teacher*, 10,1988, 6-63

Krishnamurti. (1989) *Krishnamurti on Education*. All India Press, Krishnamurti Foundation, India.

Marchisotto, Elena. (1992) Mathematics as a Humanistic Discipline. *Humanistic Mathematics Network Journal* #7.

Meece, J.L., Wigfield, A., & Eccles, J.S. (1990) Predictors of Math Anxiety and Its Influence on Young Adolescents' Course Enrollment Intentions and Performance in Mathematics. *Journal of Educational Psychology*, 82, 60-70.

Rosenthal, Bill. (1993) "Toward a Philosophy of Humanistic Mathematics." *Humanistic Mathematics Network Journal* #8.

Schiefele, Ulrich & Csikszentmihalyi, Mihaly. (1995) Motivation and Ability as Factors in Mathematics Experience and Achievement, *Mathematics Education*, 26 (2), 163-181.

Seagal, Sandra & Home, David. (1997) *Human Dynamics: A New Framework for Understanding People and Realizing the Potential in Our Organization*, Pegasus.

Skemp, Robert. (1971) *The Psychology of Learning Mathematics*, Palmer Press.

White, Alvin M. (1974) Humanistic Mathematics: An Experiment. *Education*, 95 (2), 128-131.

White Alvin. (1976) A New Paradigm for the Mathematics Classroom, *International Journal of Mathematical Education in Science and Technology*, 7 (2), 243-246.

Zaslavsky, Claudia. (1991) Multicultural Mathematics Education for the Middle Grades, *Arithmetic Teacher*, 38 (February) 8-13.

"If we knew what we were doing, it would not be called research, would it?"

--Albert Einstein

Mathematics Is an Art: The Story of a One-Time Course

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A - alpha, beginning, dreaming, imagining
Λ - lambda, letting, lighting, creating
Ω - omega, ending, finishing, completeing

PREVIEW A LA AN OUTLINE -

A. Imagination Comes First
- α. Imagining Ways to Encourage Mathematical Activity
- λ. Let There Be a Grant Proposal
- ω. The Final Count

Λ. Let There Be a Creative Math Course
- α. Exercising the Imagination
- λ_1. Let There Be a Number Whose Absolute Value is -1
- λ_2. Let There Be a Number Whose Square is Zero
- λ_3.Let There Be a Number System for Every Point in the Real Plane
- ω. Finally, an Application

Ω. The Finale
- α. Imaginative Conjectures
- λ. Let There Be a Story About a Universe
- ω. A Timeless Finish

A. IMAGINATION COMES FIRST

To create a work of art, there is that chaotic period beforehand when imagining runs rampant, myriad possibilities cross one's mind, and questions pose themselves one after the other. This has in itself been the theme for some works of art.

There is a drawing by Goya of himself sleeping at his drawing board, with bats, cats, and other monsters of his imagination swirling above him [1, Fig. 172]. At his side is a sketch of Minerva, the goddess of art and science. A similar drawing [1, Fig. 174], originally meant as a frontispiece for his *Caprichos* actually appeared as Capricho 43 and has inscribed on it, "El sueno de la razon produce monstruos" (The sleep of reason produces monsters). In this one, an owl, representing Minerva, proffers Goya a pencil. The imagination dreams up images, irrational and fleeting, providing resources for reason to shape and form before the actualization of a figure on paper. Together, they produce works of art.

The overture of Haydn's oratorio *The Creation* depicts the night of chaos before the first day of creation. This is the time when God the artist lets his imagination run rampant, and he considers the myriad possibilities, like "Should I make space Euclidean or non-Euclidean? And what about the Uncertainty Principle? Might that make things more interesting?" This period ends of course when the chorus breaks in with "Let there be light," which characterizes so well the onset of activity in actually creating a work of art.

α. Imagining Ways to Encourage Mathematical Activity

The idea of mathematics as an art raises a lot of monstrous questions. Most people don't think of math as art. Most math courses do very little to help students think of mathematicians as artists. Is that the way it should be? If a college offers courses in creative writing, in music composition, in painting and in sculpture, why can't it offer one in creative mathematics? Is mathematics really so different from the usual arts that students must wait until their third year in graduate school before being explicitly encouraged to be creative? Is it really the best thing to limit the role of college math students to studying the the great works of the masters (usually without mentioning the masters)? Such imaginings engendered a certain jealousy of colleagues who teach the usual arts, and this jealousy prompted a dream of a course offering students a chance to create some mathematics on their own. Further, this course was imagined to be part of an overall program, which in this dream was called the "Aleph Program." But for such a dream to be realized, support is needed, which suggests a grant, which in turn unleashes more questions of who to turn to and how to actualize on paper what one is dreaming.

λ. Let There Be a Grant Proposal

A grant was requested from the National Science

Foundation under its LOCI (Local Course Improvement) program in 1980. The following is a quote from the narrative part of the application.

> *Man ist was er isst* - Feuerbach.
>
> Feuerbach liked to play with words, while at the same time saying something serious with them. A particular objective of the Aleph Program is to encourage playing around with mathematics while at the same time not forgetting that math has great potential for serious use (if necessity is the mother of invention, then playing around is the father). In fact, its aim is to treat mathematics as a living language, and its goal is to encourage a small number of appropriate students to be fun-loving serious poets in that language. To press this point a bit further, Feuerbach's statement, when actually spoken, is twosided. It is like an element in a two-dimensional space which has a projection on a joking dimension which says "man ist was er ist" (one is what one is), and also a projection on a serious dimension which says "man ist was er isst" (one is what one eats). The Aleph Program likes the joke, and further, to achieve its goal, it takes this serious side, maps it homomorphically to "mind is what it learns," and proposes that quite a different learning process should be provided for minds which are learning a language with the intent of writing poetry than for those who are simply learning it to be fluent enough to function in the marketplace.

ω. The Final Count

Two panels of five persons each evaluated the grant proposal. One person initially gave it a 1, the lowest score possible, deeming it too lighthearted to be taken seriously. Another person gave it a 7, the highest score possible, considering it a gamble, but one definitely worth taking. The other scores fell in between, resulting in an average that actually was big enough for the proposal to be in the group of grants granted.

Λ. LET THERE BE A CREATIVE MATH COURSE - FALL SEMESTER, 1981

"Para 40-Mathematical Thinking and Writing" is the name of the course, where "Para" is short for "Paracollege." The Paracollege is a small college within St. Olaf College which offers an alternative route, via tutorials and examinations, to a BA degree. It also encourages students and faculty in all disciplines to come up with, and then try out, new ideas. Para 40 was a new idea intended for first year college students, and four of them - Paul Borman, Eric Heppner, Beth Nelson, and Ken Olstad - ventured to sign up for it.

α. Exercising the Imagination

The goal was to produce some mathematical works of art. To start with there was a chaotic period of casting about for ideas with various lures, and then, once there was a strike, there were further questions about how to land it. For several years it had been the custom for a small group of Paracollege students to construct something using an old haymow rope during the first week of school. This year the structure was a "merry-go-round", which was essentially a triangle made of three wooden beams which was hung by means of the rope from an overhanging bough fifty feet above the ground. When the class first met I told them I had no preconceived ideas of what we would end up doing, but that ideas for math systems were lurking everywhere, and they should learn to be on the lookout for them. Since they had helped construct the merry-go-round, I asked if they could imagine some ways to use math in describing it. They had lots of ideas, from the angles of the triangle and the tension in the ropes to its behavior as a pendulum when it was swung back and forth. It was its behavior as a spinning object which interested us most, however, and the succession of the triangle's vertices as they passed by, *A,B,C,A,B,C*.... suggested a counting system in which the passing of three vertices gives the same result as the passing of zero vertices. Modulus 3 arithmetic was thus the result and they checked to see if it had the properties which later on they would learn are subsumed under the name "ring." This system thus had many of the same properties as the system of integers. The fact that 1+2=0 in this system meant that 2=-1, and this prompted Beth to raise the question whether absolute value made any sense here. We were not able to make any sense of absolute value for the modulus 3 system, but at this point Paul remembered he had been interested in the absolute value function in high school, particularly in its similarity to the squaring function, since for ordinary numbers both functions always produced nonnegative values. He knew that interesting things had happened

when somebody imagined a number whose square is -1. He accordingly wondered what might happen if he assumed there were a number whose absolute value is -1. That made me wonder too, and at that point I felt we had a strike, namely that we had something on the line worth playing with.

Sensing it would be good to have some experience in embedding a system in a bigger one, I asked the students to generalize what they had done in defining modulus 3 arithmetic to modulus n arithmetic. I then asked them to look specifically at the modulus 2 system and see if they could find a bigger system which had the properties of a ring and which had the modulus 2 system embedded within it. Using exhaustive methods, they painstakingly discovered it can't be embedded in a 3-element system. Continuing these methods, they considered 4-element systems and came up with 4 different ways to do it. In looking at these 4 different systems, with the hope of discovering a better way to arrive at them, they discovered that each could be thought of as adding to 0 and 1 (from the modulus 2 system) an element q and an element $1+q$. The 4 systems were then produced by letting q^2 be in turn each of the 4 elements in the system. Furthermore, the 4 elements of the system can be expressed in the form $x+qy$, for $x, y \in \{0, 1\}$. Also, in looking at the 4 systems, it was quickly discovered that the system with $q^2=0$ and the one with $q^2=1$ were really the same system, since the q and $1+q$ in the first system behave just like the $1+q$ and q respectively in the second system. Thus the idea of "having the same structure," later to be called "isomorphism," first reared its comely head. To test that they had a good method for embedding, I chose the 4-element system where $q^2=1+q$ (some day they'll find out a technical name for it is "GF(2^2)") and asked them to embed it. They of course found 16 different 16-element rings. Furthermore, Beth went through the 16 systems and found that, up to isomorphism, there were only 3 different systems, just as in the previous case. With these exercises completed, I felt we were ready to start the students' first work of art.

λ_1. Let There Be a Number Whose Absolute Value is -1

After the initial creative thrust of bringing a number into being, a next important and fun step is to name it. Paul, whose idea it was, had started taking Russian, and he thought an esoteric symbol, like the Russian tverdy enak, was appropriate for such a strange number. This designation didn't last very long, however, for it was not on the usual typewriter or computer keyboard, and the next symbol used was the dollar sign. Later on (when hyperbolic trigonometry appeared) the symbol was changed to "h," and to keep the same symbol throughout, it will be introduced here. Thus h is the number such that

$$|h| = -1.$$

Since the adjective "imaginary" is used for the number i, the students searched for an appropriate adjective for h and came up with "hallucinatory." The next question was whether the real numbers could be embedded in a bigger system which had h in it. The students guessed, and then showed, that there were rings with elements $x+qy$, for $x, y \in \Re$(eal), where q^2 could equal any element in the ring. They knew that one of these systems was well-known, namely the system of complex numbers, in which case q is given the name "i" and $i^2=-1$. They then wondered if one of these q's could be interpreted as h. The next problem was thus to figure out what element in the ring h^2 might be. The students went off to the library and looked up the properties of the absolute value function for real numbers. One property was $x^2 = |x|^2$. If one assumes this to hold for h, the result is

$$h^2 = |h|^2 = (-1)^2 = 1$$

Having no other argument available for what h^2 might be, other than the aesthetic one that if $|i|=1$, $i^2=-1$ then why not $|h|=-1$, $h^2=1$, we adopted this one. We thus had a system of numbers $x+hy$, for $x, y \in \Re$, which Paul dubbed the "ineptitude numbers." Later on, to parallel the complex number terminology, they became the "perplex numbers."

To get ideas of what to do with these numbers, the students did some checking of what things were done with the complex numbers. One thing was finding inverses. For the complexes, $(x+iy)^{-1}=(x-iy)/(x^2+y^2)$, so that every complex has an inverse except $0+i0=0$, namely the point where $y=x=0$. Using the same method for the perplexes,

$$(x+hy)^{-1} = \frac{x-hy}{x^2-y^2},$$

so that every perplex has an inverse except $x \pm hx$, namely the points on the lines $y=\pm x$. In time it was found that these numbers without inverses essentially characterize the perplexes. In particular, they divide the x, y plane up into 4 quadrants where x^2-y^2 is positive in the left and right quadrants and negative in the top and bottom quadrants. Further, since absolute value was extended in the complex case by $|x+iy| = (x^2+y^2)^{1/2}$, it was suspected that x^2-y^2 should be involved in the perplex case. For $0+h1=h$, this quantity is $0^2-1^2=-1$. One can't, however, just take the square root of this, since the result would not be -1. Various ways were tried to arrive at this result, and after some time a way was found to do this. One of the criteria was to find a function which would work for both the complexes and perplexes. The function $^{1/\ddot{2}}$, called the "funny square root" by the students, and defined for x real by

$$x^{1/\ddot{2}} = (\operatorname{sgn} x)|x|^{1/2},$$ (sgn x) is the sign function,

satisfied this criterion and was chosen. It is the inverse of the function $\ddot{2}$, called the "funny square," defined for x real by $x^{\ddot{2}}=(\operatorname{sgn} x)|x|^2$. Absolute value is then extended to perplexes by

$$|(x+hy)| = \left(x^2-y^2\right)^{1/\ddot{2}}.$$

With absolute value defined, one can look at curves of constant absolute value. For the complexes these are circles, and the circle $|x+iy|=1$, namely $x^2+y^2=1$ is of particular interest. By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and since $\cos^2\theta + \sin^2\theta=1$, it follows that $e^{i\theta}$ is on this circle. For the perplexes, the curves of constant absolute value are hyperbolas, and the hyperbola $|x+hy|=1$, namely $x^2-y^2=1$, was guessed to be of particular interest. Looking up the infinite series method for arriving at Euler's formula, Ken was able to come up with the following result,

$$e^{h\theta} = \cosh\theta + h\sinh\theta,$$

and since $\cosh^2\theta-\sinh^2\theta=1$, it follows that $e^{h\theta}$ is on the right hand arm of the hyperbola $x^2-y^2=1$. Although there were lots of ideas of things to do with the perplexes, it was along about this point that the students couldn't refrain from starting a second work of art.

