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Introduction: This paper treats the structure of those lattice-ordered rings which are subdirect sums of totally ordered rings—the *f-rings* of Birkhoff and Pierce [4]. Broadly, it splits into two parts, concerned respectively with identical equations and with ideal structure; but there is an important overlap at the beginning.

D. G. Johnson has shown [9] that not every f-ring is *unitable*, i.e. embeddable in an f-ring which has a multiplicative unit; and he has given a characterization of unitable f-rings. We find that they form an equationally definable class. Consequently in each f-ring there is a definite l-ideal which is the obstruction to embedding in an f-ring with unit. From Johnson's results it follows that such an ideal must be nil; we find it is nilpotent of index 2, and generated by left and right annihilators.

Tarski has shown [13] that all real-closed fields are arithmetically equivalent. It follows easily that every ordered field satisfies all ring-lattice identities valid in the reals (or even in the rationals); and from a theorem of Birkhoff [2], every ordered field is therefore a homomorphic image of a latticeordered ring of real-valued functions. Adding results of Pierce [12] and Johnson [9], one gets the same conclusion for commutative f-rings which have no nonzero nilpotents. We extend the result to all zero f-rings, and all archimedean f-rings. We call these homomorphic images of f-rings of real functions *formally real* f-rings.

Birkhoff and Pierce showed [4] that f-rings themselves form an equationally definable class of abstract algebras, defined by rather simple identities involving no more than three variables. The same is true for unitable f-rings. However, no list of identities involving eight or fewer variables characterizes the formally real f-rings. The conjecture is that "eight" can be replaced by any n, but we cannot prove this.

We call an element e of an f-ring a *superunit* if $ex \geqq x$ and $xe \geqq x$ for all positive x; we call an f-ring *infinitesimal* if $x^2 \leqq |x|$ identically. A totally ordered ring is unitable if and only if it has a superunit or is infinitesimal. A general unitable f-ring is a subdirect sum of two summands, L, I, where L is a subdirect sum of totally ordered rings having superunits (we say *L has local superunits*) and I is infinitesimal. The summand L is unique.

We call a maximal l-ideal M in an f-ring A *supermodular* if A/M has a superunit. The supermodular maximal l-ideals of A, in the hull-kernel topology, form a locally compact Hausdorff space $\mathscr{M}(A)$. If A

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is infinitesimal, of course, the space is empty. If A has local superunits, each supermodular maximal l-ideal contains a smallest supermodular primary l-ideal G, and the intersection of all these ideals G is zero.

In case the intersection of the supermodular maximal l-ideals of A is zero, we call A *supermodular semisimple* or *S-semisimple*. This implies there are no nonzero nilpotents. Hence a commutative S-semisimple f-ring is a residue class ring of an f-ring of realvalued functions modulo an intersection of supermodular maximal l-ideals. The ideal structure of any S-semisimple f-ring A is fairly closely bound to the structure space $\mathscr{M}(A)$; the correspondence between direct summands and open-closed sets is imperfect in the noncompact case, but even then the ideals which are kernels of closures of open sets in $\mathscr{M}(A)$ are precisely the annihilator ideals. This correspondence between enough subsets of $\mathscr{M}(A)$ to form a base for the topology and a family of ideals determined by the ring structure alone leads to a result on reordering: If A and B are two S-semisimple f-rings which are isomorphic rings, then $\mathscr{M}(A)$ and $\mathscr{M}(B)$ have dense homeomorphic subspaces, and if $\mathscr{M}(A)$ or $\mathscr{M}(B)$ has no nonempty open totally disconnected subset, then any ring isomorphism must preserve order.

The contents are as follows. §1, The characterizations of unitable f-rings; §2, Idempotents; §3, l-semisimple and archimedean rings; §4, General formally real rings; §5, Supermodular ideals; §6, Structure space and reordering.

The defining identities for f-rings [4] are

$$x^+y^+ \wedge x^- = 0\ ; \quad y^+x^+ \wedge x^- = 0\ ,$$

where $x^+ = x \vee 0$, $x^- = (-x)^+$. For unitable f-rings,

$$[x \wedge y \wedge (x^2 - x) \wedge (y - xy)]^+ = 0\ ;$$
$$[x \wedge y \wedge (x^2 - x) \wedge (y - yx)]^+ = 0\ .$$

The left side of the first of these two identities is a right annihilator in any f-ring, and symmetrically.

The results in §2 are, first, that the idempotents of a unitable f-ring are central; and further results on idempotents of any f-ring which emphasize how nearly unitable they all are. The central idempotents of any ring are known [7] to form a Boolean ring under the operations $(x-y)^2$, xy; the idempotents of any f-ring are closed under these operations, but need not commute. Indeed, if they commute with each other, they are central. A totally orderedring admits a ring direct sum decomposition $A \oplus B$ where all the idempotents are in A, either all nonzero idempotents are left units for A or all are right units, B is a zero ring, and the direct sum is lexicographically ordered; but no such decomposition exists in general.

In arbitrary f-rings one has

$$xy^+ = [xy \wedge (x^2y + y)] \vee [0 \wedge (-x^2y - y)]$$

and similar identities which permit the reduction of any ring-lattice polynomial to an equal lattice polynomial in ring polynomials. It follows that any identity is equivalent to a conjunction of identities of the form $[P_1 \wedge \cdots \wedge P_m]^+ = 0$, where the P_i are ring polynomials. (Naturally, this does not apply to the identities which *define* f-rings.) These lemmas have helped us to get a nine-variable identity which is valid in the reals and not a consequence of valid identities in fewer variables; but we cannot see how to go further. The lemmas are interesting also because they lead to a proof that ordered fields are formally real f-rings which seems to us much easier than Tarski's proof. Artin has proved in two or three pages in [1] a theorem which yields by a short sequence of easy steps the conclusion that any identity $[P_1 \wedge \cdots \wedge P_m]^+ = 0$ which is valid in the rationals is valid in every ordered field. This is just what is needed to complete the proof.

The theorem that a lattice-ordered zero ring is formally real solves a problem of Birkhoff [3, Problem 107]. The decomposition of a unitable f-ring A into a subdirect sum of summands L, with local superunits, and I, infinitesimal, specializes in case A is archimedean; L has no nilpotents and I is a zero ring. Since each archimedean f-ring A is known to be commutative [4], this decomposition shows that A is formally real.

§4 includes three characterizations of free formally real f-rings. Two are in terms of piecewise polynomial functions and of totally ordered (free) rings of polynomials. Further, a free commutative f-ring modulo its l-radical (the l-ideal of all nilpotents [4]) is free formally real.

The results of §§5, 6, have already been described, except for the theorem that given a unitable f-ring there is a unique smallest f-ring with unit containing it.

1. The characterizations of unitable f-rings.

A *lattice-ordered ring* is an (associative) ring that is also a lattice in which $x \geqq 0$ and $y \geqq 0$ imply $xy \geqq 0$. We will regard lattice-ordered rings as abstract algebras (in the sense of Birkhoff [3]) with five operations $+, -, \cdot, \vee$, and $\wedge$.

Recall from [4] that a class of abstract algebras is said to be *equationally definable* if it can be characterized by means of a (possibly infinite) list of identities. Birkhoff and Pierce show in [4] that lattice-ordered rings form an equationally definable class of abstract algebras (defined by means of finitely many identities).

It is clear that every equationally definable class $\mathscr{A}$ of abstract algebras contains all subalgebras, homomorphic images, and subdirect products of elements of $\mathscr{A}$. We will also need below the nontrivial converse of this last statement, proved by Birkhoff in [2].

An *f-ring* is a lattice-ordered ring A such that for all $x, y, z \geqq 0$ in A, $y \wedge z = 0$ implies $xy \wedge z = yx \wedge z = 0$. This class of lattice-ordered rings, which contains all totally ordered rings, was introduced by Birkhoff and Pierce in [4], where they showed that f-rings are equationally definable by means of finitely many identities. We will consider only f-rings below, and we will almost always follow the notation and terminology of [4]. In particular, if A is a lattice-ordered ring, $A^+ = \{a \in A : a \geqq 0\}$, $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. The kernel of a homomorphism (preserving all five operations) is called an *l-ideal*. It may be characterized as a ring ideal I such that $x \in I$ and $|y| \leqq |x|$ imply $y \in I$. By a unit of A, we mean a multiplicative identity element. The notation $>$, and the term *strictly positive*, will be reserved for totally ordered rings.

Another important characterization of f-rings is: a lattice-ordered ring is an f-ring if and only if it is a subdirect sum of totally ordered rings [4]. Thus, any identity valid in all totally ordered rings is valid in any f-ring, and an identity is valid in a particular f-ring if and only it if is valid in all totally ordered homomorphic images.

We call an f-ring *unitable* if it can be embedded in an f-ring with unit. Unitable f-rings have been investigated by D. G. Johnson [9], whose results include a characterization and the theorem that every f-ring having no nonzero nilpotents is unitable. We shall not use Johnson's characterization; we note that the quantifiers in it appear in the pattern, universal, existential, universal. The main point of this section is a characterization of unitable f-rings by means of certain identities, examination of which will sharpen Johnson's result.

The identities are

$$\begin{aligned} &[x \wedge y \wedge (x^2 - x) \wedge (y - xy)]^+ = 0 \;; \\ &[x \wedge y \wedge (x^2 - x) \wedge (y - xy)]^+ = 0 \;. \end{aligned} \tag{1.0}$$

We prove first

1.1. *Every unitable f-ring satisfies the identities* (1.0).