λ_2. Let There Be a Nonreal Number Whose Square is Zero

In the language of elements q, $q^2=-1$ gave the complexes and $q^2=1$ the perplexes. Eric in particular was interested in the case $q^2=0$. The original symbol selected for this case was "Z" and it was called "Zarph," but later on (when linear trigonometry appeared) the symbol was changed to "ℓ" and it was called "ludicrous." Thus ℓ is the number such that

$$\ell^2 = 0 \cdot$$

In no time at all the following results appeared.

$$(x+\ell y)^{-1} = \frac{(x-\ell y)}{x^2},\ |x+\ell y| = \left(x^2\right)^{1/\ddot{2}} = |x|,$$
$$e^{\ell\theta} = 1+\ell\theta.$$

Numbers on the line $x=0$ have no inverses. Curves of constant absolute value are vertical lines, and $|x+\ell y|=1$, namely the lines $x^2=1$ are of particular interest, with $e^{\ell\theta}$ being on the line $x=1$. Because we weren't aware of any known trigonometric functions being defined on this line, we defined the "linear cosine" and "linear sine" to be

$$\cos\ell\theta = 1,\ \sin\ell\theta = \theta.$$

With this much done, there was curiosity about other q-systems, and, instead of treating them individually, Paul wondered if we couldn't produce them en masse. A somewhat more technical term for this would be "to generalize," and a generalization, somewhat like a gallery show with all the works on the same theme, might itself be considered a work of art.

λ_3. Let There Be a Number System for Every Point in the Real Plane

For the point (a, b) in the real plane, $\Re_q$ is the *q-extension of the reals*, or simply the *q-number system*, where

$$\Re_q = \{x+qy \mid x,y \in \Re\},$$ with $q^2= a+qb$.

At this point "q" stood for "quixotic," namely "romantic without regard to practicality." Later, it was realized it could stand for "quadratic," since the param-

eters a, b define a unique quadratic form zz^* for each number system.

It is convenient in this section to always let $z=x+qy$ and $z'=x'+qy$. Explicit definitions of *addition* and *multiplication* then are

$$z+z'=(x+x')+q(y+y'),$$
$$zz'=(xx'+ayy')+q(xy'+yx'+byy').$$

For zz' to be real, $xy'+yx'+byy'=0$, i.e. $x'=(-y/y')(x+by)$. A choice which motivates the next definition then is $y'=-y$, $x'=x+by$. The *q-conjugate* of z is z^*, where

$$z^*=(x+by)+q(-y), \text{ so that } zz^*=x^2+bxy-ay^2.$$

The following properties were checked to hold for the q-conjugate.

$$(z+z')^*=z^*+z'^*,\ (zz')=z^*z'^*,\ z^{**}=z.$$

The *multiplicative inverse*, if it exists, is

$$z^{-1}=\frac{z^*}{zz^*}$$

It accordingly fails to exist when $zz^*=0$, namely when $x^2+bxy-ay^2=0$. If $y=0$ then $x=0$ as expected. If $y\neq 0$, dividing by y^2 gives

$$\left(\frac{x}{y}\right)^2+b\left(\frac{x}{y}\right)-a=0, \text{ so that}$$
$$\frac{x}{y}=\frac{1}{2}\left(-b\pm\left(b^2+4a\right)^{\frac{1}{2}}\right)=\frac{1}{c_\pm},$$

which describes 2 lines, $y=c\pm x$, with slopes c_+ and c_- This provides three different cases. If $b^2+4a>0$, the 2 lines lie in the x, y plane, and they can be any two nonhorizontal lines passing through the origin. If $b^2+4a=0$, the 2 lines merge into 1 with slope $c_+=c_-=-2/b$. If $b^2+4a<0$, the slopes are no longer real, so the 2 lines "leave the x, y plane," intersecting it only at the origin. From our earlier experience with finite systems, students expected that, up to isomorphism, there might be only three systems among all the q-extensions. With a great deal of effort, Eric succeeded in showing that all the q-extensions with $b^2+4a>0$ were (ring)-isomorphic to the h-extension. Using the same technique it soon followed that all with $b^2+4a=0$ were isomorphic to the ℓ-extension and those with $b^2+4a<0$ were isomorphic to the i-extension. This however did not diminish a continued interest in all the q-extensions. The *q-absolute value of z* is

$$|z|=(zz^*)^{1/2}, \text{ with properties } |zz'|=|z||z'|,\ |z^*|=|z|,$$
$$|z^{-1}|=|z|^{-1}.$$

A curve of particular interest is

$$|z|=1 \text{ namely}$$
$$x^2+bxy-ay^2=(x+\frac{1}{2}by)^2-\frac{1}{4}(b^2+4a)y^2=$$
$$(x-y/c_+)(x-y/c_-)=1$$

It always passes through the numbers 1 and -1. If $b^2+4a<0$ it is an ellipse. If $b^2+4a=0$ it is two parallel lines with slope $-2/b$. If $b^2+4a>0$ it is a hyperbola with asymptotes being the two lines $y=c_\pm x$, namely the lines $|z|=0$. In this last case $|z|$ can also be negative and the curve $|z|$ same asymptotes as $|z|=1$, but "orthogonal" to it.

In order to generalize the concept of unit circle from the complex case, it seemed that $|z|=1$ and $|z|=-1$ should both be included, which prompted the next definition. The absolute value of the q-absolute value of z, called the *radial function of z* is

$$p(z)=\|z\|, \text{ with properties } p(zz')=p(z)p(z'),$$
$$p(z^*)=p(z),\ p(z^{-1})=(p(z))^{-1},\ p(p(z))=p(z).$$

The curve $p(z)=1$ is the *q-unit curve*, and the set of points on this curve is

$$U=\{z\mid p(z)=1\}.$$

It follows that if z, z' are in U so is their product, as well as their inverses. The system $U(\cdot)$ consisting of U together with multiplication is then a group. For $b=0$, $c_\pm=\pm a^{-1/2}$, and there is symmetry about both the x and y axes. Fig.1 indicates how these change with the pa-

rameters a and $c_{\pm}$. As b varies from zero, the axial symmetry is lost and the curve gets more and more skewed.

In order to generalize trigonometry now from the complex case, angles have to be assigned in some way to points on the unit curve. When $q=i$ we know that $e^{i\theta}$, $0<\theta<2\pi$, describes the points on the unit curve, and θ can be interpreted as arc length. Similarly, when $q=\ell$, then $e^{\ell\theta}$, $\theta \in \Re$, describes the righthand line of the unit curve, and θ again can be interpreted as arc length. When $q=h$, then $e^{h\theta}$, $\theta \in \Re$, describes the points on the unit curve in the righthand quadrant, but θ cannot be interpreted as arc length in this case. Looking up information about hyperbolic trigonometry, the students found that θ can be interpreted as twice the area swept out by a ray from the origin as its outer point moves from e^{h0} to $e^{h\theta}$. Since this interpretation also works for the i and ℓ cases, it suggests a way to generalize to q for that continuous part of the unit curve passing through $z=1$. The fact that $e^{q(\theta+\theta')}=e^{q\theta}e^{q\theta'}$ holds for the cases $q = i,\ \ell,\ h$, however, suggests a more general approach, since e^{q} acts like an isomorphism, with which the students had already had some experience.

To assign a unique angle to each element in U and to allow for addition of angles, let $\Phi(+)$ be an isomorphic copy of $U(\cdot)$. The elements of Φ are *angles*. The *angle function* α is an isomorphism from $U(\cdot)$ to $\Phi(+)$, so that, for $z,\ z' \in U$,

$$\alpha(zz') = \alpha(z) + \alpha(z').$$

In particular, then, for 0 the additive identity of $\Phi(+)$,

$$\alpha(1) = 0.$$

Further, specify a particular element $\mu \in U$ as the *unit measure for angles* and define an element $1 \in \Phi$ by

$$\alpha(\mu) = 1.$$

Then let μ be the inverse mapping α^{-1} by means of the following notation, for $\phi \in \Phi$,

$$\mu^{\phi} = \alpha^{-1}(\phi), \text{ so that } \mu^{\alpha(z)} = z.$$

It follows that

$$\mu^{\phi+\phi'} = \mu^{\phi}\mu^{\phi'},\ \mu^{0} = 1,\ \mu^{1} = \mu.$$

For $q=i,\ \ell,\ h$, the choice for μ is of course e^{q}. But e^{q} for the general case is on the unit curve only for b=0. To remedy this for the general case, we set

$$\mu = e^{-(b/2)}e^{q}.$$

Now we can define the *q-cosine*, *q-sine*, and *q-tangent* as functions cosq, sinq, and tanq from Φ to $\Re$ by

$$\cos\mathrm{q}\,\alpha(z) = x,\ \sin\mathrm{q}\,\alpha(z) = y,\ \tan\mathrm{q}\,\alpha(z) = \frac{y}{x}.$$

The *q-Euler formula* then is

$$\mu^{\alpha(z)} = z = x + qy = \cos\mathrm{q}\,\alpha(z) + q\sin\mathrm{q}\,\alpha(z)$$

From this, in the usual way, the angle addition formulas are found to be

$$\cos\mathrm{q}(\phi + \phi') = \cos\mathrm{q}\,\phi\cos\mathrm{q}\,\phi' + a\sin\mathrm{q}\,\phi\sin\mathrm{q}\,\phi',$$

$$\sin\mathrm{q}(\phi + \phi') = \mathrm{sinq}\,\phi\,\mathrm{cosq}\,\phi' + \mathrm{cosq}\,\phi\,\mathrm{sinq}\,\phi' + b\sin\mathrm{q}\,\phi\sin\mathrm{q}\,\phi',$$

$$\tan\mathrm{q}(\phi + \phi') = \frac{\mathrm{tanq}\,\phi + \mathrm{tanq}\,\phi' + b\,\mathrm{tanq}\,\phi\,\mathrm{tanq}\,\phi'}{1 + a\,\mathrm{tanq}\,\phi\,\mathrm{tanq}\,\phi'}.$$

Generalization of the usual trigonometric identities follows in similar fashion.

Finally, to have q-polar coordinates for all z with $p(z)\neq 0$, note that

$$p[z/p(z)] = p(z)/p[p(z)] = p(z)/p(z) = 1,$$

so $z/p(z) \in U$. The *extension of the angle function* α to $z \in \Re_q$ is then defined by

$$\alpha(z) = \alpha[z/p(z)].$$

Thus, for $p(z)\neq 0$,

$$z = p(z)[z / p(z)] = p(z)\mu^{\alpha[z/p(z)]} = p(z)\mu^{\alpha(z)}.$$

It took a while for me to realize that q-number systems were not new, but have been around since the time of Cauchy, since they can be considered as $\Re[x]/(x^2\text{-}bx\text{-}a)$, namely as the quotient ring of the ring of polynomials $\Re[x]$ modulo the polynomial $x^2\text{-}bx\text{-}a$. The q-definitions of absolute value and angle, however, were, to the best of my knowledge, new, and it is they which make possible the many q-generalizations from classical results in areas such as trigonometry, geometry, and physics.

One idea generates more ideas and as we worked during the semester ideas of things to do came faster than we could do them. For example, we wondered how many of all the different things done with the complexes might have q-analogues. We concentrated on q-trigonometry, but what about q-geometry, q-analysis, and q-Hilbert space? We also wondered about embedding q-number systems in bigger systems. Can we have i and h in the same system, and maybe ℓ as well? We did check out that there are q-extensions if one replaces the real field by an arbitrary field, and that each extension has its own trigonometry, but then we wondered what happens if one starts with a ring which isn't a field. We also wondered if the ℓ-number system would be good for doing calculus, since ℓ can be interpreted as an infinitesimal. This has since been done by some other students, and in that context the numbers came to be called "ethereal numbers" [2]. Other creative work on q-geometry has since been done by students, and [3] is a byproduct of that.

With the increasing degree of generalization and attendant abstractness in our work, the students did show various signs of restlessness. Already with the perplexes Eric and Ken were saying things like "Okay, we've invented a number system, but so what, what good is it?" I was somewhat taken aback. I thought the beauty and elegance of the perplex number system was ample reward for our large investment of time and energy. Thinking of it as a work of abstract art with no need to describe something in the real world was somewhat soothing for them, but not really satisfying. In varying degrees I think the students' aesthetic instincts leaned toward representational art, so I urged them to be on the lookout for possible applications of our q-number systems.

ω. Finally, an Application

Toward the end of the semester the results obtained by the class were presented at one of the math department's weekly colloquia. One of the figures shown was the unit curve for the perplexes with its hyperbolas and their $|z|=0$ asymptotes. Lynn Steen commented that these asymptotes reminded him of the light cone in special relativity theory. That was just the hint needed for an application, and excitement rose when Taylor and Wheeler's book *Space-Time Physics* was discovered to use hyperbolic trig functions in an elegant way to express the equations of special relativity [4]. The perplexes suddenly became a very natural language for this theory of physics [5]. Generalization, however, had become a habit of thought, and if the perplexes corresponded to one kind of physics, then each q-number system should correspond to a specific kind of physics. The following interpretation of $\Re_q$ will be called *q-physics*, and in this section let $z \in \Re_q$ be $z=t+qy$, where t is interpreted as a time coordinate and y as a position coordinate which is a function of time. Then z, as a function of time, traces out a curve called a *world line*. Differentially speaking, $dz=dt+qdy$. Let $d\tau$ be the radial coordinate of dz, called the *proper time*, and when $d\tau\neq 0$, let 0 be the angular coordinate of dz, called the velocity parameter. Thus,

$$dz = dt + qdy = d\tau\mu^{\phi} = d\tau(\cos q\,\phi + q\sin q\,\phi),$$

giving,

$$dt = d\tau\cos q\,\phi,\;\; dy = d\tau\sin q\,\phi, \text{ so that}$$

$$v = \frac{dy}{dt} = \frac{\sin q\,\phi}{\cos q\,\phi} = \tan q\,\phi,$$

where v is the *velocity*. For *rest mass* m_0, the *mass* m and *momentum* p are defined as

$$m = m_0\cos q\,\phi,\;\; p = m_0\sin q\,\phi = mv.$$

The "light cone" is determined by the lines $y=c_{\pm}t$, so that the slopes $c_{\pm}$ are interpreted as the velocities of light in the positive and negative directions along the

y-axis. For $b=0$, the "light cone" is symmetric with regard to the two directions, i.e. $c_{\pm}=\pm a^{-1/2}$, and for the specific cases $q=h$, ℓ, i one has respectively $c_{\pm}=\pm 1$, $\pm\infty$, $\mp i$.