Proof. It suffices to prove (1.0) in an f-ring with unit, and even in a totally ordered ring with unit, since the image of a unit under a homomorphism is a unit for the image. But then it suffices to prove that if x, y, and $x^2 - x$ are strictly positive, neither $y - xy$ nor $y - yx$ is strictly positive. Since $x^2 > x > 0$, x must be greater than 1, $xy \geqq y$, and $yx \geqq y$. This completes the proof.

For the converse of 1.1, it again suffices to consider totally ordered rings. In any f-ring A, we call an element e a *superunit* if $ex \geqq x$

and $xe \geqq x$ for all positive x. On the other hand, we call A *infinitesimal* if $x^2 \leqq |x|$ identically. The infinitesimal f-rings form an equationally definable class, since $x^2 \leqq |x|$ if and only if $(x^2 - |x|)^+ = 0$.

1.2. *A totally ordered ring satisfying* (1.0) *either is infinitesimal or has a superunit.*

Proof. If the ring is not infinitesimal, one has $x^2 = |x|^2 > |x|$ for some x, and then (1.0) says that $|x|$ is a superunit.

The following will be useful below.

1.3. *Let x and y be elements of any totally ordered ring A.*

(i) *If $xy > y > 0$ or if $yx > y > 0$, then $(2x)^2 > 2x$.*

(ii) *If $x^2 > x > 0$, and $xy = 0$ (resp. $yx = 0$), then $zy = 0$ (resp. $yz = 0$) for all $z \in A$.*

Proof of (i). If $x^2 \geqq x \neq 0$, then $2x^2 > x^2 \geqq x$, so $4x^2 > 2x$. Hence, we may assume that $x^2 < x$. Suppose that $xy > y > 0$. Then $0 \leqq (x-x^2)y = x(y - xy) \leqq 0$. If $x^2 \leqq x - x^2$, then $x^2y \leqq (x - x^2)y = 0$, whence $x^2y = 0$. But then $y < xy = (x - x^2)y + x^2y = 0$. It follows that $x^2 > x - x^2$, or $2x^2 > x$. Thus $(2x)^2 = 4x^2 > 2x$. We proceed similarly if $yx > y > 0$.

Proof of (ii). We give the proof in case $xy = 0$. We may assume without loss of generality that y and z are in A^+.

Write $z = (zx \wedge z) + w$, where $w = z - zx \wedge z \geqq 0$. Since $0 \leqq zx \wedge z \leqq zx$, and $zxy = 0$, we have $(zx \wedge z)y = 0$.

$(zx \wedge z)x = zx^2 \wedge zx = zx$ since $x^2 > x > 0$. So $wx = 0$. We must have $w < x$; for, if $w \geqq x$, then $0 = wx \geqq x^2$. Hence $0 \leqq wy \leqq xy = 0$. Thus, $zy = (zx \wedge z)y + wy = 0 + 0 = 0$.

1.4. *Every infinitesimal f-ring is unitable.*

Proof. We may consider only the totally ordered infinitesimal rings A. Let B denote the set of all ordered pairs (n, a), where n is an integer and $a \in A$. We define addition of elements of B coordinatewise and multiplication by $(n, a)(m, b) = (nm, nb + ma + ab)$, where the products nb, ma, are defined by repeated addition in A. We order B lexicographically, i.e., so that (n, a) is positive if $n > 0$ or if $n = 0$ and $a \geqq 0$ in A. To see that a product of positive elements $(n, a), (m, b)$, is positive, observe first that it is trivial when n and m are both zero or both nonzero. In case $m = 0$, the product is $(0, nb + ab) \geqq (0, b + ab)$.

Suppose that $b + ab < 0$. Then $b \neq 0$, and since $(0, b)$ is positive, $b > 0$. Also, $a < 0$; so we may write $a = -c$, where $c > 0$. Thus, we have $cb > b > 0$. By 1.3, this implies that $(2c)^2 > 2c$, contrary to the assumption that A is infinitesimal. Thus, B is a totally ordered ring with unit (1,0) that contains A.

REMARK. This proof, like almost everything else in this paper, applies with trivial modifications to algebras over the reals or any other ordered field.

The conclusion of the next lemma is Johnson's necessary and sufficient condition for a totally ordered ring to be unitable [9] (in slightly different language).

1.5. *In a totally ordered ring A having a superunit, for each positive integer n and each positive ring element a, either $ab \geqq nb$ and $ba \geqq nb$ for all positive b or $ab \leqq nb$ and $ba \leqq nb$ for all positive b.*

Proof. Suppose that the conclusion is false for some n and a. Then, by 1.3

(i), $(2a)^2 > 2a > 0$; we shall show also that a (and hence $2a$) is a proper zero divisor, whence by 1.3

(ii), no $z \in A^+$ can be a superunit. This contradiction will establish 1.5.

Suppose for example $a^2 - na \geqq 0 > ab - nb$. Multiplication by the positive elements b and a yields $a^2b - nab \geqq 0 \geqq a^2b - anb = a^2b - nab$; and a is a proper zero divisor. Similarly if $ab - nb > 0 \geqq a^2 - na$, $a(ab - nb) = 0$. With the parallel computations involving ba, this shows that unless all $ab - nb$ and $ba - nb$ have the same sign as $a^2 - na$, a is a proper zero divisor.

We shall just sketch the embedding of a totally ordered ring A with a superunit e in a totally ordered ring B with unit. Johnson has done essentially this in detail [9; III. 2.5, III. 3.1]. One forms as usual a ring B_0 of all ordered pairs (n, a), n an integer, a in A; but one must then divide out the ideal of all (n, a) such that both $ab = -nb$ and $ba = -nb$ for all b in A. The resulting ring B has, like B_0, a unit; and it still contains A isomorphically. By 1.5, one can order B by calling (n, a) positive if $nb + ab$ is positive for all positive b in A.

This completes the proof of

1.6. THEOREM. *Unitable f-rings form an equationally definable class of abstract algebras, characterized among the f-rings by the identities (1.0). Also, the unitable f-rings are precisely the subdirect sums of infinitesimal totally ordered rings and totally ordered rings having superunits.*

1.7. THEOREM. *In any f-ring,*

$$z[x \wedge y \wedge (x^2 - x) \wedge (y - xy)]^+ = 0$$

and

$$[x \wedge y \wedge (x^2 - x) \wedge (y - yx)]^+ z = 0$$

identically. Hence an f-ring whose left and right annihilator ideals vanish is unitable. More generally, the homomorphic image of an f-ring under any homomorphism whose kernel contains the left and right annihilator ideals is unitable.

Proof. To prove the identities in suffices to consider totally ordered rings. Here if $[x \wedge y \wedge (x^2 - x) \wedge (y - xy)]^+$ is not zero, we have $x^2 > x > 0$, and $y > xy$. So $x^2y \geqq xy$, and $xy \geqq x^2y$, whence $x(y - xy) = 0$. Thus, by 1.3 (ii), $(y - xy)$ is a right annihilator of A. But $0 \leqq [x \wedge y \wedge (x^2 - x) \wedge (y - xy)]^+ \leqq (y - xy)$, so the bracketed expression is also a right annihilator of A. The rest of the theorem is evident.

2. Idempotents.

2.1. *The only idempotents of a totally ordered ring with unit are* 0 *and* 1. *Hence all idempotents of a unitable f-ring are central.*

Proof. If x is an idempotent, then so is $1 - x$, and $x(1 - x) = 0$. Since $x(1 - x)$ lies between x^2 and $(1 - x)^2$, one of these squares is zero; but x and $1 - x$ are idempotents, so $x = 0$ or $x = 1$. The rest of the theorem is immediate.

A. L. Foster [7] and D. R. Morrison [11] have pointed out that the central idempotents of any ring are closed under the operations $(x - y)^2$ and xy, and form a Boolean ring under these operations. We go on to examine the idempotents of an arbitrary f-ring, to throw some light on the obstructions which may prevent an f-ring from being unitable.

D. G. Johnson's example [9] of a nonunitable f-ring is essentially as follows. The ring consists of all ordered pairs of integers, added coordinatewise, multiplied by $(a, b)(c, d) = (ac, ad)$, and ordered lexicographically. All $(1, b)$ are idempotent zero divisors. For nonzero idempotents e and f, $ef = f$; $(e - f)^2 = 0$.

A simple modification of Johnson's example yields a nonunitable commutative f-ring generated by two elements. This example will also introduce a general construction. If A is any totally ordered ring, and B is a totally ordered zero ring, let $A \oplus B$ denote the totally ordered ring which is the ring direct sum of A and B, ordered lexicographically. A product of positive elements is positive, since its coordinate in B cannot cause any trouble. If J is the ordered ring of integers, and Z is the ordered group of integers made into a zero ring, then $J \oplus Z$ is commutative and nonunitable, since its unique idempotent $(1, 0)$ is a zero divisor.

We pause to note that there are plenty of nonunitable ordered rings without idempotents. If S is the semigroup on two generators, x, y, with all x^n distinct but $xy = yx = y^2 = 0$, the ordering $\cdots x^{n+1} > x^n > \cdots > x > y$ induces a natural ordering on the semigroup ring which makes it nonunitable.

Our results are as follows.