The elements μ^{ϕ} are the *q-Lorentz transformations*, such that z and z' are related by

$$z = z' \mu^{\phi},$$

where the reference frame for z' moves relative to that for z with velocity $v=\tan q\,\phi$. A transformation μ^{ϕ} followed by $\mu^{\phi'}$, results in $\mu^{\phi}\mu^{\phi'} = \mu^{(\phi+\phi')}$. For the corresponding velocities,

$$v = \tan q\,\phi,\ \ v' = \tan q\,\phi',\ \ v'' = \tan q(\phi+\phi'),$$

the *addition rule for velocities*, from the addition formula for tangents, is

$$v'' = \frac{v + v' + bvv'}{1 + avv'} = \frac{v(1 - v'/c_+) + v'(1 - v/c_-)}{1 - vv'/c_+c_-}.$$

If $v'=c_+$, then $v''=c_+$, and if $v=c_-$, then $v''=c_-$. Because of the symmetry for v and v', if either one is $c_{\pm}$, then $v''=c_{\pm}$. One thus has the interesting result, that, *for each q-physics, the speed of light in a given direction will be the same in every inertial frame*.

Ω. THE FINALE

A lot had happened in the course. There was a surge of mathematical creativity, culminating in q-number systems. Then the wish to find a use for these systems was fulfilled by means of q-physics. On the last day of class some literary creativity was sparked when Eric came up with an idea for a story.

α. Imaginative Conjectures

All the different q-physics provided much puzzlement, especially when $b\neq 0$ and the light speed is faster in one direction than another. To try and get a better understanding, we first went back to the special cases of $q=h$, ℓ, i and restricted attention to angles which correspond to traveling forward in time at speeds less than the speed of light. A tabulation of some results is given in Figure 1.

h-physics is special relativity physics with units chosen so that light speed is unity. The distinguishing characteristic here is that m increases with $|v|$. The $e^{h\phi}$, $\phi\in\Re$, are the classic Lorentz transformations. It was rather a surprise to discover that ℓ-physics is Newtonian physics, where m is independent of $|v|$. That seemed to make sense, however, for an infinite light speed. The $e^{\ell\phi}$, $\phi\in\Re$, are the Galilean transformations. i-physics, as far as we know, was a physics no one had thought of before. It is, in a sense, the complement of h-physics in that m here decreases as $|v|$ increases, actually approaching zero as $|v|$ approaches infinity. The $e^{i\phi}$, $-\pi/2<\phi<\pi/2$, thus have no name originating from physics. They are of course the ordinary Euclidean rotations through the angle ϕ. That $c_{\pm}=\mp i$ was intriguing, and we puzzled over what that could mean. Might light traverse a real distance in an imaginary time, or an imaginary distance in a real time? Whatever it might mean, we ended up conjecturing that we as sentient beings could only perceive light if it traversed a real distance in real time. We then broadened our scope some and considered all the cases where $b=0$, so now there is a q-physics for each value of a, namely for each point on the real line. This was something to ponder over. The centerpoint,

q	$c_{\pm}$	v	m	p	ϕ
h	± 1	$\tanh\phi$	$m_0\cosh\phi$	$m_0\sinh\phi$	$-\infty<\phi<\infty$
ℓ	$\pm\infty$	$\tan\ell\,\phi=\phi$	$m_0\cos\ell\,\phi=m_0$	$m_0\sin\ell\,\phi=m_0\phi$	$-\infty<\phi<\infty$
i	$\mp i$	$\tan\phi$	$m_0\cos\phi$	$m_0\sin\phi$	$-\pi/2<\phi<\pi/2$

Figure 1
A tabulation of some results.

with $c_{\pm}=\pm\infty$, is Newtonian. Things get more and more relativistic as light speed decreases to the right. To the left we conjectured perceptual darkness, and at light speeds of zero, i.e $a=\pm\infty$, we further conjectured no motion, assuming nothing can travel faster than light. Eric then came up with his conjecture that the real line is a time line. This stimulated more ideas. This time line needn't be coordinated with ordinary clock times. Clock times speed up and slow down relative to each other and thus also to this time line. But such a time line means that the kind of physics changes with time. There is a beginning, call it "alpha," at $-\infty$, and there is an end, call it "omega," at ∞. In addition, there is a very special centerpoint, let's call it "lambda," corresponding with the dictum "Let there be light." This cosmological scenario accordingly suggests a little story.

λ. Let There Be a Story About a Universe

> In the beginning there was nothing. The speed of light was nothing, and since no thing can move faster than light, there was no motion. Then the speed of light edged away from zero, but it was imaginary, and there were also slight motions, but they too were imaginary. Nevertheless, this imaginary commotion continued to build, gaining momentum, accelerating more and more into a phantastic frenzy, moving off toward imaginary infinity, at which point, suddenly
>
> there was light !
>
> It was the big bang of a white hole exploding into reality the imagining which had gone on before. Now there was real light, light traveling instantaneously from one place to another, with motions to match, and the motion was Newtonian. And the light and motion interacted, bringing forth day and night, earth and firmament, planetary merry-go-rounds and galactic vortices. Then the speed of light decreased slightly from infinity, motion took on a tinge of relativity, and complexity evolved unabatedly. At one point motion became genuine Einsteinian, but the speed of light waned inexorably, and as aeons passed, everything slowed down more and more, heading off into a black hole where finally everything came to a stop, including light, motion, and even time, so that the end was like the beginning.

ω. A Timeless Finish

In a relativistic universe clock times slow to a stop for things which reach light speed, and when the speed of light is zero, everything travels at light speed, so at the omega point clock time is frozen. The time line may keep on going, but when it reaches the omega point, the universe, as a work of art, is finished. Eric's story, like Haydn's oratorio, is itself a work of art, which, at one level, tells how another work of art, a universe, was made. At another level, however, it is an analogy which can be interpreted as depicting the creative process involved in making any work of art, with the initial chaotic period devoted to the imagination, the point of inspiration when imagined things start becoming real, and the final moment when time becomes frozen for that particular work of art and it is finished (at which point it can be put on a record, or hung on a wall, or put on a bookshelf, and serve as a resource for other imaginations to draw upon).

REFERENCES

[1] Lopez-Rey, J. *Goya's Caprichos, Vol. 2*. Princeton, Princeton University Press, 1953.
[2] Fjelstad, P. "Limitless Calculus via Ethereal Numbers," *UMAP Journal*. 15 (1995): 161-174.
[3] -. "Extending the Pythagorean Theorem to Other Geometries," *Mathematics Magazine*. 69 (1996): 222-223.
[4] Taylor, E. and Wheeler, J. *Space-Time Physics*. San Francisco, W.H. Freeman, 1966.
[5] Fjelstad, P. "Extending Special Relativity via the Perplex Numbers," *American Journal of Physics*. 54 (1986): 416-422.

A Reflection on the Word
Remembering the Word "Word" Is Reflexive

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Also to be remembered:
logos - the Greek word for word;
Logos - in Greek philosophy, reason, thought of as constituting the controlling principle of the universe, and as being manifested by speech -Webster's Dictionary;
man - a human being, whether male or female (prob. IE base men - to think, seen also in L. mens, mind) -Webster's;
Man - a reflection of the Logos.

1 In the beginning was the Word, and the Word was with the Word, and the Word was the Word.
2 That is all there was. There was not even any nothing.
3 But the Word, by nature, is a Creator; so the Word said, Let there be nothing.
4 So now there was something, namely nothing, and the Word was with nothing, and the Word was nothing.
5 But the Word, by nature, is a Creator; and the Word said, Let nothing be called 0.
6 So now there was more than nothing, namely nothing and its name, and the Word said, Let this new something, which is not nothing, be called 1; and the Word was with 1, and the Word was 1.
7 But the Word is a natural Creator, and 0 and 1 were called 2, and 0, 1 and 2 were called 3, and 0, 1, 2 and 3 were called 4.
8 A pattern thus emerged and, instead of naming each natural number, the Word prescribed an algorithm and attempted the creation of all natural numbers by saying, Let the sum total of natural numbers be called ω; and lo and behold - it happened.
9 Note well: ω is not a natural number; also, naming is ipso facto creating, provided no contradiction ensues.
10 The Word looked at the natural numbers and saw that they were good; and the Word was with ω, and the Word was ω.
11 But the Word continued creating natural things, calling them into existence.
12 The light was called Day and the darkness Night; the dry land was called Earth, and the gathering together of the waters were called Seas.
13 That which was created in the image of the Word, reflecting the Word's naming capability, was called Mankind; two kinds, that is, called Man and Woman.
14 The Word looked at what was done thus far and saw that it was very good, and then, instead of continuing in this manner, prescribed the rhythm of creation and attempted the creation of everything by saying, Let the sum total of everything be called Ω; and lo and behold - nothing happened.
15 If everything had come into being, completing the creation process, then besides everything there would be its name, so everything would not be everything, a contradiction.
16 With this the Word rested, content to know there were transcendent entities which could be thought but not named; but the Word's reflection in the natural world continued the creation process.
17 Mankind called into being numbers that were negative, fractured, irrational, and imaginary, as well as ones that were whole, rational, real and complex.
18 Mankind also produced things called prejudice, slavery, oppression, and demonology, as well as tolerance, freedom, cooperation and theology.
19 The Word watched as creation continued on at an accelerating pace, working its way toward everything, and although no name could be given it, the Word saw that everything was very good, and the Word was with everything, and the Word was everything.

Calculus for the Liberal Arts: A Humanistic Approach

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BRIEF SUMMARY
At the Humanistic Mathematics Sessions at the Mathematics Association of America meetings in Orlando in January 1996 I presented an outline of the approach I planned to take in introducing Calculus into a Liberal Arts Mathematics course. I followed that plan of action in four classes in spring and summer 1996. This paper is a report on the success of this endeavor containing some of the essays written by students in the spring courses.

INTRODUCTION
The Calculus has played an enormous role in the development of modern science and technology. It still constitutes the primary college mathematics course for those continuing in mathematics and other technical fields. However, we rarely touch on the Calculus in the courses offered for general education purposes, the so called Mathematics Appreciation or Liberal Arts Mathematics Courses. Liberal arts students should be exposed to this important branch of mathematics in a positive way, one which they will find relevant and interesting. It has had an enormous impact on their lives and is the cornerstone of modern mathematics and science. Nevertheless, the Calculus is usually ignored in general education courses. Some argue that it is impossible to do Calculus without sufficient algebra skills, and many or most liberal arts students are lacking those skills. It has been deemed unproductive to force these students into yet another attempt to learn algebra, and therefore calculus is not available to them. This argument, while it once had some merit, is really not compelling today. With computer algebra systems readily available, there is no reason liberal arts students with weak algebra backgrounds must be denied an opportunity to understand calculus concepts or to use calculus concepts to solve problems. Also, if the goal is that students appreciate the Calculus, rather than master it, then it is not necessary that they, themselves work the problems or go through the algebra. This is one place where perhaps it might be appropriate for mathematics to be a "spectator sport."

There still, however, remains the question of interest and a conceptual framework. Why should students who do not plan to use the Calculus in their fields have any interest in it? Algebra and Calculus are rarely presented in a social or historical context. This makes it difficult for liberal arts students to find them relevant. Calculus can be presented in terms of its role in the Enlightenment and its historical and cultural significance. Then students from all disciplines would have a framework of interest in which to work. They should see the Calculus as one of the most profound intellectual achievements of the modern world. They can and will appreciate its significance. They should no longer have to ask, "What is Calculus?"

This paper is a progress report on my attempt to introduce the Calculus in our Liberal Arts course, Mathematics and Culture. Using an essay on the Calculus which I prepared while on sabbatical last fall, I led three sections of students through a three week tour of the Calculus this Spring and one section this summer. I do not claim to have perfected the approach yet. I would like to incorporate more use of technology so that students can solve Calculus problems themselves, and I expect to be able to do that this fall. However, in the three sections I taught last spring, access to a computer lab was limited. Nevertheless, I feel that the overall experience was successful.

THE APPROACH
The first thing we discussed were four problems that are central to the motivation for the Calculus, Zeno's

Paradox (infinite sums), the area problem, the tangent problem and optimization problems. Zeno's paradox in particular is an interesting problem to debate with liberal arts students. After that we digressed into the foundations, or "preliminaries." Interestingly enough, the "preliminaries," or preCalculus topics: Functions, the real number system, etc. were developed after Newton and Leibnitz had published the Fundamental Theorem of Calculus. This fact and the reasons for developing the real number system, functions, and limits are discussed as well. At this point the students are asked to write down three questions about what we've covered so far. I put these questions on an overhead and we go over them in class. The questions they had at this point include (some have been paraphrased as they were essentially repeated by more than one student):

- How can one master calculus without a strong mathematics background? [The answer is of course that you can't "master" calculus without the background, but you can "appreciate" it.]
- How can I use Calculus in my daily life? Why should it be important to the average person? What real life questions will Zeno's paradox help me solve?
- Why is anyone interested in things like Zeno's paradox? Especially if it is a non-issue due to the definition of limit & infinite sums? Why is it a paradox? We figured out when Achilles would catch the tortoise; Zeno was wrong; why are we even discussing it?
- What is the significance of learning sin, cos and tan...?
- What is the significance of Calculus to this course?
- Why are these "problems" so crucial to Calculus? They don't seem life-threatening if they weren't solved.
- Something that occurred to me about the problems we did was that they were more interesting than I believed math problems could be. Math is not too bad when you incorporate more than just numbers. Are we going to examine more problems like these?"
- Can we do some "hands on" exercises to illustrate the exercises? [This is why I hope to be able to incorporate some lab exercises in the future. There is one other "hands on" optimization exercise I'd like to use but didn't in the interest of time]
- Is Calculus a product of mathematics, or is it a discovery of mathematicians? Is it a part of the world, or did mathematicians develop it as a means of problem solving?
- Now that we have real numbers, have we become lazy and started to take them for granted?
- I do not understand how we can add measurements that do not exist into problems, only to take them out for the purpose of being unable to divide by 0. They were not taken out mathematically, but simply disregarded. Mathematics is a subject that requires great precision, and this invalidates the entire finding. How can you be sure if an answer is mathematically correct when some of the problem was not only unknown but unreliable? [Bravo! That's exactly why we need the preliminaries.]
- What are some real world applications of Calculus?
- Zeno was a philosopher; are any of the other problems philosophical in nature?
- Explain why the diameter and the circumference of a circle could not be compared. What are the difficulties in assigning numbers to geometric quantities? What does incommensurate mean? How can you compare two areas if you don't know one?
- Why does the area of a rectangle have a curved graph?
- Just how small is infinitely small? How do you visualize an infinitesimal?
- What is meant by a rigorous definition?
- Are there any practical uses of logarithms?
- How exactly are limits applied by using the definition?
- What did the Greeks think pi was if it wasn't a number? What is B? How is it calculated?
- Would Zeno's problem still be a paradox if the endurance of the competitors changed midway through the race? Depends on how it is set up...What if the tortoise gets a 20m head start?...Why could everyone deal with the time travel paradoxes but have so much trouble with Zeno's paradox? Is there a shortage of non-mathematical paradoxes?
- Why is there no real number such as an infinitesimal? What is an infinitesimal?
- Please prove that 1-1+1-1+1.... = 1/2. Doesn't everything in mathematics always have a right answer? If you put this into a computer you would always get 0 or 1; why is it not possible to refute this? Is this actually equal to anything? Was Euler on crack?
- What is the difference between natural and whole

numbers other than the fact that natural numbers start with 1 and whole numbers start with 0? What's so special about 0?
- Why did it take so long to begin using irrational numbers in the real number system?
- What does it mean to approach 0, and does approaching 0 relate with the notion of infinity?
- If a function is not continuous, can you make it continuous mathematically?
- What exactly is the method of exhaustion? How do you find the area between the circle & the triangle?
- On the tangent problem, how do we pick the point? Won't we get a different answer if we use a different point?
- In the area problem, why can't you just find the area using $A = \pi d^2/2$?
- How do we know the sequences 2, 2.1, 2.14,... and 3, 3.1, 3.14, 3.141,... do not have rational limits?
- What are the properties of a logarithm function?
- What are sequences which converge within themselves? Why did Cauchy define real numbers this way?