2.2. THEOREM. *Let A be a totally ordered ring, let x and y be nonzero idempotents of A, and let n be any integer. Then*

(i) *$xy = x$ or $xy = y$. Indeed, if $xy = x$ (resp. $xy = y$), then $xz = x$ (resp. $xz = z$) for any idempotent z of A.*

(ii) *$(x - y)^2 = 0$.*

(iii) *$(n + 1)x - ny$, $x + y - xy$, and $x + n(xy - yx)$ are idempotents.*

If A contains only finitely many idempotents, then it contains at most two, and they are in the center; otherwise the idempotents do not commute with each other.

2.3. COROLLARY. *If x and y are idempotents of an f-ring A, and n is any integer, then xy, $(x - y)^2$, $x + y - xy$, and $x + n(xy - yx)$ are idempotents. If all the idempotents of A commute with each other, then they are in the center of A.*

2.4. THEOREM. *Every totally ordered ring can be decomposed into a direct sum $A \oplus B$, lexicographically ordered, so that all the idempotents are in A, either all nonzero idempotents are left units for A or all are right units, and B is a zero ring. If there is a nonzero idempotent, then the decomposition is unique.*

Proof of 2.2. Since $y^2 = y$, $(2y)^2 > 2y$, and $(x - xy)2y = 0$. So by 1.3 (ii), $(x - xy)x = 0$, i.e., $xyx = x$. Then $xyxy = xy$, so that xy is an idempotent e between $x^2 = x$ and $y^2 = y$; and so is $yx = f$. But $ef = xyx = x$, and $fe = y$. Since ef and fe must be between e and f, e and f must be x and y in some order.

Suppose that $xy = x$, and let z be any idempotent in A. If z lies between x and y, then xz lies between $x^2 = x$ and $xy = x$, so $xz = x$. Similarly, if $xy = y$, and z lies between x and y, then $xz = z$. If $xy = y$ and x lies between y and z, we may use this latter argument to conclude by contradiction that $xz = z$. The remaining cases are similar. This completes the proof of (i).

Since $(x - y)^2 = x - xy - yx + y$, (i) immediately implies (ii). Further routine ring computations yield (iii).

Clearly, if $x \neq y$, then $\{(n + 1)x - ny : n = 1, 2, \cdots\}$ is an infinite family of distinct idempotents; also, (i) shows that no two distinct nonzero idempotents commute. Hence, it remains only to prove that if there is just one nonzero idempotent, then it is central.

Computation shows that if e is a nonzero idempotent, then so is

$e + xe - exe$ for any x. That it is idempotent we leave to the reader; it is not zero since $e(e + xe - exe) = e$. Similarly, $e + ex - exe$ is a nonzero idempotent. If e is the only nonzero idempotent, this implies $xe = exe = ex$. (This is a pure ring computation and doubtless well known). This proves 2.2.

The corollary follows at once.

Proof of 2.4. If 0 is the only idempotent, taking $B = 0$ satisfies the conditions asserted. Otherwise let e be a nonzero idempotent. By 2.2, nonzero idempotents are multiplied either by the rule $xy = x$ or by the rule $xy = y$. Suppose it is $xy = x$. Then define A as the set of all a such that $ae = a$, B as the set of all a such that $ae = 0$. Obviously A and B are left ideals. Now recall that for all $x, e + ex - exe$ is a nonzero idempotent. Therefore $(e + ex - exe)e = e + ex - exe$, which means $exe = ex$. Then for a in A, the product $axe = aexe = aex = ax$; that is, A is a two-sided ideal. B is not only a right ideal but a left annihilator, by (1.3) (ii) and $(2e)^2 > 2e$. Evidently $A \cap B = 0$; also, any x in the ring is the sum of $xe \in A$ and $x - xe \in B$. Then the given ring is the direct sum of A and B. Moreover, B is a zero ring, since it left annihilates itself. Finally, every positive element a of A exceeds every element b of B, since $ae = a > be$. In A, e is a right unit, and so is every other nonzero idempotent since these idempotents multiply by the rule $xy = x$. Finally, for uniqueness, any subset of the ring on which e is a right unit must be contained in A, and any subset annihilating e must be contained in B.

One might conjecture, seeing 2.4, that every commutative totally ordered ring is the direct sum of a unitable ordered ring and a zero ordered ring. However, it is easy to describe a nonunitable example which has no proper direct sum decomposition. If F is the free commutative ring on three generators, and I the ideal generated by xz, yz, z^2, and $x^2 - y^3 - z$, then one can show that F/I is not a direct sum. On the other hand, every element of F/I can be written uniquely as $P(x, y) + nz$, where the polynomial P is a sum of monomials $mx^a y^b$, $b \leqq 2$, $a + b \geqq 1$. Define the *degree* of such a monomial as $3a + 2b$ if $m \neq 0$; define the degree of nz as $0 (n \neq 0)$, and the degree of 0 as -1. Then define $P(x, y) + nz$ to be positive if its term of highest degree has a positive coefficient. One can verify that this makes F/I a nonunitable ordered ring.

3. l-semisimple and archimedean rings. We define a *formally real* f-ring as a member of the smallest equationally definable class which includes the ordered real field. Thus, an f-ring is formally real if and only if it is a homomorphic image of a lattice-ordered ring of real-valued functions. The following arguments will show (and it is easy to show directly) that one would get the same class if the reals were replaced by the rationals in the definition. It seems worth while also to define a *strongly real ring*, as a ring which is ring-isomorphic with at least one formally real f-ring and is not ring-isomorphic with any f-ring which is not formally real. We shall see that this latter concept is a direct generalization of the (standard) concept of a formally real field.

3.1. LEMMA. *In any f-ring,*

$$x^2y^+ - xy^+ + y^+ \geqq 0 .$$

Proof. This is equivalent to an identity, since $a \geqq 0$ means $a \wedge 0 = 0$; hence it suffices to consider totally ordered rings. We may write y instead of y^+. Then in case $x^2 \geqq x$, we have $(x^2 - x)y + y \geqq 0$ as required. In case $y \geqq xy$, $x^2y + (y - xy) \geqq 0$. If neither of these is the case, we have $xy > y$ and $x > x^2 \geqq 0$. Hence $xxy \geqq xy$, and $(x^2y - xy) + y \geqq 0$.

3.2. LEMMA. *In any f-ring.*

$$xy^+ = [xy \wedge (x^2y + y)] \vee [0 \wedge (-x^2y - y)] .$$

Proof. In case $y \geqq 0$, $xy \leqq x^2y + y$ by 3.1, and $0 \geqq -x^2y - y$ obviously. Then the proposed formula reduces to $xy \vee (-x^2y - y)$. Applying 3.1 to $-x$ and y, we find $-(-xy) \geqq -x^2y - y$, and the equation·reduces to $xy^+ = xy$, which is correct in this case. The case $y < 0$ is similar.

Evidently there are corresponding formulas expressing x^+y, xy^-, and x^-y as lattice polynomials in ring polynomials in x and y. Using further formulas from [4; 57] (such as $x^+(y \vee z) = x^+y \vee x^+z$) and the fact that the lattice operations in a lattice-ordered group distribute over each other [3; Ch. XIV] reduces the following theorem to an exercise; we omit the details.

3.3. THEOREM. *If A is an f-ring and S a subring, then the least sublattice of A which contains S is also a subring of A.*

3.4. COROLLARY. *If S is a ring of real-valued functions on a set, the least lattice of functions which contains S is also a ring.*

3.5. COROLLARY. *Every word (element) in a free f-ring is a lattice polynomial in ring polynomials in the generators.*

3.6. COROLLARY. *Every identity in f-rings is equivalent to a conjunction of identities of the form $(P_1 \wedge \cdots \wedge P_m)^+ = 0$, where the P_j are ring polynomials.*

Now Artin has proved a theorem [1; Satz 3] more general than the following statement. Let E be an ordered field consisting of the field of rational functions $Q(x_1, \cdots, x_n)$ over the rational number field Q, in any ordering. Let $f_1, \cdots, f_m$ be elements of E, with each $f_j > 0$ in E. Then there exists a rational n-tuple $q = (q_1, \cdots, q_n)$ such that the functional values $f_j(q_1, \cdots, q_n)$ are all finite and strictly positive.

From Artin's theorem and 3.6 there follows

3.7. *Every ordered field which is a pure transcendental extension of the rationals satisfies all lattice-ordered ring identities which are valid in the rationals.*

Proof. Since an ordered field is an f-ring, 3.6 applies. Now the negation of an identity of the form in 3.6 is the conjunction of a set of inequalities $P_j(x_1, \cdots, x_{n_j}) > 0$. In a pure transcendental extension of Q, the x_1's are rational functions of finitely many independent transcendentals y_k. For each x_i, either $x_i > 0$ or $-x_i > 0$. By Artin's theorem, rational values q_k can be substituted for the y_k so that (in particular) all the x_i are finite and all the P_j are strictly positive in Q. Then the conjunction of the inequalities $P_j(x_1, \cdots, x_{n_j}) > 0$ is satisfiable in Q, so that its negation is not valid.

But now

3.8. *Every ordered field is formally real.*

Proof. Since any single identity involves only finitely many free variables, it suffices to consider a finite extension F of Q. Any such F is a simple algebraic extension of a pure transcendental extension; let $F = Q(x_1, \cdots, x_n)\ [\alpha]$. It suffices to show that there is an ordering on $D = Q(x_1, \cdots, x_n)[x_{n+1}]$ for which some homomorphism h of D onto F is order-preserving. (For it is immediate from 3.7 that D is formally real.)