What was most interesting in the discussions of these questions was the way that students would start to answer each other's questions. In one class a particularly lively discussion ensued about the "why do I have to know this?" questions. In any event, the questions clearly show that the level of understanding varies from student to student but that all of them could ask questions, and most of the questions were about the material and reasonable. Taking time to address all these questions increased the comfort level of the class at a time when some students were beginning to get desperate about what might be expected of them.

After the discussion on their questions, we go over the Derivative and the Integral including a discussion of how they help solve three of the problems (Zeno was dealt with in the preliminaries). During the course of this discussion we "prove" a few theorems, including Rolle's theorem (then the Mean Value theorem is stated and used) and the Fundamental Theorem of Calculus.

RESULTS
At the conclusion of the Calculus section of the course students are given the following assignment:

It is now the year 2002. You are hosting a party for a mixed group of neighbors, colleagues, friends and some of their adult or almost adult children. One of those present is an older gentleman, Ben, who is well respected and whose good opinion you value. An 18-year-old college freshman, Jean, is also present. Ben asks Jean what (s)he is studying that semester, and (s)he replies that (s)he is taking English, History, Psychology, Computer Programming and Calculus. Ben then says, "You know, I've known a lot of people who say they are studying Calculus, but I've never quite figured out what Calculus is; what is it?" You notice that Jean is rather uncomfortable and doesn't seem to be able to answer. As the Host(ess) you want to jump in and help out by answering. So, How do you respond to "What is Calculus?" Remember that you want to make a good impression on Ben and that he will not appreciate your interruption if your answer is flip or doesn't really satisfy his curiosity.

Responses pretty much fell into three categories. The first group says something about Calculus being an important branch of mathematics or something engineers use or something else along those lines. They address the importance of the Calculus to our technology or our society but don't really get into what it is. They give reasonable answers, but their responses could just as easily apply to almost any branch of mathematics. The second, and largest, group have very good responses, showing that they realize that Calculus involves change and/or going through the four problems. They are pretty much parroting back what they've read or been told, but they are doing so in a reasonable manner. Their essays clearly show that they have picked up an appreciation for what Calculus is all about. The third group surprised me. They really were able to describe Calculus, or some aspect of it. in their own words. I asked some of them if I could share their work with you.

Lilly McCready wrote:

WHAT IS THE CALCULUS?
There is not one concise definition of calculus that everyone agrees on. I think that calculus is a series of functions and relationships which are defined in terms of one another. These functions or relationships are generally changing, too. This is why Newton first called calculus "fluxions" because the variables are constantly changing.

Calculus is unlike basic elementary mathematics in

that it does not usually entail a lot of actual numbers or simple arithmetic. Instead, symbols are used to represent numbers. One reason for this is that the numbers are always changing. By using a variable, it can represent any number that we choose. Another reason that symbols are used instead of numbers is because often times in calculus, the numbers one is dealing with are extremely large or small. One other reason that symbols might be used is because one might not know what number is wanted. The purpose of the function may be to find out the number that you need. Other times a graph might be drawn to explain a certain function or relationship between two variables instead of doing computations.

The calculus may not appear in your daily life, but it is in life. In a world where things are constantly changing, the calculus is useful. Since calculus offers a different way to deal with relationships that are either independent or interdependent of one another, it can help with the value of the dollar, exports going up and imports becoming more expensive and even the space shuttle taking off.

Amy Bauersfeld wrote:

WHAT IS CALCULUS: A SIMPLE DEFINITION WHILE EATING COCKTAIL WEENIES

"Sorry to interrupt you, Jean, but I couldn't help but overhear your discussion about calculus. It got me nostalgic for this class I once took in college at good old Salisbury State. When I was a sophomore, Ben, my teacher asked me that very same question: What is calculus? At the beginning of the class I had not the slightest clue. But as the class progressed I discovered that calculus was a system that made solving complex mathematical problems simpler. For the most part I learned that calculus is a blending of geometry, algebra, and arithmetic. It involves a method of using symbols to represent numbers in an equation to solve for unknown factors.

As far as the history of calculus goes, it's been around for something like two thousand years. If I remember correctly Sir Isaac Newton and this other guy Wilhelm von Leibnitz were the mathematicians who simplified the process of the Calculus. You have to remember that this was before the advent of the modern real number system or the use of infinity or infinitesimals which I'm sure you're studying now, Jean, in your class.

Newton and Leibnitz, in my opinion, must have been geniuses to develop a system with such utility and simplicity. I don't know where we would be today if calculus had not been developed. If you think about it, calculus has been used to discover some of the greatest technologies of the twentieth century, nuclear power being just one of them. Without the usage of calculus equations, the physics behind the power would never have been discovered. That's only one thing off the top of my head where I know calculus is put to use, but I'm positive that there must be a million more applications out there.

Rebecca Hudson concentrated on the problems:

WHAT IS CALCULUS?

Calculus is the study of four problems and the questions that these problems bring to mind. The first of these problems is called Zeno's Paradox. It deals with the fact that if you want to move from one point to another point, you must first travel half the distance between the points, then half the remaining distance, and so on. If this is true, then you can never reach your destination. This is called Zeno's Paradox.

The second problem is the Area Problem. The area of any region with straight line borders can easily be found by dividing the region into rectangular and triangular divisions. The problem that Calculus confronts is how to find the area of a region with curved borders.

The third problem is the Tangent or Velocity Problem. A tangent is a line that intersects a circle at only one point. If an object was moving along the edge of a circle and suddenly broke loose, the object would run along a tangent. Another property of the tangent of a circle is that it is perpendicular to the radius. Calculus helps us determine what to do when we have curves instead of circles.

The last problem that Calculus deals with is called the Optimization Problem. These are problems in which it is necessary to find the largest or smallest possible value of something. For example, suppose you were given a certain amount of material and asked to construct the largest structure possible. Calculus allows us to find the many possible dimensions that

such a structure might have.

Jennifer Taylor had seen some Calculus before this class. Nevertheless, her response showed that she had gained from our discussion as well.

WHAT IS CALCULUS?

If I were hosting this party, I would pleasantly ask if I could join the interesting conversation. Once accepted, I would give my opinion of calculus to get Jean off the spot like any good hostess would do. I would explain to Ben that my experience with calculus has led me to the belief that calculus is a type of mathematics that helps you solve complex real life problems. For example, calculus is divided into two sections, differential calculus and integral calculus. These sections help solve problems a step beyond what algebra and other types of mathematics allow us to do.

The main purpose of differential calculus is to find the rate at which a known, but varying, quantity changes. For example, if we wanted to know the speed at which a plane travels over a distance of 500 miles in one hour, we could use simple algebra to find the answer (R = D/T). Differential calculus comes into play when the rate, or in this case, the speed of the plane does not remain constant over the 500 miles or one hour. Differential calculus allows us to determine the speed of the plane at any moment in time.

Integral calculus, on the other hand, is used to find a quantity, knowing the rate at which it is changing. One example is when you want to determine the distance a bullet has gone at a given time, knowing the rate at which the speed is decreasing. Integral calculus is often used in geometry in finding areas of curved objects.

The history of calculus is also helpful in understanding its function. Some of the first ideas of calculus began with Archimedes, a Greek mathematician, who formulated methods of finding the volume and surface area of spheres.

Calculus is the answer to questions that other types of mathematics cannot answer. The concepts of calculus allow us to perform many important problem solving skills, and is used in many scientific and engineering fields.

Lauren Michener, a particularly articulate communication arts major, had a unique and very creative response to the question.

CALCULUS

"My dear friend Ben, let me interject and try to answer this question for Jean. When I was younger my school teacher taught us a poem, which although may be slightly childish, it effectively and pleasantly answers the question, 'What is calculus.' I happen to have a copy of it here in my apron, for I was tutoring my niece just before the guests began to arrive. Let me read it to you:
'What is calculus?,' a teacher might say
The student replies, 'It will ruin my day
It doesn't light up and it doesn't talk back
It doesn't play ball, not good for a snack
It's not on T.V. and it's not found in stores
All that it is is a terrible bore!'
The teacher looked grim and to this she replied
'Yes, but it's in your book, just look inside.
Calculus has been used for quite a long while
Since the 17th century it's been the style
To solve computations it was used first
Or to calculate problems that were the worst
This method of solving was a fabulous creation
So grand it was done in special notation
Such as logic or symbolic that also was seen
And for people who knew it this all was real keen
Other methods of calc were created by von Leibniz
These were differential and integral, boy what a whiz!
Then while trying to find a "universal characteristic"
That could unify all thoughts that were mathematic
For symbolic logic he laid the foundation
Next calculus grew and with it exploration
Parts of math and physics got new application
And theoretical basis went under examination
It all was accepted but not 'till centuries passed
And four basic problems were the start to calc class
These inspired the need for this powerful math tool
That today every student can learn in their school
For technicians, engineers, and teachers the same
Calculus is more cool than any light-up game
In the words of Dr. Shannon I'll repeat this dedication
"It is a beautiful monument to human imagination."'"

"Bravo! Bravo! That was an absolutely exquisite explanation of calculus! I understand completely now. It was created in the seventeenth century as a means of creating or solving computations or calculations in

a special notation such as that of logic or symbolic. Those must be the numbers and letters I see scribbled across mathematicians' notebooks today. That symbolic logic was not created until von Leibniz, that clever fellow, attempted to find one "universal" way of expressing mathematical thought. All of these systems and styles were then lumped together to get what we know today as calculus. And there are four problems which tormented the minds of mathematicians, and because calculus was the only man fit for the job, he was created and called upon. My, calculus is a glorious thing! It is used everyday, isn't it? Even in places and ways I'm sure I was not previously aware of. Where is this Dr. Shannon; I want to learn more about calculus."

"She is over by the Diet Cokes. I'm glad I could help, and I'm sure you'll love the study of calculus."

CONCLUSION

In conclusion, it is both possible and rewarding to include a discussion of the Calculus in a liberal arts mathematics course. In the course of our Department's review of the Calculus sequence, while we were discussing the importance of the Calculus in the context of the course description it struck me that if we really do believe that Calculus is one of the greatest intellectual achievements, then we should not reserve an appreciation of its development to only a select few. If we avoid Calculus in our Mathematics Appreciation courses, then we do our students and our discipline a disservice.

Ode to Mathematics

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A revised version of a poem by Serge J. Zaroodny

The world has far too many boring books;
For books, par excellence, are very static;
But of this statics, by its very looks,
Most boring is the book that's mathematic.

Long laughed at human logic human tongue;
With logic tends it to be causalistic
So to untangle simple truths from wrong
One must resort to methods symbolistic.

Oh, why is't so that language must conceal
A simple thought in cloudy definition?
Why can't we lucid truth at once reveal
Without disguising it by erudition?

Cheer up! That quantum jump, sweet understanding's jerk--
It only comes to those who do the work.

Divisibility: A Problem Solving Approach Through Generalizing and Specializing

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ABSTRACT

This paper describes a divisibility rule for any prime number as an engaging problem solving activity for preservice secondary school mathematics teachers.

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My students, preservice secondary school mathematics teachers holding majors or minors in mathematics or science, were raised to believe that there were some "neat" divisibility rules for numbers like 2, 3, 5, 9, 10, 100, some considering the last digit or digits and some considering the sum of the digits. They have also heard of some "weird" and totally unuseful divisibility rules for 7, 11 and maybe even 13. Usually, the former are introduced, and at times even proved, in junior high school. The latter are mentioned briefly without a proof, or omitted altogether. In the AS (After Sputnik) era of growing dependence on calculating machines, who could possibly be interested in divisibility rules?

The curiosity of one student generated an interesting investigation that I wish to present here. This student discovered, in fact found on the internet, a divisibility rule for 7, and wondered why it worked. I blessed her curiosity and suggested that the class work on it. The results went far beyond our original intentions: a divisibility rule for any prime number has been derived and proved. More than the mathematical exercise, I wish to share the exciting mathematical investigation and experimentation in which the students engaged.

I will present the results as a problem solving activity that started with collecting data through observation and incorporated several rounds of implementing a "What if not?" strategy (Brown & Walter, 1990). I will present the results as students' engagement in generalizing and specializing (Mason, 1985) and will conclude with a brief discussion on the relevance of such an activity as well as several ideas for possible extensions.

PROLOGUE

Consider the divisibility rule for 3: A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Let's prove it for a 4 digit number.

Consider an expanded notation of a 4 digit number written with digits a,b,c and d from left to right.

$$1000a + 100b + 10c + d = (999a + a) + (99b + b) + (9c + c) + d$$

Applying associativity and commutativity of addition, this equals

$$(999a + 99b + 9c) + (a+b+c+d)$$

The first addend in this sum $(999a + 99b + 9c)$ is always divisible by 3. The second addend $(a+b+c+d)$ is the sum of the digits. Therefore, the number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Even though this proof refers to a four digit number, it gives a general idea how the proof can be extended to a number with n-digits. The strategy used in this proof is representing a number as a sum of two addends. The divisibility of one component is obvious. The divisibility of the second component determines the divisibility of the number. A similar strategy will be applied in the following proofs.

DIVISIBILITY BY 7 - INTRODUCING THE ALGORITHM

Divisibility of a number by 7 can be determined us-

ing the following recursive algorithm:

1) Multiply the last digit of the number by 2.
2) Subtract the product in (1) from the number obtained by deleting the last digit of the original number.
3) Continue steps 1 and 2 until the divisibility of number obtained in (2) by 7 is "obvious." The original number is divisible by 7 if and only if the number obtained in step (2) is divisible by 7.

Examples of implementation:

(a) Is 86415 divisible by 7?
86415 —> 8641 - (5x2) = 8631
8631 —> 863 - (1x2) = 861
861 —> 86 - (1x2)= 84
84 —> 8 - (4x2) = 0
Yes, 0 is divisible by 7, therefore 86415 is divisible by 7.

(b) Is 380247 divisible by 7?
380247 —> 38024 - (7x2) = 38010
38010 —> 3801 - (0x2) = 3801
3801 —> 380 - (lx2) = 378
378 —> 37 - (8x2) = 21
21 is divisible by 7, and, therefore, 380247 is divisible by 7. (We could continue one step further to get a zero).

(c) Is 380245 divisible by 7?
380245 —> 38024 - (5x2) = 38014
38014 —> 3801 - (4x2) = 3793
3793 —> 379 - (3x2) = 373
373 —> 37 - (3x2) = 31
31 is not divisible by 7, and, therefore, 380247 is not divisible by 7.

When examples similar to the above were presented in class, the immediate response for many students was a desire to try it out, to carry out the algorithm on numbers of their choice and verify divisibility with a calculator. This generated a large body of evidence to suggest that the algorithm "works." This also generated two related questions:

1) Why does this work?
2) Why does this work for 7?

The first question is drawn by the desire to understand the algorithm and to prove that it determines divisibility by 7 for any natural (or integer) number. The second question is drawn by the desire to determine the special place of the number 7 in the algorithm. Specializing on 7 in turn invites generalization: Does it work for 7 only? Will the algorithm work for another number? For which numbers will it work? How can the algorithm be modified to work for another number?

WHAT IF NOT 7?

While experimenting with other numbers, a lucky trial by one student prompted a conjecture, that exactly the same algorithm can be applied to determine divisibility by 3. This conjecture has been supported by several examples, however, no other number was found for which the above algorithm can be applied to determine divisibility. To encourage further investigation I suggested the following variation.

DIVISIBILITY BY 19 - VARYING THE ALGORITHM

Divisibility of a number by 19 can be determined by the following algorithm:

1) Multiply the last digit of the number by 2.
2) Add the product in (1) to the number obtained by deleting the last digit of the original number.
3. Continue steps 1 and 2 till the divisibility of number obtained in (2) by 19 is "obvious." The original number is divisible by 19 if and only if the number obtained in step (2) is divisible by 19.

Examples of implementation

(a) Is 15276 divisible by 19?
15276 —> 1527 + (6x2) = 1539
1539 —> 153 + (9x2) = 171
171 —> 17 + (1x2) = 19
19 is divisible by 19, and, therefore, 15276 is divisible by 19.

(b) Is 12312 divisible by 19?
12312 —> 1231 + (2x2) = 1235
1235 —> 123 + (5x2) = 133
133 —> 13 + (3x2) = 19
19 is divisible by 19, and, therefore, 12312 is divisible by 19.

For convenience of reference in further discussion, we shall name this algorithm a *trimming algorithm*.

WHY-QUESTIONS TO PONDER

Experimenting with the two variations of the trim-

ming algorithm presented above there are (at least) two questions that arise:

1) Why is the last digit multiplied by 2?
2) Why does the algorithm involve subtraction in case of 7 and addition in case of 19?

DIVISIBILITY BY 17 - ANOTHER VARIATION

A different variation on the trimming algorithm can be used to determine divisibility by 17. In this case we multiply the last digit by 5 and subtract the product from the "trimmed" number:

EXAMPLES:

(a) Is 82654 divisible by 17?
82654 —> 8265 - (4x5) = 8245
8245 —> 824 - (5x5) = 799
799 —> 79 - (9x5) = 34
we may stop here or continue one step further
34 —> 3 - (4x5) = -17
Conclusion: 82654 is divisible by 17.

(b) Is 17456 divisible by 17?
17456 — > 1745 - (6x5) = 1715
1715 — > 171 - (5x5) = 146
146 — > 14 - (6x5) = -16
Conclusion: 17456 is not divisible by 17.

REPHRASING THE WHY-QUESTIONS

The similarities among the three algorithms are obvious. However, the last variation suggests rewording of the first question:

1) How is the multiplier of the last digit of the number determined? (Why was it 2 in case of 19 and 7 and 5 in case of 17?)

The second question remains basically the same:

2) Why does the algorithm involve addition in some cases and subtraction is others?

DIVISIBILITY BY 7 - A SPECIFIC "GENERIC" PROOF

After experimenting with a variety of examples the students became convinced that the algorithms do indeed represent a divisibility rule. However, they were still seen as some magic tricks. The interest in WHY (they work) took over from the initial excitement of HOW they work.

Let us prove the divisibility algorithm for 7.

Consider any natural number n. If N is the number obtained from n by deleting the last digit a, we can always represent n as $10N+a$. (Example: 3456 = 10 x 345 + 6) We are interested in connecting our original number n and the number obtained by the algorithm, namely, N-2a. In fact, we would like to prove that n is divisible by 7 if and
only if N-2a is divisible by 7.

Applying simple arithmetic we get:

$$10N + a = 10(N - 2a) + 20a + a$$
$$= 10(N - 2a) + 21a$$

The last addend (21a) is divisible by 7 for any digit a. Therefore n is divisible by 7 if and only if N-2a is divisible by 7. Now we can treat the "new" number (N-2a) as the number for which divisibility by 7 has to be established using the same method.

WHAT IF NOT 7?

What if divisibility by a prime number p is in question? Separate proofs, similar to the above, can be developed for a variety of numbers. Inviting students to develop these proofs and discuss similarities among them may help in generalizing to attain an algorithm which determines divisibility of a number by any prime p.

DIVISIBILITY BY *P* - GENERALIZING THE ALGORITHM

In order to construct an algorithm to determnine divisibility by a prime number p we are looking for a natural number k such that $10k\pm1$ is divisible by p. Then,

$$10N + a = 10(N \mp ka) \pm 10ka + a$$
$$= 10(N \mp ka) \pm (10k \pm 1)\ \mathrm{a}$$

If $10k\pm1$ is divisible by p, then $(10k\pm1)a$ is divisible by p for any digit a. Therefore, $10N+a$ (which is our number n) is divisible by p if and only if $N \mp ka$ (the number obtained by applying the algorithm) is divisible by p.

SPECIALIZING: DIVISIBILITY BY 17

For example, to determine divisibility by 17 we looked for a number of the form $10k\pm1$ divisible by 17. We found 51. Therefore, k=5. This is the number used in the trimming algorithm to establish divisibility for 17. Since 51 has the form $10k+1$, the number obtained in the algorithm should be of the form N-ka, therefore the algorithm involves a subtraction of a product of

the last digit by 5.

SPECIALIZING: DIVISIBILITY BY 31

What is divisibility rule for 31? 31 itself differs by 1 from the closest multiple of 10. Therefore, k=3 and the algorithm involves subtraction.

Example:

Is 4185 divisible by 31?
4185 —> 418 - (50) = 403
403 —> 40 - (30) = 31
Conclusion: Indeed, 4185 is divisible by 31.

SPECIALIZING: DIVISIBILITY BY 13

What is the divisibility rule for 13? We find 39 as a multiple of 13 that differs by 1 from a multiple closest to 10. Therefore k=4 and the algorithm involves addition.

Example:

Is 4173 divisible by 13?
4173 —> 417 + (3x4) = 429
429 —> 42 + (9x4) = 78
78 —> 7 + (8x4) = 39
Conclusion: 4173 is divisible by 13.

EXISTENCE PROOF

We believe that so far the why questions (1) and (2) raised earlier have been answered. Now it is time to wonder whether it is possible to find an appropriate trimming algorithm to determine divisibility by any prime.

The mathematical answer is no. However, the "human" answer is--almost. Such an algorithm can be determined for all the primes except 2 and 5. (However, since 2 and 5 have well known divisibility rules, we will focus on the other primes.)

The existence of a trimming algorithm for p depends on the existence of a multiple of p which is larger by 1 or smaller by 1 than a multiple of 10. It is obvious that such a multiple does not exist for 5 and 2 which are factors of 10. Let us prove its existence for all other primes.

Formally, let's prove that for any prime p, $p \neq 2, 5$, there exist natural numbers k and m such that $|mp-10k|=1$.

Let us consider the last digit x of p. The possibilities are 1,3,7 or 9, since this digit cannot be even or 5. If x=1 or x=9 the prime itself differs by 1 from the closest multiple of 10. In this case m=1 and k is determined accordingly. If x=3, let m=3, then the last digit of mp is 9 and the number mp is smaller than the closest multiple of 10 by 1. If x=7 , let m=3, then the last digit of mp is 1 and the number mp is bigger than the closest multiple of 10 by 1. Therefore if x=3 or x=7, then m=3 and k is determined accordingly.

In summary, for any prime p, $p \neq 2, 5$, it is possible to determine a divisibility rule based on a trimming algorithm.

NUMBER THEORY CONNECTION

In a number theory text (e.g. Long, 1987, p. 98) the following can be found as an exercise:

(a) If p is a prime and $(p, 10)=1$, prove that there exist integers k and y such that $yp=10k+1$.
(b) Let $n=10a+b$. If p is a prime with $(p, 10)=1$, prove that p/n if and only if $p/(a-kb)$, where k is determined in (a).

However, without a concrete experience the relationship between this exercise and divisibility rules may not be apparent to many students.

FOR FURTHER INVESTIGATION

In Polya's tradition, the fourth step in problem solving is "looking back" (Polya, 1988). This involves searching for alternative solutions or solution paths, generalizing solutions and exploring situations to which the problem or the method of solution can be applied. We have presented one level of looking back at the problem of divisibility by 7 by exploring the divisibility algorithm for any prime. Further, "looking back" at the general divisibility rule, we can extend our investigation by asking several "what if not" questions.

- What if not primes? Can a similar algorithm be used or modified to determine divisibility by a composite number? What properties of primes were used in our proof? For what composite numbers can the algorithm be applied or modified?
- In all the above examples we have chosen the smallest multiple of p that was bigger by 1 or smaller by 1 than a multiple of 10. However, in our proof there was no reference to the choice of the smallest k. So, what if not the smallest? Will

the algorithm still work? In other words, is the algorithm, for which existence is proved above, unique?
- Which familiar divisibility rules can be seen as special cases of the general algorithm?

A COMMENT ON USEFULNESS

There is a common trend in mathematics education to focus on "applicable" mathematics, on mathematics that is related to "real life situations." From this perspective, there is a danger of labeling divisibility rules as "unuseful."

I believe that usefulness, together with mathematical beauty, is in the eye of the beholder. For me, a problem that attracts students' interest and curiosity, that generates an engaging investigation, that invites students to make conjectures and test conjecture--is most useful. I believe that an engaging mathematical investigation is useful for all learners, and the excitement of mathematical investigation is especially useful for individuals planning for a teaching career. I hope that my students feel the same.

BIBLIOGRAPHY

Brown, S. and Walter, M. (1990). *The Art of Problem Posing*, Second Edition. Hillsdale, NJ: Lawrence Erlbaum.
Long, C. T. (1987). *Elementary Introduction to Number Theory*, Third Edition. London: Prentice Hall.
Mason, J. (1985). *Thinking Mathematically*. Wokingharn, England: Addison Wesley.
Polya, G. (1988). *How to Solve It*, Second Edition. Princeton, NJ: Princeton University Press.

Excerpts from Ivan's Commandments for Himself

Ivan Niven

Ivan Niven of the University of Oregon, author of Mathematics of Choice and several texts on number theory, died on May 9, 1999, at age 83. At a memorial service in his honor, the program included the item below, which was found among his personal papers.

Thou shalt make an unceasing effort to see the world as it truly is, not as a product of your desires, not as a work of your imagination, not as a matrix of your special interests, but as an external reality that is no respecter of persons.

Thou shalt not deliberately misstate or misrepresent another's position by exaggeration, by quotation out of context or by confusing a statement and its converse. Neither shall thou attempt to destroy another's position by harping on some error or minor defect that in no way affects his principal contention.

Thou shalt not claim to know more than thou knowest. Thou shalt judge the merits of a proposal in terms of its own worth, irrespective of the proponents thereof.

Thou shalt not exalt trivial matters, nor claim as primary what is at best secondary. For who but a foolish person will resign from his church, his political party or his club because of one or two speeches or occurrences not to his liking.

Thou shalt have the grace to concede a point without going into a huff, without claiming that wasn't what you said, or meant to say, and without saying, "Didn't you know I was only kidding?"

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The Role of Values in Mathematics Education

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A cursory look at the proceedings of the last four ICME's international conferences reveals an increasing interest in the role of cultural factors in mathematics education. This paper attempts to enrich these discussions in two directions. First, to apply the contrastive analysis approach in bringing out the role of values in mathematics education; and, second, to go in some details into the role of values in the various aspects of mathematics education. The paper thus consists of three main parts. The first part presents episodes from two distinct cultures to be used as examples for illustrative and contrastive purposes. The second part contrasts the episodes from different perspectives. The third part discusses and analyses in some details the potential role of values in mathematics education as well as computer education.

EPISODES

Episode One

"And unto you belongeth a half of that which
your wives leave, if they have no child;
but if they have a child then unto you
the fourth of that which they leave, after
any legacy they may have bequeathed, or debt
(they may have contracted), hath been paid.
And unto them belongeth the fourth of that
which ye leave if ye have no child, but if
ye have a child then the eighth of that which
ye leave, after any legacy ye may have
bequeathed, or debt (ye may have contracted),
hath been paid. And if a man or a woman
have a distant heir (having left neither
parent nor child), and he (or she) have a
brother or a sister (only on the mother's
side) then to each of them twain (the
brother and the sister) the sixth, and if
they be more than two, then they shall be
sharers in the third, after any legacy that
may have been bequeathed or debt (contracted)
not injuring (the heirs by willing away more
than a third of the heritage) hath been paid.
A commandment from Allah. Allah is Knower,
Indulgent." (Surrah IV, 12). (Pickthall, 1970).