Let h be any ring homomorphism of D onto F. The kernel H of h is a principal ideal generated by a prime element p of D. Every element of D can be written uniquely in the form ap^m, for some nonnegative integer m and some $a \notin H$. Let $ap^m > 0$ if and only if $h(a) > 0$ in F. Evidently this does it.

After we had proved 3.8, Dana Scott pointed out to us that the result can be deduced from Tarski's decision method for the algebra of real-closed fields [13]. That deduction is shorter, but it does not give the additional detail about the rationals.

Previously known results enable us to push 3.8 considerably further. Obviously every ordered integral domain, being a sub-f-ring of an ordered field, is formally real. From Johnson's work [9], the same holds for totally ordered rings without unit which are commutative and *l-semisimple* in the sense of [4], i.e. without nonzero nilpotents. But Pierce has shown [12] that an l-semisimple f-ring is a subdirect sum of l-semisimple totally ordered rings. Note finally that l-semisimplicity is independent of the ordering; and we have

3.9. THEOREM. *Every commutative l-semisimple f-ring is strongly real (and in particular, formally real).*

For the next result we shall use the fundamental theorem concerning linear homogeneous inequalities over an ordered field, which may be stated in either of the following ways. Let Σ be a finite system of linear homogeneous strict inequalities; then Σ has a solution if and only if the inequality $0 > 0$ is not deducible from Σ by addition and multiplication by positive scalars. Or, let S be a convex cone; then S has an interior point if and only if the polar cone S^* contains no nonzero linear subspace. The theorem is due to Farkas, [6]; one can of course find it in several places in the Annals Study on linear inequalities [10].

3.10. THEOREM. *Every zero f-ring is strongly real; in particular, it is formally real.*

Proof. Evidently it suffices to treat the totally ordered case. Suppose the theorem false. Then as in 3.9 we have a homomorphism $h: A \to E$ of a free f-ring into a totally ordered zero ring taking some word $M = \Lambda_s P_s \in A$ to a positive element $h(M)$ of E. Here the P_s are polynomials in the generators $y_1, \cdots, y_n$ of A, and supposedly $g(M) \leqq 0$ for all homomorphisms g of A into the reals. Since $h(P_s(y_1, \cdots, y_n)) = P_s(h(y_1), \cdots, h(y_n)) > 0$, there are nonzero first degree terms in each P_s. Write $P_s(y_1, \cdots, y_n) = u^s \cdot \bar{y} + P_s'(\bar{y})$, where u^s is an integral n-vector and P_s' has no first degree terms. Consider the cone generated by the u^s in n-space over the rationals. It contains no nonzero linear subspace, i.e. 0 has no nontrivial positive representation as $\Sigma c_s u^s$; for if it did, we could first convert the rational coefficients to integers by multiplying by the least common denominator, and then deduce $\Sigma c_s h(P_s) = 0$ in E. Consequently the polar cone has interior points $\bar{q} = (q_1, \cdots, q_m)$. For such a $\bar{q}$, or any positive scalar multiple of it, $u^s \cdot \bar{q} > 0$; for a sufficiently small scalar multiple, $u^s \cdot \bar{q}$ exceeds the quadratic $|P_s'(\bar{q})|$, and $P_s(\bar{q}) > 0$. Then map A into the reals, sending each generator y_j to q_j; each P_s goes to a real number > 0, and the contradiction completes the proof.

It is obvious that every lattice-ordered zero ring satisfies the defining identities for f-rings, given in [4] and reproduced in the introduction to this paper. Since every vector lattice becomes an f-algebra under the zero multiplication, 3.10 solves Problem 107 of [3].

3.11. THEOREM. *Every archimedean f-ring is a subdirect sum of an l-semisimple and a zero f-ring, and therefore is formally real.*

Proof. To begin with, an archimedean f-ring A is commutative [4]. Let M denote the set of all elements m of A such that $|m| \leqq xy$ for some x and y in A; let N denote the set of all $a \in A$ such that $a^2 = 0$. By definition M contains *all* products. It also contains sums of its own elements, since $xy + uv \leqq |xy| + |uv| \leqq (|x| + |u|)(|y| + |v|)$. Then M is an l-ideal; and evidently A/M is a zero f-ring. To complete the proof we need only show that N is the l-radical of A and that $M \cap N = 0$; it will then follow that A is a subdirect sum of the semisimple f-ring A/N and the zero f-ring A/M.

To show that N is the l-radical, it suffices to show that there are no positive nilpotents of index 3. That is, for any $x > 0$ such that $x^3 = 0$, $x^2 = 0$ also. Since A is archimedean, it suffices to show $nx^2 \leqq x$ for all positive integers n.

The implication, if $x^3 = 0$, then $nx^2 \leqq x$, is valid in arbitrary f-rings. The proof again reduces to the totally ordered case, and we leave it to the reader.

Finally suppose that $z \in M \cap N$. This means that $z^2 = 0$ and $|z| \leqq xy$ for some x, y in A. It suffices to consider the case when x, y and z are positive. If $u = x + y$, then $u^2 \geqq xy > 0 = (nz)^2$ for any positive integer n. Hence $nz \leqq u$, since this holds in any totally ordered homomorphic image of A. Since A is archimedean, $z = 0$.

We conclude this section with two remarks about 3.11. First, not every archimedean f-ring is strongly real. An example is the direct sum $A = R \oplus R'$ of the real field R and the zero ring R' having the same additive group. A can be ordered both lexicographically and coordinatewise. In the latter order, it is archimedean. In the former order, it is not unitable by 1.1. Thus A is not strongly real. Second, "subdirect sum" cannot be strengthened to "direct sum" in 3.11. To construct an example consider the additive group of those continuous real functions on [0, 1] which are piecewise polynomial and vanish at 0. Form the direct sum of two copies of this, both ordered by functional values, one with natural multiplication, one with zero multiplication. Consider the subset consisting of those pairs of functions (f, g) for which f and g have the same derivative at 0. One can verify that this archimedean f-ring is not a direct sum of a zero and an l-semisimple summand.

4. General formally real rings. Birkhoff's theorem on equationally definable classes [2] yields the following.

THEOREM (*Birkhoff*). *Every formally real f-ring [f-algebra] is a homomorphic image of an f-ring [f-algebra] of real-valued functions.*

4.1. COROLLARY. *Every formally real f-ring can be embedded in a commutative f-ring in which whenever* $|x| \leqq |y|$, *x is a multiple of y.*

Proof. It suffices to show that any formally real f-ring B with unit can be so embedded. Then B is a homomorphic image of an f-ring A of real-valued functions, by a homomorphism with kernel K, and we may assume that A has a unit. Relative to some representation of A as a family of functions, let A^* be the set of all functions g such that $|g| \leqq f$ for some f in A. If K^* is the smallest l-ideal in A^* that contains K, then $K^* \cap A = K$. Thus, A^*/K^* is an f-ring B^* containing B. Finally, if $|x| \leqq |y|$ in B^*, let x_0, y_0 be representative in A^* of the cosets x, y. Then $x_1 = (x_0 \wedge |y_0|) \vee (-|y_0|)$ is another representative of x, and since $|x_1| \leqq |y_0|$, x_1 is a multiple of y_0 in A^*.

A commutative ring is called a *valuation ring* if for any two elements, at least one is a multiple of the other. Since the property in 4.1 is preserved under homomorphic images, we have at once

4.2. COROLLARY. *Every formally real f-ring can be embedded in a product of totally ordered commutative valuation rings.*

We do not know whether conversely every totally ordered commutative valuation ring is formally real.

4.3. COROLLARY. *Every formally real f-ring can be embedded as a sub-ring in a formally real f-algebra.*

This corollary follows from Birkhoff's theorem in the same manner as 4.1. We have no idea how large a class of f-rings can be embedded in f-algebras—conceivably all of them.

These corollaries can be established without using Birkhoff's general theorem, going by way of

4.4. *A free formally real f-ring is an f-ring of real-valued functions; specifically, if the number of generators is m, the f-ring is the smallest lattice of functions on a product of m real lines which contains the polynomials in the coordinate functions with integral coefficients and zero constant term.*

Proof. In view of 3.4, the indicated set of functions forms an f-ring generated by the coordinate functions x_α. It remains only to observe that no word in the x_α vanishes as a function unless it vanishes for all real values of the x_α, and thus in all formally real f-rings.

4.5. THEOREM. *The free formally real f-ring on a set of generators is the residue class ring of the free commutative f-ring on the same generators modulo its l-radical.*

Proof. Consider the natural homomorphism h of the free commutative f-ring C upon the corresponding free formally real f-ring F. By 3.9, C modulo its l-radical is formally real; hence the kernel of h is contained in the radical. By 4.4, the kernel contains the radical.

At this point we need to recall some details of the proof that an integral domain in which 0 is not a sum of squares can be totally ordered. We number the lemma for convenience, but the proof is in many standard references, cf. [14].

Recall that a semiring $P(+,\cdot)$ is a set of elements P with two binary operations such that $P(+)$ and $P(\cdot)$ are semigroups, and $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in A$.