Were it not for the archaic English language, one would have thought that the above quotation is from a tax code of a modern Western country. Surprisingly, the quotation is from a religious book more than 1350 years old: the *Glorious Qur'an*. For Muslims the *Qur'an* is not simply a revelation, but the very words of God, embodied in an immutable text in Arabic. The above quotation from the *Qur'an* is intended here to illustrate three points relevant to the issues at hand. First, the level of details and the degree of precision illustrate the extent to which the *Qur'an* establishes a complete system for civil laws and social and political institutions. In fact, until now, the inheritance laws in effect in Islamic courts adhere fully to the precise and explicit commandments in the *Qur'an*. Second, the quotation illustrates the sophisticated use of numbers in communicating precise quantitative concepts in such a systematic way that they can be easily transformed into a flow-chart with almost all possible options covered. Third, the quotation illustrates the view of the *Qur'an* and Islam of the function of mathematics as an important knowledge in so far as it contributes to utilitarian purposes. However, the mode of thinking in mathematics, like all other forms of knowledge, has to remain subservient to the Islamic mode of thinking whose ultimate purpose is to know God through His Book.

Episode Two

Early in my career in the late sixties and the early seventies, I had the experience of extensively observing mathematics instruction in the classrooms in the Sudan, Saudi Arabia, and Palestinian camps in Lebanon, Syria, Jordan, West Bank and Gaza. Invariably, the teacher, the students, and the textbooks were engaged in what seemed to me a "well-rehearsed act" in which the actors and the set changed but the roles and characters remained essentially intact. The teacher presents the lesson to the class by talking and writing on the board while the students are silent and perhaps listening. Occasionally, the teacher asks a ques-

tion to which the students respond by raising their hands to indicate their ability to answer. The teacher picks one student, listens to the answer, determines its correctness and moves on with his presentation. Towards the end of the class, the students are asked to solve problems from their textbooks (most often on the board). The textbook is the determinant of what is to be taught, and the teacher is the interpreter of the textbook and the judge of the correctness of what is learned.

Episode Three

In his dialogue "Meno" (Jowett,1937) Plato presents his ideas about knowledge, teaching, and learning using an example from mathematics. The persons in the dialogue are Meno, Socrates, and a slave of Meno (referred to as Boy). The illiterate "Boy" learns certain mathematical conclusions through the answers elicited by Socrates' questions. The dialogue proceeds as follows:

(1) The "Boy" learns that the area (size) of a square of side two feet is four (square) feet
(2) To the question about a square of double area (8 square feet), The "Boy" conjectures that it should have double the side i.e. four feet
(3) Socrates make him recollect or "discover" that such a square actually has a side of less than 3 and greater than 2.
(4) The dialogue then proceeds as follows:

"***Soc***. Do you see, Meno, what advances he has made in his power of recollection? He did not know at first, and he does not know now, what is the side of a figure of eight feet: but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows.
Men. True.
Soc. Is he not better off in knowing his ignorance?
Men. I think that he is.
Soc. If we have made him doubt and given him the 'torpedo's shock' have we done him any harm?
Men. I think not.
Soc. We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double space should have a double side.
Men. True
Soc. But do you suppose that he would ever have enquired into or learned what he fancied that he knew, though he was really ignorant of it, until he had fallen into perplexity under the idea that he did not know, and had desired to know?
Men. I think not, Socrates.
Soc. Then he was the better for the torpedo's touch?
Men. I think so.
Soc. Mark now the farther development. I shall only ask him, and not teach him, and do you watch and see if you find me telling or explaining anything to him, instead of eliciting his opinion. Tell me, boy, is not this a square of four feet which I have drawn?
Boy. Yes.
Soc. And now I add another square equal to the former one?
Boy. Yes.
Soc. And a third, which is equal to either of them?
Boy. Yes.
Soc. Suppose that we fill up the vacant corner?
Boy. Very good.
Soc. Here, then, there are four equal spaces?
Boy. Yes.
Soc. And how many times larger is this space than this other?
Boy. Four times.
Soc. But it ought to have been twice only, as you will remember.
Boy. True.
Soc. And does not this line, reaching from corner to corner, bisect each of these spaces?
Boy. Yes.
Soc. And are there not here four equal lines which contain this space?
Boy. There are.
Soc. Look and see how much this space is.
Boy. I do not understand.
Soc. Has not each interior line cut off half of the four spaces?
Boy. Yes.
Soc. And how many spaces are there in this section?
Boy. Four.
Soc. And how many in this?
Boy. Two.
Soc. And four is how many times two?
Boy. Twice.
Soc. And this space is of how many feet?
Boy. Of eight feet.
Soc. And from what line do you get this figure?

Boy. From this.
Soc. That is, from the line which extends from corner to corner of the figure of four feet?
Boy. Yes.
Soc. And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal?
Boy. Certainly, Socrates." (pp. 363-365)

Episode Four

Between 1970 and 1973 I was in the United States working for my Ph.D. at the University of Wisconsin, Madison. During this period I had a chance to visit elementary school classes in Madison. What struck me most in the mathematics classes, which were using the then experimental instructional materials of the Developing Mathematical Processes Project (DMP), the "chaos" in contrast to the "law and order" that I experienced in Lebanon and Palestinian schools. Children roamed around, talked to each other, had fun, and occasionally engaged in some mathematical activities. The teacher seemed to have assumed the role of an organizer whose main responsibility was to structure the environment for the children to learn and occasionally engage them in some mathematical activities. From my perspective at that time, whatever mathematics learning was taking place in the American Schools was different from whatever learning I had experienced.

CONTRASTS

Values as Psychological Constructs

By contrasting the two classroom episodes (episodes two and four) one observes (at least from the perspective of the author) that the type and sequence of actions and interactions performed by the players (students, teachers, instructional materials) is quite different. In episode two the teacher is the major determinant of instruction, the students are the recipient audience, and the textbook is a concrete definition of the tasks to be explained by the teacher or performed by the students. The tacit assumption, on the part of both the teacher and the class, is that mathematics is knowledge that is possessed by the teacher and is to be transmitted to the students who thus are expected to possess it i.e. learn it.

In episode four, the teacher in contrast determines the setting of learning but not instruction. The students select what to do with the learning environment organized by the teacher and hopefully learn the mathematics injected in the environment and as intended by the teacher. Mathematics seems to be some interesting tasks we do because they are there around us and because they are fun to do.

If this account of the two episodes is reasonably valid, the differences in the scenarios may be partly accounted for by the beliefs and values of the players in each episode. These values and belief are contributors to, if not determinants of, the actions and interactions in classroom instruction. The two episodes reflect different values related to the nature of learning and teaching, nature of mathematics, objectives of learning/teaching mathematics, role of instructional materials and learning environment, and above all, "who" determines the legitimacy of truth and validity of mathematical knowledge.

In these two episodes, values may be looked at as psychological constructs that students and teachers have formed as a result of cumulative individual and collective contextualized experiences. Thus values may be considered regardless of their historical development. The claim is that, even if values are detached from their cultural history, they do impact mathematics instruction in specific and definite ways.

Values as Cultural Products

By contrasting the two historical episodes (episodes one and three), quite different patterns of discourse emerge. One may attribute the differences in the two discourses to the differences in their contexts, the discourse in episode one being from a religious book (the *Qur'an*) after Christ and in episode two from a philosophy book (the *Dialogues of Plato*) before Christ. Nevertheless, each of the two discourses is in its own context a value-capturing sample of the greatest books in the Islamic and Greek cultures. After all, Islam for the Arabs was what philosophy had been for the Greeks.

A close analysis of these two samples of discourses reveals differences in the value-systems in which they are embedded. Specifically, the values encompass differences regarding the nature of learning, knowledge, and mathematics. In episode one, knowledge is fixed and final whereas, in episode three, it is a continual dialectical process of thinking. In episode one, the

truth has the finality and authority of the Divine whereas, in episode three, it has the tentativeness and fragility of human reasoning. In episode one, learning is the act of receiving knowledge as expressed by the words of God, whereas, learning is a continual testing of hypotheses in episode three. Mathematics comes as concepts and techniques which are useful in life and in executing the commandments of God in episode one. In episode three, however, mathematics comes out as a medium and vehicle for questioning and reasoning.

In these two episodes, values may be regarded as shared meanings which had captured in certain periods in history the collective experience of a culture. Thus values may be considered as cultural products of the past regardless of their subsequent impact on the value-systems of the present. The question arises as to the extent and form of this impact in mathematics education.

Relation of Values as Cultural Products to Values as Psychological Constructs

In addition to contrasting values as psychological constructs or cultural products, one may also focus on the relationship between values as cultural products and values as psychological constructs. My claim is that the impact of values as cultural products on values as psychological constructs is strong enough to be observable. The few episodes provided earlier do not warrant any inference but may be used to illustrate this relationship.

There is a close affinity, I suggest, between the values reflected in episode one and those reflected in episode two. The values of the finality of truth as determined by the Divine in the form of an immutable text and of the function of mathematics as an instrumental knowledge to utilitarian purposes in episode one are echoed in episode two in the form of valuing the textbook as the repository of mathematical knowledge, the teacher as a determinant of knowledge, and mathematics as a body of knowledge to be transmitted by those who possess it to those who do not. In the same manner, in episode three, the values of tentativeness of truth, of the dialectical nature of learning, and of mathematics as a medium for coping with reality resonate in episode four, perhaps in an exaggerated and confused ways, in valuing "chaos" in the American classroom, the role of the teacher as an organizer, and mathematics as a vehicle to cope with reality. These should not imply, however, that the values operating in mathematics classrooms can be accounted for solely in terms of values as cultural products. However, their impact cannot be neglected with the understanding that values are hybrids which have resulted from complex interactions of the value-systems of different cultures. The metaphor is that of a wave (cultural values) which, as it travels in space and time, interact with other waves (other cultures) and produce new waves having some from each source but more of the primary source.

EXAMPLES OF VALUES IN MATHEMATICS EDUCATION

Many attempts have been recently made to identify values that impact mathematics education. Bishop (1988), using the four components of culture as defined by White (1995), has identified three pairs of complementary values relating to Western mathematics corresponding White's sentimental, ideological, and sociological components. The first pair relates to the two values of control (power of mathematics to offer feelings of security and control) and progress (development of knowledge through mathematics). The second complementary pair of values which belong to the ideological component is rationalism (logic as a criterion of mathematics knowledge) and objectism (power of mathematics in using symbols to deal with abstracts entities as if they were objects). Openness-mystery is the third complementary pair of values which belong to the sociological component.

Others have attempted broader social-cultural values that may impact mathematics education. Swadener and Saedjadi (1988) illustrate how mathematics education may promote the values implied in the five fundamental principles of the *Panca Sila* which is the foundation of the national values in Indonisia. Jurdak (1989) identified some of the values of Arab-Islamic culture which may act as cultural carriers or barriers in mathematic education.

CONJECTURES

In recent years there has been a tendency towards more microscopic description of mathematics education. The macro descriptions of each of the goals, content, methods of instruction, and evaluation of mathematics education are giving way to more analytic descriptions in such a way that not only larger "blocks" are being broken down into smaller and finer

“units” but also new relationships among these units are being identified and refined.

The micro description of mathematics education presents a more explicit and powerful method for analyzing and investigating the role of values in mathematics education. The impact forces which are not apparent at the macro level become so at the micro level. Thus, the role of values, for example, seems marginal if mathematical content is defined as terms, concepts, and skills but would be greatly enhanced if content is defined as points in multidimensional taxonomy consisting of subject matter, teacher intention, time allotment, and order of presentation, all of which are value-loaded.

An attempt will be made therefore to identify and analyze the role of values on mathematics education using the micro descriptions which exist in the literature. A similar attempt will be made regarding computer education.

Goals and Values

The goals of mathematics education reflect values regardless of their macro or micro description. After all, goals are primarily value-judgements as to what is important in learning/teaching mathematics. I advance a speculation that values not only determine, to some extent, the goals of mathematics education but also play a major role in prioritizing these goals. This speculation is almost a truism, yet we tend to ignore it because, perhaps, of the predominant belief almost everywhere, that mathematics education is value-free. The goals in the public perception are determined by the experts (mathematicians and teachers) who are knowledgeable about their field and know why and what to teach in mathematics.

If the goals of mathematics education are not affected by values, how could one account for differences, say, between the goals of mathematics education in the NCTM Standards (NCTM,1989) and Yemen except to say that these goals are reflections of the needs of two different societies whose valuing of certain “purposes” of mathematics education (purpose is used here in the sense of Niss (1981)? How could one explain why reading in the U.S. takes priority over mathematics, and the priority is reversed in Japan and Taiwan (Stigler, Lee, and Stevenson,1987) whereas religion takes priority over mathematics in Saudi Arabia? One plausible explanation is that the needs and beliefs of these different cultures result in different degrees of valuing mathematics in relation to other areas.

Content and Values

The multidimensional definition of mathematical content in the context of instruction has helped to clarify the role of values in content decisions. A three-dimensional taxonomy (general intent, nature of material, operation) which had been suggested to describe the content of elementary school mathematics was used to investigate the existence of national elementary school mathematics curriculum in the U.S. (Freeman *et al*, 1988). The large variability in the content (as defined) has challenged the commonly held belief about the existence of a national curriculum. A refinement and a generalization of this definition to all school mathematics will most likely produce a tool powerful enough to pick up even smaller differences. This in turn discredits the commonly held conviction about the universality of school mathematics content across different cultures.

Much of the variability of content in school mathematics across cultures can be explained by the value-systems of the latter. Content decisions that involve not only the selection of mathematical skills and concepts are bound to be value-mediated decisions. For example, a content decision to teach fractions for skill building vs. problem solving (general intent) is embedded in a value-system about what mathematics and its teaching are.

Cross-cultural research, meager as it is, provides support to the speculation about the role of values in content decisions in mathematics. Grade-placement of addition and subtraction topics in the U.S. elementary textbooks was found to be different than in Japanese, Chinese, or Soviet textbooks (Fuson, Stigler, & Bartsch, 1988). This is essentially a content decision which reflects differences in beliefs and values about what is possible to learn at certain ages. Likewise, the content decision to embed school mathematics in out-of-school setting or de-contextualize it from real life applications reflects a difference in the purpose of teaching mathematics and this in turn is highly value-mediated.

Teaching Methods and Values

Teaching methodology is bound to be affected significantly by values. Because teaching methodology is in essence a complex interaction among teachers, students, and materials, it requires decision making which is value-mediated.

Research which has focussed on the lesson structure has provided us with a more detailed picture of what goes in classroom instruction. The reconstruction of lessons in terms of segments, routines, scripts, and agendas (Lienhardt & Greeno, 1986; Putman, 1987; Yinger, 1980) has enriched our understanding of instruction in the context of expert/novice contrasts. The basic assumption in these studies is that teaching is a complex cognitive skill that rests on lesson structure knowledge and subject matter knowledge. I believe the picture is not complete without considering the complex value-system associated with the lesson structure (cultural and social values) and knowledge structure (values associated with subject matter i.e. mathematics in this case). Any curriculum or class script cannot be fully understood without reactivating the value-system which mediated the many and complex decisions that take place in a class script. The teaching episodes I described earlier are examples of how cultural values (for example, the finality or tentativeness of knowledge) and subject matter values (nature of mathematics, utilitarian or way of thinking) help us understand the dynamics of decision-making in classroom instruction.

Cross-cultural research supports the hypothesized critical role of values in explaining variations in methods of teaching mathematics. The Michigan Studies (Stigler, Lee, and Stevenson, 1987) report, for example, that classroom organization (whole class, group, or individual) and teacher leadership in the U.S. differ significantly from those in Japan or Taiwan. These differences may be accounted for partly in terms of the values attached to leardership and team solidarity in the three different cultures.

Evaluation and Values

As the term indicates, values play a major role in all aspects of evaluation: who, what, and how. Cultural-social values influence the roles of teachers, parents, school, and state in evaluation. What is evaluated is contingent on what is intended (goals) and this in turn is significantly dependent on mathematical as well as cultural-social values. Social values also affect how evaluation is conducted. The teaching episodes described earlier illustrate how social-cultural values (locus of authority) mediate the "who" in the evaluation process. One should note also that evaluation is closely related to goals, methodology, and content decisions. As such evaluation in mathematics education is subject to be influenced by values in the same manner.

Computer Education and Values

Bishop (1988) had the following to say about the role of values in computer education:

> "There is even more of a pressing need today to consider values because of the increasing presence of the computer and the calculator in our societies. These devices can perform many mathematics techniques for us, even now, and the arguments in favor of a purely mathematical <u>training</u> for our future citizens are surely weakened. Society will only be able to harness the mathematics power of these devices for appropriate use if its citizens have been made to consider values as part of their education." (p. 181).

The role of values in computer education is readily apparent in the goals-component, and consequently it manifests itself in the other components: content, methodology, and evaluation. Ralston (1992) identifies two distinct ways in which the computer can be used in classroom instruction: an "electronic blackboard" by the teacher or an interactive tool by the students. These two functions are closely related to social-cultural values such as control-autonomy, passivity-activity, and imitation-exploration (Burkhardt & Fraser, 1992; Noss, 1988).

A value-system which regards knowledge as a construction of human beings interacting in a social context are likely to embrace the objectives of autonomy of learners to take initiatives and explore possibilities. Such a value-system is likely to adopt the function of the computer as an interactive tool. Consequently, the remaining components will be affected accordingly. Content decision (selecting, using and organizing software) will be done in such a way to capitalize on the initiative and activity of the individual students to explore alternatives on their own. Teaching methodology is likely to reduce the control of the teacher over

instruction and will change his role from a model to be imitated to a facilitator of learning. Evaluation is likely to be more intrinsic and personal, and less extrinsic and judgemental.

On the other hand, a value-system which regards knowledge as final as determined by "authorities" are likely to embrace the objectives of transmitting knowledge by proper control of the environment in order to maximize efficiency. The function of the computer as an "electronic blackboard" is likely to be favored by such a value-system. Consequently, content decisions, teaching methodology, and evaluation will exhibit less autonomy and exploratory activities on the part of the students as they use the computer.

CONCLUDING REMARKS

The examples and conjectures I have provided are not intended to promote culturo-centrism. On the contrary, my intention is to call attention to the critical role of values in mathematics education in bringing about cross-cultural dialogue. It is only by better understanding of the role of values in mathematics education that we can capitalize on differences in values as keys to open windows for interaction among different culture. Contrasts and conjectures will hopefully provide the impetus for further research in this area.

REFERENCES

Bishop, A.J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics*, **19**, 179-191.

Burkhardt, H. and Fraser, R. (1992). An overview. In Cornu, B. and Ralston, A. (Eds.), *The influence of computers and informatics on mathematics and its teaching*. Paris: Unesco.

Freeman, D.J., Kuhs, T.M., Porter, A.C., Floden, R.E., Schmidth, W.H.,& Schwille, J.R. (1983). Do textbooks and tests define a national curriculum in elementary school mathematics? *Elementary School Journal,* **83**, 501-513.

Fusion, K., Stigler, J., & Bartsch, K. (1988). Grade placement of addition and subtraction topics in Japan, Mainland China, the Soviet Union, Taiwan, and the United States. *Journal for Research In Mathematics Education*, **19**, 449-456.

Jowett, M.A. (Translator into English). (1973). *The dialogues of Plato*. New York: Random House.

Jurdak, M.J. (1989). Religion and language as cultural carriers and barriers in mathematics education. In C. Keitel (Ed.), *Mathematics, education, and society.* Science and Technology Education, Document Series No. 35 (pp.12-14). Paris: UNESCO.

Lienhardt, G., & Greeno, J.G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology,* **78**, 75-95.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards.* Reston, VA: NCTM.

Niss, M. (1981). Goals as a reflection of the needs of society. In Morris, R.(ED.), *Studies in Mathematics education*, **Vol. 2** (pp. 1 - 21). Paris: UNESCO.

Noss, R. (1988). The computer as a cultural influence in mathematics learning. *Educational Studies In Mathematics*, **19**, 251-268.

Pickthal, M.(Translator into English). (1970). *The meaning of the Glorious Coran*. Beirut: Dar Al-Kitab Allubnani.

Putman, R. (1987). Structuring and adjusting content for students: A study of live and simulated tutoring of addition. *American Educational Research Journal*, **24**, 1348.

Ralston, A. (1992). The impact of computers and computer science on the mathematics curriculum. In Cornu, B. & Ralston, A. (Eds), *The influence of computers and informatics on mathematics and its teaching.* Science and Technology Education, Document Series No. 44 (pp 19-32). Paris: UNESCO

Stigler, J., Lee, S., and Stevenson, H. (1987). Mathematics classroom in Japan, Taiwan, and the United States. *Child Development*, **58**, 1272-1285.

Swadener, M. & Soedjadi, R., (1988). Values, mathematics, and the task of developing pupils' personalities: An Indonisian Perspective. *Educational Studies in Mathematics*, **19**, 193-208.

White, L.A. (1995). *The evolution of culture.* New York: McGraw-Hill

Yinger, R.J. (1980). A study of teacher planning. *Elementary School Journal*, **80**, 107-127.

Book Review: *A Tour of the Calculus* by David Berlinski

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A Tour of the Calculus David Berlinski. Vintage Books, 1997.

At first reading this book comes across as a continuing commercial for the marvels of mathematics. The author is so eager to dazzle you with spectacular displays of wit and imagination that the stated goal—to develop mathematical ideas from the real numbers up through the fundamental theorem of the calculus—is often obscured rather than illuminated by the fireworks. Referring to the mathematician's inclination toward abstraction, he notes that, "This turning away from various particulars requires great discipline. Revisiting the facts, the mathematician must *resist* the tug of those rich, very voluptuous descriptions of reality that the novelist or the physician might favor..." (p. 63). Unfortunately the author didn't resist very hard, and he is aware of it; a few pages later he writes, "So much for purple prose, the stuff beginning to wear, even on me..." (p. 69).

On going through the book a second time, however, I could see that it may well be useful to the teacher of mathematics, supplying a flavor and richly colorful human dimension to mathematical developments. The author's deft use of language and frequent flights of metaphor give added insight into mathematical ideas, and extended digressions often contain biographies and rich, imaginative reconstruction of the lives, personalities and thinking of major mathematicians.

For example, leading up to the definition of the slope of the line tangent to a curve at a point, he remarks that, "The mathematician intent on donating the line's slope to a curve, and the skeptic bemused by the fact that lacking a slope the tangent line is mathematically undetermined, may be satisfied alike by a procedure that assigns a slope to the tangent line, the neutral idea of an assignment conveying, I think, the odd commingling of discovery and definition that is involved in any mathematical advance" (p. 181). Again, "In nature, some things are close (the lion and the tiger, cats both) and some things far apart (the tiger and the flatworm, different animals, different phyla even), the concept of *distance* one of the crucial, if generally hidden and obscure, instruments by which we assess the world and find our way within it" (p.18).

His soaring rhetoric and flair for the dramatic are illustrated in the following comment on Newton's towering work. "The *Principia* is the supreme expression in human thought of the mind's ability to hold the universe fixed as an object of contemplation: it is difficult to reconcile its monumental power with a number of humanly engaging but anecdotal accounts of its composition: the disheveled and halfdressed Newton, so the stories run, his crumb-filled wig askew, shambling about the evil-smelling room in which he lived and worked, muttering to himself, his thin lips half forming words, stiff with attention or slack and slumped indifferently on his unmade bed, entirely absorbed, forgetting to eat and sleeping in weak, disorganized fits, an apple rotting on the desk, the *Principia* taking shape in stages, vellum sheets piling up on the wooden desk" (p. 5).

Berlinski says that, "I have written this book for men and women who wish to understand the calculus as an achievement in human thought." I do not recommend it for someone who is not already well acquainted with the calculus. Such a person will not grasp its core ideas here. There are too few examples, the lengthy digressions mentioned above may be offputting for the novice, and some of the mathematical discussions are confusing, even misleading, The author seems at times to sacrifice accuracy for imagery. I doubt that the manuscript was reviewed prior to publication by another mathematician.

An example of notational confusion is offered by a proof of the Mean Value Theorem (p. 212). A function $y = f(t)$, $a \le t \le b$, is under consideration. An accompanying figure indicates clearly that b denotes the right-hand endpoint of a t-interval. It is then recalled (floridly) that, "straight lines submit to the discipline of an equation expressing their innermost nature: $y = mt$

+ b, where m is the line's slope and b its y-intercept..." Suffice it to say that the ensuing discussion employs b in both its meanings.

I was even more disturbed by an extended discussion (pp. 179-183) in which "slope" and "curvature" appeared to be used interchangeably. Was I being overly critical? Then comes this sentence: "Curvature is assessed at a point by reference to the slope of a line tangent to the curve at that point, the curve acquiring its slope at second hand, it is true, but acquiring nonetheless a slope and so a number embodying and then expressing its curvature."

The mathematician will find here amusing anecdotes, charmingly expressed, and rich metaphors, to enliven mathematical ideas. But if he or she wishes to assign readings from this book to students of calculus, my advice is to select such readings carefully.

Book Review: *Mathematical Reflections* by Peter Hilton, Derek Holton and Jean Pedersen

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Mathematical Reflections. Hilton, Peter; Holton, Derek; and Jean Pedersen. Springer, 1997.

I believe the following book is particularly suited to study by those who are or will be teachers of math and science: *Mathematical Reflections* by Peter Hilton, Derek Holton, and Jean Pedersen. After 8 chapters of delightful math, the short 9th chapter contains some very useful and thought provoking reflections on "how math should be done" and "principles of mathematical pedagogy." The following comes from a section on the special role of geometry in secondary and undergrad math.

"In fact, geometry and algebra, the two most important aspects of math at these levels, play essentially complementary roles. Geometry is a source of questions, algebra is a source of answers. Geometry provides ideas, inspiration, insight; algebra provides clarification and systematic solution."

"Thus it is particularly absurd to teach geometry and algebra in separate watertight compartments... Geometry without algebra leaves the student with questions without answers, and hence creates frustration; algebra without geometry provides the student with answers to questions nobody would ask, and hence creates boredom and disillusion. Together, however, they form the basis for a very rich curriculum, involving discrete and continuous math."

In arguing against the separation and restriction to synthetic geometry methods, they point out that completing geometric problems by Euclidean means almost always exploits some clever trick, an ingenious construction—so, roughly speaking, each problem "requires its own special idea. And none of us is bright enough to function, in any aspect of our lives, with such an enormous idea-to-problem ratio; we have to make a good idea go a long way. ... So remember: All Good Ideas in Maths Show Up in a Variety of Mathematical and Real-World Contexts."

Finally, I note that Peter Hilton (mathematician, SUNY Binghampton) has collaborated with Stephen Willoughby (former NCTM Pres., Prof. Univ. AZ), Carl Bereiter (cognitive psychologist), and Joseph Rubinstein (biologist) over a period of many years to develop the series of elementary math texts called Real Maths for K-8 originally published by Open Court, and its latest version now called Math Explorations and Applications published by SRA/McGraw-Hill currently available for K-6. I'd like to hear commentary on this series from those who know it well. Also, can someone point me to reviews of this series, with comparison to others?

Tea Tasting and Pascal Triangles

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INTRODUCTION TO THE PHYSICAL SITUATION

R. A. Fisher is generally considered to be the most famous statistician of all time. He is responsible for such familiar statistical ideas as analysis of variance, Fisher's exact test, maximum likelihood, design of experiments and hypothesis testing, among others. He was very prolific, but as James Newman kindly put it, "Fisher is not an easy writer." [1] Even today, almost four decades after his death, people on opposing sides of a statistical argument can quote Fisher to prove their respective points. Fisher was indeed very interested in what might be termed the philosophy of statistics; that is, to paraphrase the famous aphorism in management science, how to do things right statistically and how to do the right statistical things.

Fisher, when asked what he did for a living, would reply he was a scientist because he felt that statistics more properly belonged to the sciences and not to mathematics. Indeed, he was never a professor of statistics but instead was a professor of eugenics (at London University)–the overwhelmingly negative connotations of eugenics came years later with the rise of nazism–and then a professor of genetics (at Cambridge University). Despite his reputation as someone difficult to interpret unambiguously, one of his philosophical contributions to the annals and history of the discipline is both well known, and, according to Newman, "a model of lucidity and required no mathematics other than elementary arithmetic. It demands of the reader the ability to follow a closely reasoned argument, but it will repay the effort by giving a vivid understanding of the richness, complexity and subtlety of modern experimental method."

Fisher, in his *Design of Experiments*, entitled this contribution "The Principles of Experimentation, Illustrated by a Psycho-Physical Experiment." [2] Newman, in his four-volume *magnum opus, The World of Mathematics*, reprinted it in its entirety under the name by which it is perhaps more commonly known: "Mathematics of a Lady Tasting Tea." Fisher uses the unlikely example of a woman who claims to be able "to discriminate whether the milk or the tea infusion was first added to the cup" in order to elucidate which design principles are essential and which others are but auxiliary. He discusses why she should be informed of how many cups of each there are and why three cups of milk first and three cups of tea first would not be convincing even should she be able to get all six correct: with three and three, the chance of identifying all six merely by guessing would be 1 in 20 which is *not less than .05*–to Fisher, "It is usual and convenient for experimenters to take 5% as a standard level of significance." Whereas, with four of each, the chance of identifying all eight merely by guessing would be 1 in 70 which *is* less than .05. Fisher could have set this up within his context of significance testing where the null hypothesis would be

H_0: The number of correct answers is due to chance alone.