4.6. LEMMA. *Let A be a commutative ring without proper divisors of zero, and P a semiring contained in A, containing all nonzero squares, and not containing* 0. *Then A can be totally ordered so that every element of P is positive.*

4.7. THEOREM. *Let G be a free commutative ring, E a formally real totally ordered ring, and h a ring homomorphism of G into the ring E. Then there is a total ordering of G such that h is order-preserving.*

Proof. By 4.6, it suffices to show that the smallest subsemiring P of G containing all nonzero squares and all g such that $h(g) > 0$ in E does not contain 0. The elements g of the free ring are polynomials in the generators x_α; let us write $g = g(\bar{x})$. For any finite set of positive elements $h(g_i)$ of the formally real ordered ring E, one can find real numbers r_α such that the polynomial functions g_i all take positive values at the point $\bar{r}$ with coordinates r_α. Moreover, all g_i are positive on a neighborhood of $\bar{r}$. Further, a nonzero polynomial of the form f_j^2 is nonnegative everywhere, and vanishes only on a nowhere dense set. Hence any sum of products of the polynomial functions g_i and f_j^2 is strictly positive almost everywhere near $\bar{r}$, and 0 is not in P.

4.8. COROLLARY. *The free formally real f-ring on any set of generators is a subdirect product of totally ordered free commutative rings on the same set of generators; specifically, if C is the free com-*

mutative ring, embedded in the product P of all totally ordered rings isomorphic with C, then the smallest sub-f-ring of P which contains C is free formally real.

This brings us to the promised example to show that identities in eight variables or fewer cannot characterize formally real f-rings. The argument involves the following identity in nine variables.

$$[x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 \wedge z_1 \wedge z_2 \wedge z_3 \wedge (x_1z_1 - y_1z_2) \wedge (x_2z_2 - y_2z_3) \wedge (x_3z_3 - y_3z_1) \wedge (y_1y_2y_3 - x_1x_2x_3)]^+ = 0 .$$

There is no difficulty in verifying this identity for real numbers.

First we describe the essential features of our argument. Starting with nine generators x_i, y_j, z_k, which will not be allowed to satisfy the above identity, we construct a nilpotent totally ordered algebra A. The products of two or more of the generators all lie in a formally real ideal I; moreover, the subalgebra generated by I and any eight of the generators is formally real. Finally, any subalgebra of A generated by eight or fewer elements can be thrown into one of these by an automorphism of A.

The construction involves something which may reasonably be called the *semigroup algebra with zero* of a semigroup S with zero. This is the set of all formal linear combinations of nonzero elements e_i of S, added formally and multiplied by $e_ie_j = e_k$ or $e_ie_j = 0$ according to the multiplication in S. Moreover, we shall define a total ordering of S, and induce the lexicographic ordering on A. (That is, $\Sigma\alpha_ie_i > 0$ if the largest e_i whose coefficient is nonzero has a positive coefficient.) One may verify that sufficient conditions for this to make A an ordered algebra are that S is commutative, 0 is the smallest element of S, and if $x > y$ in S then either $xz > yz$ or $xz = yz = 0$.

We now describe the elements, the order, and the multiplication of S. The elements will be named unambiguously by expressions $e(n)$, for 79 particular integers n. The first nine values of n are 10000, 10020, 10080, 10260, 10800, 11123, 16000, 16081, and 16322. The other seventy are 20000, 20020, 20040, 20080, 20100, 20160, 20260, 20280, 20340, 20520, 20800, 20820, 20880, 21060, 21123, 21143, 21203, 21383, 21600, 21923, 22246, 26000, 26020, 26080, 26081, 26101, 26161, 26260, 26322, 26341, 26342, 26402, 26582, 26800, 26881, 27122, 27123, 27204, 27445, 30000, 30020, 30040, 30060, 30080, 30100, 30120, 30160, 30180, 30240, 30260, 30280, 30300, 30340, 30360, 30420, 30520, 30540, 30620, 30780, 30800, 30820, 30840, 30880, 30900, 30960, 31060, 31080, 31123, 31140, and 31143. These are the sums of numbers from among the first nine which do not exceed 31143. Multiplication is defined by $e(m)e(n) = e(m + n)$ so long as $m + n$ is in the list; beyond this, $e(31140)$ (not $e(31143)$) is the zero element 0 of S, and $e(m)\ e(n)$ is defined to be $e(31140)$ if $m + n > 31143$. Finally, the order of S is decreasing in the order of the list except that $e(31123) > e(31143) > 0 = e(31140)$; that is, $e(31143)$ is displaced one place.

One may verify that S satisfies the conditions indicated above which make the semigroup algebra with zero of S a totally ordered algebra A. On the other hand, A is not formally real. To see this let $y_1, y_2, x_1 x_2, x_3, y_3, z_1, z_2, z_3$, in that order, be the elements of A corresponding to the first nine elements of S. (That is, y_1 is the expression $1 \cdot e(10000)$, and so for the others.)

The integers $10000, \cdots, 31143$ are so chosen that (among other things) every nonzero element of S can be factored uniquely into prime factors. This makes it possible to construct certain automorphisms of A in the following manner. Let us relabel the generators $y_1, \cdots, z_3$ in order as $s_1, \cdots, s_9$. For any i among $1, \cdots, 9$, let t_i be any element of A of the form $\alpha s_i + r$, where α is a strictly positive scalar and $|r|$ is smaller than any strictly positive scalar multiple of s_i. There is a unique automorphism p of A which sends s_i to t_i and leaves the other eight generators fixed. To prove this, observe that since $s_1, \cdots, s_9$ generate A, one can write a formula for p; and one can then verify that p is an order-preserving automorphism.

For $i = 1, \cdots, 9$, let B_i be the subalgebra of A which is the semigroup algebra with zero of $S - \{S_i\}$. We conclude next that any eight elements $g_1, \cdots, g_8$ of A can be sent into some B_i by an order-preserving automorphism. For this, evidently, we need only consider those g_j which, written as $\Sigma \alpha_{ij} e_i$, have nonzero coefficients attached to some of the generators $s_1, \cdots, s_9$; and if the indices $i(j)$ of the largest s_i involved in g_j are all different, then eight applications of the preceding paragraph will suffice. But by a simple elimination we can replace $g_1, \cdots, g_8$ with elements $h_1, \cdots, h_8$ which satisfy the latter condition and generate the same linear subspace (a fortiori the same subalgebra) as $g_1, \cdots, g_8$.

It remains to show that each of $B_1, \cdots, B_9$ is formally real. For each B_i we shall construct a totally ordered cancellation semigroup T_i, and then a totally ordered integral domain D_i mapping homomorphically upon B_i. We shall use the following property of our description of S, which is clear from the list above. With exceptions, any two different integers, i, j, which are indices of elements $e(i)$, $e(j)$ of S differ by more than 12. The exceptions are 26080, 26081; 26342; 27122, 27123; and 31140, 31143. They correspond to the products appearing in the last displayed identity.

The semigroup T_1 is generated by eighteen elements which we designate as $f(n)$, for numbers n as follows: $10020, \cdots, 16322$; $19992, \cdots, 25996, 26318$; 26081; 29988. Here the first group of numbers are the indices attached to $s_2, \cdots, s_9$ in S. The others, except 26081, correspond to the ten additional primes in the semigroup $S - \{s_1\}$, and are the indices they would have if the index of s_1 were changed from 10000 to 9996. Hewever, we leave $26081 = 10000 + 16081$ alone. Now T_1 consists of all expressions $f(n)$ which can be generated from these by $f(i)f(j) = f(i + j)$, and is ordered by $f(i) > f(j)$ if and only if $i < j$. Because of the ordering, it is clear that the semigroup algebra of T_1 is an integral domain D_1. (This is the usual semigroup algebra, since T_1 has no zero.) Moreover, lexicographic ordering makes D_1 an ordered algebra; since it is an integral domain, it is formally real. Finally, to define a homomorphism of D_1 upon B_1, map each of the generators $f(n)$ to that nonzero $e(m)$ in B_1 for which $|m - n|$ is a minimum. We omit details of the verification, but note that the conceivable difficulties at 26081 and 31143 do not arise because $e(26081)$ is an annihilator in A and because it is $f(31139)$ that goes to $e(31143)$.

For the construction of T_i, $i = 2$ or 6, use the same device of subtracting 4 from the index of s_i with an exception for 26342, 27123, respectively. The cases $i = 3, 4, 5$ differ only in that one adds 4 instead of subtracting. For T_7, add 4 to the index of s_3 throughout except in s_3s_7. (There are only fourteen n's: 10000, 10020, 10084, 10260, 10800, 11123, 16081, 16322, 26000, 26020, 26080, 26260, 26800, 27123.) For T_8, do the same with s_4, and for T_9, with s_5. This concludes the description of the example A.

One might naturally inquire what are the properties of the smallest equationally definable class of f-rings, which includes the integers. One obvious and very restrictive identity is $x^2 \vee x = x^2$. It seems worth pointing out further that every unsolvable Diophantine equation yields an identity for the integers. For example, the implication $x^2 = 2y^2 \Rightarrow x = 0$ yields

$$[x \wedge z \wedge (x^2z - 2y^2z + z) \wedge (2y^2z - x^2z + z)]^+ = 0 .$$

5. Supermodular ideals. We shall want some lemmas on general f-rings. In any latticeordered ring (or group), for any subset S, we define the *polar set* $S^\perp$ to be the set of all x such that for every s in S, $|x| \wedge |s| = 0$. It is trivial to verify that $S^{\perp\perp} \supset S$, $S^{\perp\perp\perp} = S^\perp$, and $S^\perp$ is always a convex sublattice and additive subgroup.

5.1. *A lattice-ordered ring is an f-ring if and only if every polar set is an l-ideal.*

The verification is trivial. Observe that for two polar ideals, H, K, $(H + K)^\perp$ is exactly $H^\perp \cap K^\perp$; $(H \cap K)^\perp$ contains $H^\perp + K^\perp$, generally properly.

5.2. *For any two l-ideals, I, J, in an f-ring, a maximal l-ideal* *which contains $I \cap J$ contains I or J.*

Proof. Suppose this is false for the maximal l-ideal M in the f-ring A. Then $A = M + I = M + J$. For any $x \in A$, there are an $i \in I$ and a $j \in J$ such that $x \equiv i \equiv j \pmod{M}$. Then $|x| \equiv |i| \wedge |j| \equiv 0 \pmod{M}$ since $I \cap J \subset M$, contray to the assumption that M is proper.

Following the usage of [9], an f-ring is called *l-simple* if it has no proper l-ideals and is not a zero ring.

5.3. *An l-simple f-ring has a superunit.*

Proof. Let A be an f-ring without proper l-ideals. Clearly A is totally ordered, as otherwise there would be a proper polar ideal. If A were not unitable, then by 1.6 and 1.7, it would contain both nilpotents and non-nilpotents. Then its l-radical would be proper. Hence A has a superunit or is infinitesimal. Suppose that the latter holds. Then A is a subring of a totally ordered ring B with unit 1, and $|a| < 1$ for all $a \in A$. Thus, for all positive x, y in $A, xy \leqq x \wedge y$. Hence every order-convex additive subgroup of A is an l-ideal. If x is in the smallest convex subgroup of A containing x^2, then there is a positive integer n such that $nx^2 \geqq x$. But then $(nx - 1)x \geqq 0$, whence $nx \geqq 1$, contrary to the fact that $x \in A$. Thus $x^2 = 0$, and A is a zero ring.

For future reference, we display the corollary

5.4. *An f-ring without proper l-ideals is unitable.*

The rest of the paper is concerned with unitable f-rings, though we omit "unitable" from the hypothesis when it causes no extra work to do so. Recall that a unitable f-ring is a subdirect sum of a family of totally ordered rings each of which either is infinitesimal or has a superunit. If all the summands are infinitesimal, so is the sum. We shall say that an f-ring *has local superunits* if it is a subdirect sum of (totally ordered) f-rings having superunits. Evidently the definition is the same whether we include the parenthesis or not. Now the decomposition of an archimedean f-ring given in 3.11 can be generalized as follows.

We call an l-ideal J in an f-ring A *supermodular* if A/J has a superunit.

5.5. THEOREM. *Every unitable f-ring can be represented as a subdirect sum of an infinitesimal f-ring and an f-ring having local superunits. The second summand is unique.*

Proof. Let A be a unitable f-ring, and K the intersection of all supermodular l-ideals of A. Then A/K has local superunits; and the kernel of any homomorphism of A upon such an f-ring must contain K. Next consider the polar ideal $K^{\perp}$. For any $x \geqq 0$, $(x^2 - x)^+$ is in $K^{\perp}$. For, suppose on the contrary that $(x^2 - x)^+ \wedge k$ is nonzero for some $k \geqq 0$ in K. Then it has a nonzero image under some homomorphism of A onto a totally ordered ring T that is unitable and cannot have a superunit, since the image of K is not zero. Hence T is infinitesimal, by 1.2, and the image of $(x^2 - x)^+$ is zero.

This shows that $A/K^{\perp}$ is infinitesimal. Since $K \cap K^{\perp}$ is zero, A is a subdirect sum of the infinitesimal summand $A/K^{\perp}$ and the summand A/K which has local superunits.

Next we show $S^{\perp} \subset K$, where S is the set of all $(x^2 - x)^+$ for $x \geqq 0$. Indeed, for any y not in K, there is a homomorphism h upon a totally ordered ring with superunit taking y to a nonzero image $h(y)$; for any positive x such that $h(x)^2 > h(x)$, $(x^2 - x)^+ \wedge |y|$ is nonzero. That done, consider a subdirect sum decomposition of A into an infinitesimal summand A/L and a summand A/M with local superunits. We have noted that M must contain K. On the other hand, L must contain S, and M must be contained in $L^{\perp} \subset S^{\perp} \subset K$. Hence M must be exactly K, and the proof is finished.

Concerning 5.5, we note that $A/K^{\perp}$ is the smallest infinitesimal summand which can be used, since $K^{\perp}$ is the largest l-ideal which together with K gives a subdirect decomposition.

If an l-ideal J is both supermodular and maximal, we call it supermodular maximal supermodular indifferently. This is justified by

5.6. *Every supermodular l-ideal J of an f-ring A is contained in a supermodular maximal l-ideal. In particular, every l-ideal in an f-ring with superunit is in a maximal l-ideal.*

Proof. If A/J has a superunit x, no proper l-ideal in A/J can contain x. Then by the usual Zorn's lemma argument (for rings), A/J has a maximal l-ideal. Such an ideal must an ideal must be supermodular, and its inverse image in A is the required supermodular maximal l-ideal containing J.

Later (6.11) we shall want the following lemma.

5.7. *In an f-ring having local superunits, every proper polar ideal is contained in a supermodular polar ideal.*

Proof. Let H be a proper polar ideal of an f-ring A with local superunits. Each maximal l-ideal must contain H or $H^{\perp}$, by 5.2. Since the intersection of the supermodular maximal ideals is zero, not all of them contain $H^\perp$; so one of them, say M, contains H. Let e be a positive element of A which is a superunit modulo M. Then $(e^2 - e)^+$ is not in M, hence not in H; therefore, there is $t \geqq 0$ in $H^\perp$ such that $(e^2 - e)^+ \wedge t = u \neq 0$. Let K be the polar ideal $[u]^\perp$. Since u is in $H^\perp$, K contains H. It remains to verify that there is a superunit modulo X, namely e. For any $y \geqq 0$, $(e^2 - e) \wedge (y - ey) \leqq 0$, since, by 1.6, A is unitable. Hence $(y - ey)^+ \wedge (e^2 - e)^+ = 0$, $(y - ey)^+$ is in K, and $ey \geqq y$ (mod K). Similarly $ye \geqq y$ (mod K), as required.

Between the supermodular polar ideals and the supermodular maximal ideals there is another class which we shall examine. The basic idea, from real function rings, is that of the ideal of all functions vanishing on a neighborhood of a fixed point. For a supermodular maximal l-ideal M in a unitable f-ring A, we define the *germinal l-ideal G associated with M* to be the sum of all l-ideals H such that $M + H^\perp = A$. We call an l-ideal *supermodular germinal* if it is the germinal ideal associated with some maximal supermodular ideal. (5.9 below shows that such ideals are supermodular.)

5.8. *If A is a unitable f-ring, M a maximal supermodular ideal, and x a positive element of A, then the following conditions are equivalent.*

(i) *x is in the germinal ideal G associated with M.*

(ii) *$x = a^-$ for some $a > 0$ (mod M).*

(iii) *$x = e^-$ for some e which is a superunit modulo M.*

Proof. Assume (i). Then x is a finite sum of elements $x_i \in A^+$ for which there exist elements y_i such that $x_i \wedge y_i = 0$ and $y_i > 0$ (mod M). Then $y = \wedge y_i > 0$ (mod M), and $x \wedge y = 0$, so $x \in M$. Thus $a = (y - x)$ satisfies (ii).

If (ii) holds, since A/M is l-simple, there is a $z \geqq 0$ such that za^+ is a superunit modulo M. Let $e = za^+ - a^-$. Then $e^- = a^- = x$, $e \equiv za^+$ (mod M), so (iii) holds.

Clearly (iii) implies (i).

An l-ideal is called *primary* if it is contained in a unique maximal l-ideal.

5.9. THEOREM. *A proper l-ideal of a unitable f-ring A is supermodular and primary if and only if it contains some supermodular germinal l-ideal. A has local superunits if and only if the intersection of the supermodular germinal l-ideals of A is* 0.

Proof. Let M be a supermodular maximal l-ideal, and let G be the associated supermodular germinal l-ideal. Let $x \in A^+$ be a superunit modulo M. We shall show that $x' = 2x$ is a superunit modulo G. For any positive y, $(y - x'y)^+$ generates an l-ideal H such that $H^\perp$ centains $(x')^2 - x'$. (Apply 1.6 to each totally ordered subdirect summand of A.) Also $(x')^2 - x' > 0$ mod M, so $M + H^\perp = A$, whence $H \subset G$. The same reasoning applies to $(y - yx')^+$, and shows that G is supermodular.

To see that G is primary, consider any other maximal l-ideal N, and recall that there exist x_1, x_2, such that $(x_2 - x_1)^+$ is in N but >0 (mod M) and $(x_1 - x_2)^+$ is in M but >0 (mod N). Then if H is the principal l-ideal generated by $(x_1 - x_2)^+$, $M + H^\perp = A$. Hence G contains H and cannot be contained in N. Thus, G is supermodular and primary, as is any proper l-ideal containing it.

Next consider a supermodular l-ideal J, contained in M, which fails to contain G. J cannot contain all the positive elements of G; thus for some a, $a > 0$ (mod M) and a^- is not in J. Then J and a^+ generate a proper l-ideal K. Since K contains J, it is supermodular and is contained, by 5.6, in a supermodular maximal l-ideal L. Since $K + M$ is already A, $L \neq M$, and J is not primary.