He further goes on to show that five cups of one and three of another would be inferior in the sense that guessing all correct in this instance would be 1 in only 56 and thus not as impressive. Fisher also points out that letting "the treatment of each cup be completely determined by chance, as by the toss of a coin, so that each treatment has an equal chance of being chosen" while increasing the "sensitiveness" of the experiment to 1 in 256, should be foregone because it might bewilder the subject and "deny her the real advantage of judging by comparison." In this last instance, the null hypothesis would be

H_0: Probability of success for any one cup is .5

Fisher also looked at other refinements such as avoiding or limiting possible confounding elements so as to ensure that the tea, milk and temperature were the same for each cup; however, perhaps his greatest

emphasis in this imaginary psycho-physical experiment is on the need for randomization, a concept, taken for granted today by all statisticians, which back in the 1930s when this was written, necessitated proselytizing.

The usual way this problem in all its various forms is analyzed is via combinatorics, a subject which is often difficult for beginners. Combinatorics frequently result in formulas which are sufficiently obtuse so that insight is hard to come by. Fortunately, this problem can be attacked via simulation. Furthermore, Fisher's tea tasting experiment in all its various forms leads to Pascal triangles, a subject more conducive to insight and one which has a long history in mathematics and seems to pop up in unlikely places. [3] Because the resulting Pascal triangle is most familiar for the case in which the treatment of each cup is determined by *chance*, that will be the first one simulated. The next simulation will be for *equal* cups of milk first and tea first; it will be seen that a "quadratic" Pascal triangle results. The last simulation will be for the *unequal* (but known cups of each) case; this too will result in a type of Pascal triangle.

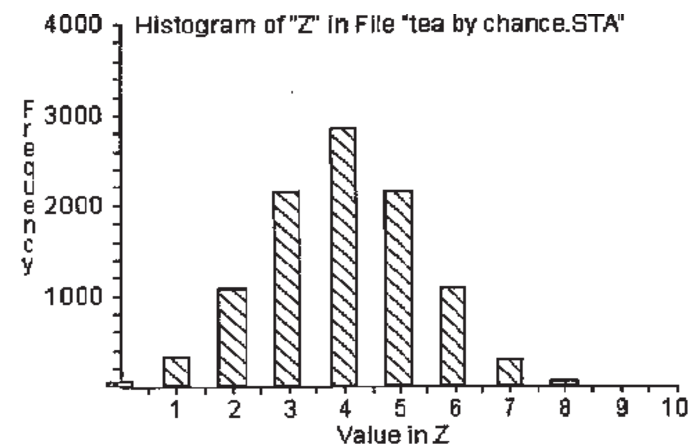
SIMULATION AND PASCAL TRIANGLES

With the ubiquity of computers, one is tempted to avoid messy combinatoric formulas by simulating the experiment, and thus not only obtaining a numerical answer, but, in addition, acquiring some clue about the underlying structure of this problem. The simulations will be carried out in *Resampling Stats*, a software package designed to do simulation in statistics. [4] The simulation could be carried out in any computer language, but what is taking place is a computerization of what one could do physically; computers, of course, merely speed things up.

NUMBER OF CUPS OF TEA OR MILK DETERMINED BY CHANCE

When the treatment of each cup is determined by chance, the following simple program could be used when the lady is presented with eight cups and is not informed of how many are milk or tea first. The "sample" command means sample with replacement so that while "deck" contains eight elements, four 1's followed by four 2's, "deck$" has eight elements with from zero to eight 1's depending on chance. The "score" command is used to keep track of her number of successes in the 10,000 trials; at the end of the "repeat" loop, the variable "z" has 10,000 elements.

```
A lady testing tea "by chance" example
concat 4 #1 4 #2 deck
maxsize z 15000
repeat 10000
  sample 8 deck deck$
  count deck$ = 1 a
  score a z
end
histogram z
count z = 8 k
divide k 10000 kk
print kk
```



KK = 0.0044

The symmetry in the histogram is clearly evident, and any general result ought to exhibit this property. Furthermore, the program is easily modified to deal with any number of cups merely by changing the "sample" command; of course, by changing the "count" command's equal sign to greater than or equal for the variable "z," one could find the probability of more than a certain number of successes instead of exactly that many. In contrast, the general combinatoric formula for the probability of getting l or more correct out of the n chances when the probability of success is .5 is

$$\sum_{k=1}^{n}\binom{n}{k}(0.5)^{n}$$

This formula, all too often, fails to reach the beginning student. As it turns out, this (binomial) formula is related to Fibonacci numbers and can be displayed as a Pascal triangle which is often shown in an isosceles shape:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Another, but less familiar way to show the Pascal triangle is to display it as a right triangle.

n/*k*	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Here, n stands for the number of cups which the lady tastes and k is the number of correct choices. For example, the probability of getting 6 or more correct out of 8 chances is given by using the proper row of either of the Pascal triangles

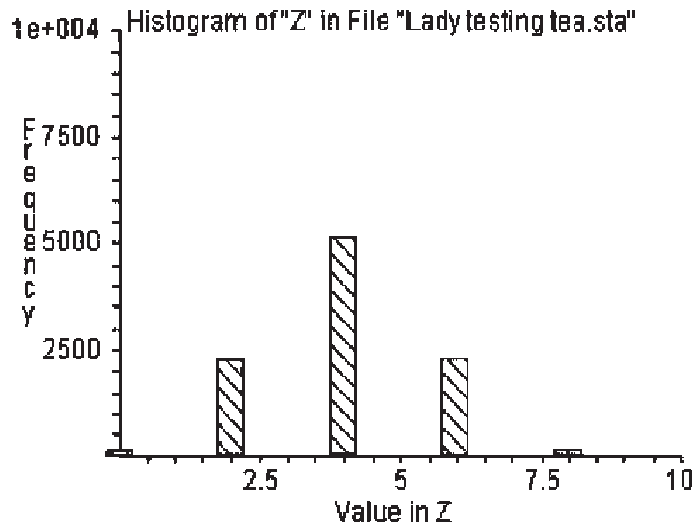
$$\frac{(28+8+1)}{(1+8+28+56+70+56+28+8+1)} =$$

$$\binom{8}{6}(0.5)^8+\binom{8}{7}(0.5)^8+\binom{8}{8}(0.5)^8=\frac{37}{256}$$

FISHER'S ORIGINAL SITUATION

In Fisher's *original* proposal the lady is informed that there is an equal number of milk or tea first–four of each in his article. The simulation given below is very similar to the *by chance* program already given. The main difference is that the "sample" command is replaced by "shuffle" and "take" which means that sampling is done without replacement.

```
A lady testing tea example
concat 4 #1 4 #2 deck
maxsize z 15000
repeat 10000
  shuffle deck deck$
  take deck$ 1,4 milk
  take deck$ 5,8 tea
  count milk = 1 a
  count tea = 2 b
  add a b c
  score c z
end
histogram z
count z = 8 k
divide k 10000 kk
print kk
```



Once again the histogram indicates symmetry. Further, only an even number of correct choices is indicated as possible. The combinatoric formula for obtaining a correct cups of milk (and as it turns out, therefore a correct cups of tea) when there are m cups of milk first and n=m cups of tea first is:

$$\frac{\binom{m}{a}\binom{n}{a}}{\binom{2n}{n}}$$

So to speak, this combinatoric formula lacks transparency. That is, while it is correct, insight is hard to come by. However, coupled with the simulation results there is a suggestion of some deeper structure. That deeper structure turns out to be what might be called a "quadratic" Pascal triangle:

$$
\begin{array}{c}
1^2 \\
1^2 \quad 1^2 \\
1^2 \quad 2^2 \quad 1^2 \\
1^2 \quad 3^2 \quad 3^2 \quad 1^2 \\
1^2 \quad 4^2 \quad 6^2 \quad 4^2 \quad 1^2 \\
1^2 \quad 5^2 \quad 10^2 \quad 10^2 \quad 5^2 \quad 1^2
\end{array}
$$

That is, each element of the usual Pascal isosceles triangle is squared. Or, the ordinary Pascal isosceles triangle is "multiplied" by itself. The same applies to "multiplying" right angle Pascal triangles to obtain:

n/k	0	1	2	3	4	5
0	1^2					
1	1^2	1^2				
2	1^2	2^2	1^2			
3	1^2	3^2	3^2	1^2		
4	1^2	4^2	6^2	4^2	1^2	
5	1^2	5^2	10^2	10^2	5^2	1^2

where there are n cups of milk first and $n=m$ cups of tea first; $2k$ is the number of incorrect choices. Hence, the solution for any equal number of milk or tea first may be found by using the corresponding row of a (quadratic) Pascal triangle. For example, when there are four cups each, the probability of getting 6 or more correct, equivalent to obtaining 2 or fewer failures is analytically given by

$$\frac{\left(1^2+4^2\right)}{\left(1^2+4^2+6^2+4^2+1^2\right)}=\frac{17}{70}$$

When there are five cups each, we just move down one row. The Pascal triangle gives a neat pictorial presentation of how much more difficult it is for the lady to guess correctly; the combinatoric formula does not reveal this so readily.

UNEQUAL NUMBER OF KNOWN CUPS OF EACH

When m is less than or equal to n but still known to the lady, the program just given is easily modified; the only substantive difference is the "deck" and the "take" commands utilize the particular m and n. The combinatoric formula, on the other hand, is quite frightening to beginners:

$$\frac{\binom{m}{a}\binom{n}{n-m+a}}{\binom{m+n}{m}}$$

The combinatoric formula and the simulation suggest some structure in this situation as well. However, for this situation of unequal m and n, things aren't quite so tidy. Nonetheless, some sort of "multiplication" is taking place. For example, when there is one more tea first than milk first, $n=m+1$, the following triangle results:

$$
\begin{array}{c}
1 \\
1 \quad 2 \\
1 \quad 6 \quad 3 \\
1 \quad 12 \quad 18 \quad 4 \\
1 \quad 20 \quad 60 \quad 40 \quad 5
\end{array}
$$

Due to the lack of symmetry, this is better viewed via the right angle Pascal triangles.

m/k	0	1	2	3	4	
0	1					
1	1	2				
2	1	6	3			
3	1	12	18	4		
4	1	20	60	40	5	(1)

where m is the number of cups of milk first and n is the number of cups of tea first. $2k$ is the number incorrect and $2m$-$2k$+1 is the number correct.

The element in the ith row in the resulting triangle in (1) is obtained from multiplying the element in the ith row and the element in the same column in the $(i+1)$th row in the ordinary right angle Pascal triangle. For example, when $m=2$ and $n=3$, the third row in (1) is constructed by multiplying the element in the third row and the element in the same column in the fourth row in the Pascal triangle.

$$
\begin{array}{cccc}
1 & 2 & 1 & \\
\text{x} & \text{x} & \text{x} & \\
1 & 3 & 3 & 1 \\
\Downarrow & \Downarrow & \Downarrow & \\
1 & 6 & 3 &
\end{array}
$$

For example, when the difference between n and m is two, then the following triangle results:

m/k	0	1	2	3	4
0	1				
1	1	3			
2	1	8	6		
3	1	15	30	10	
4	1	24	90	80	15

(2)

where m is the number of cups of milk first and n is the number of cups of tea first. $2k$ is the number incorrect and $2m-2k+2$ is the number correct.

The element in the ith row in the resulting triangle in (2) is obtained from multiplying the element in the ith row and the element in the same column in the $(i+2)$th row in the Pascal triangle. In general, the triangle which results for any m and n is to multiply the ith row by the $(i+n-m)$th row.

The lack of symmetry when m is unequal to n implies that Pascal triangle approach loses some of its visual advantage over the combinatoric formula. Nevertheless, the Pascal triangle still displays some pictorial benefit.

PHILOSOPHICAL CONCLUSIONS

Being a scientist, Fisher's main purpose in this tea tasting scenario was to illustrate the ideas behind the design of experiments when psychology was combined with the physical. On the other hand, mathematicians often have a different agenda such as showing surprising and non-intuitive interconnections. That the tasting of tea as described by Fisher should lead to a quadratic Pascal triangle is esthetically pleasing to a mind with a mathematical bent. Just as important is the fact that when Fisher's scenario is altered to allow unequal (but known total cups of each), the Pascal triangle can be easily used to determine numerical results as the number of cups change; this is in contrast to the combinatoric formula which tends to hide what is taking place and is often difficult to calculate numerically.

BIBLIOGRAPHY

1. Fisher, R.A., "The Mathematics of a Lady Tasting Tea," *The World of Mathematics*, edited by Newman, J.R., pp 1512-1521, 1456-58, 1956.

2. Fisher, R.A., "The Principles of Experimentation, Illustrated by a Psycho-Physical Experiment," *Design of Experiments*, Edinburgh, Oliver and Boyd, pp 11-25, 1949.

3. Bondarenko, B.A., *Generalized Pascal Triangles and Pyramids; Their Fractals, Graphs, and Applications*, Fibonacci Association, 1993.

4. Simon, J.L., *Resampling: The New Statistics*, Resampling Stats, Inc, 1997.

Roots

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I think that I shall never see
The square root of the number three,
A number so irrational
It cannot be conceived at all.
Squares were made for fools like me,
But only God can root a three.

Comments and Letters

I much appreciate your sending me the the *Humanistic Mathematics Network Journal*, but I have a gripe concerning Issue #19 (March 1999). Whether one prefers male or female math, hard or soft math, one cannot be a mathematician unless one respects definitions. And I submit that, by definition, the verses on pp. 42-43 are NOT limericks.

Yes, each stanza has five lines and most have rhyme scheme “aabba”—the “b” lines each one poetic foot shorter-but these stanzas do not consistently have lines for which a single foot contains one stressed syllable and two unstressed ones. And, the author is not free to move the stresses at will from their natural positions in spoken English. The author published his work at a point where he should have polished it further.

You may recall me from your having published my collection of mathematical references in literature (HMNJ #13). Periodically, “humanistic” mathematicians wonder why the world produces so littie mathematical poetry. The reason, I submit, is that most mathematicians who attempt poetry do not take it seriously. They produce doggerel and it satisfies them.

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In Future Issues...

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Yan Kow Cheong
Mathematics and Sex

Emam Hoosain
The Need for Interviews in the Mathematics Classroom

Nitsa Movshovitz-Hadar and Orit Hazzan
How to Present It?—On the Rhetoric of an Outstanding Lecturer

Barry Schiller
Using Environmental News to Help Teach Mathematics

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