If the intersection of the supermodular germinal ideals is 0, then so is the intersection of the supermodular ideals, and the ring has local superunits. Conversely, if the ring has local superunits, any nonzero positive x has a strictly positive image in a totally ordered ring T having a superunit e. The kernel of the homomorphism upon T is a primary l-ideal I contained in a maximal l-ideal M. Then x cannot be in the germinal ideal associated with M, by 5.8 (ii); for $a > 0$ (mod M) implies $a > 0$ (mod I), $a^+ \wedge x \neq 0$, and $x \neq a^-$. This completes the proof of 5.9.

We note that one can show

5.10. *In an l-semisimple f-ring, every polar ideal, and also every supermodular germinal ideal, is an intersection of prime l-ideals.*

We conclude this section by examining what happens to l-ideals on embedding in an f-ring with unit. The main point is that those maximal l-ideals which are supermodular correspond to maximal ideals of the bigger ring, and the others do not. Another interesting point is that there is a unique smallest containing f-ring with unit. This was conjectured by D. G. Johnson, who proved it in the totally ordered case [9].

Let A be a unitable f-ring and B an f-ring with unit containing A. We say that B is a *smallest* f-ring unit containing A if no proper subring of B is an f-ring with unit containing A. (This is stronger than the condition that A and 1 generate B as an f-ring.) We understand "unique" in the usual sense of an isomorphism leaving A elementwise fixed; then Johnson's result [9] is that *the smallest f-ring with unit containing a unitable totally ordered ring is unique.* It is, of course, totally ordered; and note that a totally ordered ring with unit containing A is smallest if it is generated by A and 1 (since 1 is the only nonzero idempotent by 2.1).

If I is a subset of a sub-f-ring A of an f-ring B, we let $I_B^\perp = \{x \in B: |x| \wedge |a| = 0 \text{ for all } a \in I\}$. $I_B^{\perp\perp}$ is defined similarly.

5.11. Theorem. *If A is a unitable f-ring, then*

(i) *A is contained in a unique smallest f-ring B with unit. Moreover, $B = A_B^{\perp\perp}$.*

(ii) *For each supermodular maximal l-ideal M of A, there is exactly one maximal l-ideal M' of B such that $M = M' \cap A$.*

(iii) *A is contained in no proper l-ideal of B, or in exactly one l-ideal M of B (which is maximal) according as A has a superunit or not.*

(iv) *In the latter case, if H is the germinal ideal associated with M, then $H \cap A$ is the sum of all the polar ideals I of A such that $I^\perp$ is supermodular.*

(v) *For every supermodular germinal ideal G of A, there is exactly one germinal ideal G' of B such that $G = G' \cap A$.*

(vi) *There are no maximal l-ideals of B other than those described in* (ii) *and* (iii).

(vii) *If I is a polar ideal of A, then $I = I_B^{\perp\perp} \cap A$.*

Proof. Regard A as the subdirect sum of all of its totally ordered homomorphic images T_α; embed each T_α in the unique smallest (totally ordered) ring with unit U_α containing it. Let C be the direct sum of all U_α, and B the sub-f-ring generated by A and 1. To see that B is smallest, note that a unit u for any sub-f-ring containing A must have image 1 in each U_α.

Now consider any smallest f-ring D with unit containing A. Without loss of generality we may assume that A and 1 generate D. We show next that $A_D^{\perp\perp}$ has a unit. Indeed, $D/A_D^{\perp\perp}$ is generated by 1 (if 1 is not already in $A_D^{\perp\perp}$). But $D/A_D^{\perp\perp}$ contains $A^\perp$ isomorphically. Then if $A_D^\perp$ is not zero, it has a superunit f. $A_D^\perp$ also contains $f \wedge 1$, an idempotent; and $1 - (f \wedge 1)$ is a unit in $A_D^{\perp\perp}$. Hence we may now assume $D = A_D^{\perp\perp}$.

Next, we prove the uniqueness of B.

Observe that since A and 1 generate D, for every $d \in D$ there is an $a \in A$ and an integer n such that $|d| \leqq n + a$. For any l-ideal I of A, let

$$I^0 = \{d \in D: |d| \leqq i \text{ for some } i \in I\},$$

and let

$$I^1 = \{d \in D: |d| \wedge |a| \in I^0 \text{ for all } a \in A\}.$$

It is easily verified that I^0 and I^1 are l-ideals of D such that $I^0 \cap A = I^1 \cap A = I$.

Let I_α denote the kernel of the homomorphism of A onto the ordered ring T_α. We show next the intersection of all the I^i_α is 0, for $i = 0, 1$.

If $d \neq 0$ is in all the I^0_α, then $|d| \leqq a$ for some $a \in A$, and there is a homomorphism h of D onto a totally ordered ring such that $h(d) \neq 0$. Then $h(a) > 0$, and the kernel K of h meets A in some I^0_α. Then d is not in that I^0_α. It follows also that the intersection of all the I^1_α is in $A^\perp_D$, and we showed above that $A^\perp_D = 0$.

We prove next that D/I^1_α is totally ordered (for any fixed α). It suffices to show that T_α, the image of A under the homomorphism h of D onto D/I^1_α, and 1 together generate a totally ordered ring. Every element of this latter takes the form $a + n$ for some $a \in T_\alpha$, and some integer n. Suppose that one of these elements is incomparable with 0. This cannot happen for $n = 0$. Then by change of sign if necessary, we can rewrite the element as $a - n$, where $n \geqq 1$. Since T_α is totally ordered, a^2 and na are comparable.

We show first that if $a^2 \geqq na$ then $(a - n)^- = 0$. Let a_0 be a representative of a in A. We must show $(a_0 - n)^- \in I^1_\alpha$, i.e., for any positive b_0 in A, $b_0 \wedge (a_0 - n)^- \in I_\alpha$. Let $b = h(b_0)$. We need $b \wedge (a-n)^- = 0$, and clearly it suffices to prove this for $b \geqq a$.

Since $a^2 \geqq na \geqq a$, T_α is not infinitesimal, and as in the proof of 1.5, a cannot be a zero divisor. From the same proof, then, $ba \geqq nb$ for all positive b. Then $b(a - n) = b(a - n)^+ - b(a - n)^- \geqq 0$; since $(a - n)^+ \wedge (a - n)^- = 0$, $b(a - n)^-$ must be 0. Trivially $2b(2a - 2n)^- = 0$; also, when $b \geqq a$, then $2b \geqq 2a$.

The conditions $x(y - 2n)^- = 0$ and $x \geqq y \geqq 0$ imply $x \wedge (y - 2n)^- = 0$ in all f-rings. To prove this it suffices to treat the totally ordered case. Then suppose $y \geqq n$. We have $x \geqq n$ and $x(y - 2n)^- \geqq n(y - 2n)^-$; since $x(y - 2n)^- = 0$, $(y - 2n)^- = 0 \leqq x$, and the meet is zero. In the contrary case $y < n$, $(y - 2n)^- > n$, $x(y - 2n)^- \geqq nx$, and since the product is zero, x must be zero. This proves the implication, and for the case $a^2 \geqq na$ we have $(a - n)^- = 0$.

Suppose that $a^2 < na$. For this case we shall show $b \wedge (a - n) = 0$ for all positive b in T_α, so that $(a - n)^+$ must be zero as before. From the statement of 1.5 we have $ba \leqq nb$ in this case, hence $b(a - n)^+ = 0$, Again we can assume $b \geqq a$, and this with $b(a - n)^+ = 0$ implies $b \wedge (a - n)^+ = 0$ as before. This completes the proof that D/I^1_α is totally ordered.

Since D/I^1_α is a totally ordered ring with unit generated by T_α and 1, it is U_α. Since D is the sub-f-ring of the direct sum of all U_α generated by A and 1, it is B; and (i) is proved.

In the special case that I_α as treated above is supermodular maximal, it is clear that any maximal l-ideal of $B(= D)$ intersecting A in I_α must contain I^1_α. But U_α has a unique maximal l-ideal; this ideal cannot contain T_α, since there is an element $\geqq 1$ in T_α, and its intersection with T_α is therefore 0. This proves (ii).

Let A have a superunit e, and let I be any l-ideal of B containing A. Then, since $(e \wedge 1)$ is a unit for A in I, we have $I = B$, so A is in no proper l-ideal of B. If A has no superunit, then A^0 is an l-ideal of B containing A such that B/A^0 is the ordered ring of integers. Since $B = A_B^{\perp\perp}$, A^0 is maximal, and any l-ideal of B containing A contains A^0. Thus A^0 is the unique l-ideal of B containing A. This proves (iii).

We leave the proofs of the remaining assertions for the reader.

An l-ideal I in an f-ring A is called *modular* if A/I has a unit. Modular l-ideals are extensively investigated in [9]. We have

5.12. THEOREM. *An f-ring can be embedded as an l-ideal in an f-ring with unit if and only if it is unitable and every supermodular ideal is modular.*

Proof. Necessity of the conditions is obvious.

Suppose conversely that A is unitable and every supermodular l-ideal of A is modular. Let B be the smallest f-ring with unit containing A. We show first that for a in A, $a \wedge 1$ is in A. Let K be the smallest l-ideal of A modulo which a is a superunit, i.e., the l-ideal generated by all elements $(ax - x)^-$ and $(xa - x)^-$ for $x \in A^+$. We show next that a is a unit modulo $K + K_A^\perp$. For any x in A^+, $ax - x = (ax - x)^+ - (ax - x)^-$. Here $(ax - x)^-$ is in K. To show that $(ax - x)^+$ is in $K^\perp$, it suffices to show that $(ax - x)^+ \wedge (ay - y)^-$ and $(ax - x)^+ \wedge (ya - y)^-$ vanish for all y in A^+. For these to vanish it suffices that their images vanish in any totally ordered homomorphic image of A, and from 1.5 this is easily seen. Similarly, each $xa - x$, for $x \in A^+$, is in $K + K^\perp$.

From the hypothesis, the supermodular ideal K is modular in A. Let e be a unit modulo K. Then e is, like a, a unit modulo $K + K_A^\perp$; so $e - a \in K + K_A^\perp$. Write $e - a = p + q$, with q in $K_A^\perp$ and $p \in K$. We claim that $a \wedge 1 = a + q$, which is in A. To verify this, it suffices to verify that for any homomorphism h of A upon a totally ordered ring, $h(a + q)$ is a unit if $h(a)$ is a superunit, and $h(a + q)$ is $h(a)$ if this is not a superunit. But since the image is totally ordered, the kernel must contain K or $K_A^\perp$. If the kernel contains K, $h(a)$ is a superunit and $h(p) = 0$, which implies $h(a + q) = h(e)$—a unit. In the other case $h(q) = 0$, and $h(a + q) = h(a)$ as required.

We conclude next that in B, the ring generated by A and 1 is a lattice. This is the set of all $a + m$, for a in A and m an integer. Consider $(a + m) \wedge (b + n)$, where $n \geqq m$. This is $b + m + (a - b) \wedge (n - m)$; so it suffices to show that $(a - b) \wedge (n - m)$ is in A. But this is the same as $(a - b) \wedge (n - m)(|a - b| \wedge 1)$, by an easy check reducing to the totally ordered case. We have shown that $|a - b| \wedge 1$ must be in A; so $(a + m) \wedge (b + n)$ is in A. The join $(a + m) \vee (b + n)$ is

$-(-(a+m) \wedge -(b+n))$, and is also in A.

Then B, being smallest, consists only of the elements $a+m$. If A has a superunit, then the supermodular ideal 0 must be modular, and $B=A$. If A has no superunit, and $|a| \geqq |b+n|$ for some $a, b \in A$, then $|a|+|b| \geqq |b+n|+|b| \geqq |(b+n)-b| = |n|$; so $n=0$. This shows that A is a proper l-ideal of B.

6. Structure space and reordering.

Let $\mathscr{I}$ be any family of l-ideals of an f-ring A. If $\mathscr{S} \subset \mathscr{I}$, let the *kernel* k of $\mathscr{S}$ be the intersection of all the elements of $\mathscr{S}$. If J is any l-ideal of A, let the *hull* h of J be the set of all elements of $\mathscr{I}$ that contain J. Call a subset $\mathscr{S}$ of $\mathscr{I}$ *closed* if $\mathscr{S} = h(k(\mathscr{S}))$. It is well known that if $I \supset I_1 \cap I_2$ implies $I \supset I_1$ or $I \supset I_2$ for any $I \in \mathscr{I}$, and intersections I_1, I_2 of elements of $\mathscr{I}$, then $\mathscr{I}$ becomes a topological space. (This is noted, at least for ring ideals, in [8].) The resulting topology is called the *hull-kernel* topology.

By 5.2, the set of all maximal l-ideals, the set of all modular maximal l-ideals, and the set of all supermodular maximal l-ideals of A form topological spaces with the hull-kernel topology. We call the latter the *structure space* $\mathscr{M}(A)$ of A.

6.1. THEOREM. *The set of all maximal l-ideals of an f-ring is a Hausdorff space in the hull-kernel topology. The structure space is a locally compact open subset. Indeed, each M in $\mathscr{M}(A)$ contains a supermodular ideal I whose hull is a neighborhood of M. If A has a superunit, then $\mathscr{M}(A)$ is compact.*

Proof. For Hausdorff separation of two maximal l-ideals M_1 and M_2, note that A/M_1 and A/M_2 are totally ordered. Since $M_1 + M_2 = A$, there are x_1 in M_1 such that $x_1 > 0 \pmod{M_2}$ and x_2 in M_2 such that $x_2 > 0 \pmod{M_1}$. Let $\mathscr{U}$ be the set of all maximal l-ideals M such that $x_2 > x_1 \pmod{M}$. This is an open set, since it is the complement of the hull of the principal l-ideal generated by $(x_2 - x_1)^+$. Similarly the set of all M such that $x_1 > x_2 \pmod{M}$ is an open set $\mathscr{V}$, disjoint from $\mathscr{U}$. Since M_1 is in $\mathscr{U}$ and M_2 in $\mathscr{V}$, the space is Hausdorff.

We show next that if A has a superunit e, then $\mathscr{M}(A)$ is compact. Let $\{\mathscr{F}_\alpha\}$ denote any family of closed subsets of $\mathscr{M}(A)$ with empty intersection. Then $\{k(\mathscr{F}_\alpha)\}$ together generates A. Hence e is in the smallest l-ideal containing some finite subfamily $\{k(\mathscr{F}_{\alpha_1}), \cdots, k(\mathscr{F}_{\alpha_n})\}$. But then $\bigcap_{i=1}^n \mathscr{F}_{\alpha_i}$ is empty. So $\mathscr{M}(A)$ is compact.

Now, let $M_0 \in \mathscr{M}(A)$, and choose some $x \in A$ which is a superunit modulo M_0. Let $x' = 2x$, and observe that $(x')^2 = 2xx' > x' \pmod{M_0}$. Let $\mathscr{H} = \{M \in \mathscr{M}(A) : x' \text{ is a superunit} \pmod{M}\}$. In other words, $\mathscr{H}$ is the hull of the l-ideal K generated by all $(x'y - y)^-$ and $(yx' - y)^-$, for $y \in A^+$. Then x' is a superunit modulo K. Moreover, since A/K has a superunit, $\mathscr{H}$ is compact by the above. Finally $\mathscr{H}$ is a neighborhood of M_0, for it contains the open set of all M modulo which $(x')^2 > x' > 0$ (since A/M is unitable by 5.4).

The first theorem we know of to the effect that a space of ideals of a fairly general partially ordered ring is Hausdorff is due to Gillman [8]. The proof above is similar to Gillman's argument.

Since subspaces of locally compact Hausdorff spaces are completely regular we have

6.2. COROLLARY. *The modular maximal l-ideals of an f-ring form a completely regular space in the hull-kernel topology.*

6.3. EXAMPLES. (A) *The space of all maximal l-ideals of an f-ring of real-valued functions need not be regular.* (B) *The space of modular maximal l-ideals of an f-ring of real-valued functions need not be locally compact.*

EXAMPLE A. Let $\mathscr{X}$ be the half-line $[0, \infty)$, $\mathscr{N}$ the subspace of positive integers, and A the f-ring of all continuous functions f on $\mathscr{X}$ satisfying the following condition. For some subset $\mathscr{M}$ of $\mathscr{N}$ whose complement is finite, f vanishes at each n in $\mathscr{M}$ and has finite left and right upper and lower derivates there; and all these derivates form a bounded set of real numbers.

The structure space of A as we have defined it is the Stone-Cech compactification $\beta\mathscr{X}$ of $\mathscr{X}$, less the derived set $\mathscr{D}$ of $\mathscr{N}$ (i.e., less $\mathscr{D} = \mathscr{N}^- - \mathscr{N}$). However, corresponding to each point d of D there are two different nonsupermodular maximal l-ideals, $L(d)$, $R(d)$. Observe that d is the limit of an ultrafilter $\mathscr{F}$ in $\mathscr{N}$. $L(d)$ is the set of all functions f in A whose left upper and lower derivates at points of $\mathscr{N}$ converge along $\mathscr{F}$ to 0; $R(d)$ is defined similarly by right derivates. $L(d)$ and $R(d)$ have neighborhoods disjoint from each other and from $\mathscr{N}$; however, neither can be separated from the closed set $\mathscr{N}$ by disjoint open sets.

EXAMPLE B. Let R be the real line, and A the f-ring of all continuous functions f on R such that at each rational point r of R, f takes a rational value of the form $2p/q$, q an odd integer. It is easy to show that A contains enough functions to separate points, from closed sets; and its structure space, as we have defined it, is βR. However, the subspace of modular maximal ideals is nowhere locally compact, for it includes all the irrational points of R but none of the rational points.

A homomorphism h of an f-ring A *onto* an f-ring B induces a homeomorphism of the structure space of B onto the hull of the kernel of h; if this is not completely obvious, it becomes obvious on noticing that the supermodular maximal ideals are precisely the kernels of homomorphisms onto l-simple f-rings.

We call an f-ring *supermodular semisimple*, or *S-semisimple*, if the intersection of the supermodular maximal ideals is zero; equivalently, if it is a subdirect sum of l-simple f-rings. Note that an l-simple f-ring has no proper divisors of zero. Therefore an S-semisimple f-ring is l-semisimple. From 3.9 we have

6.4. *The commutative S-semisimple f-rings are precisely the residue class rings of f-rings of real-valued functions modulo ideals which are kernels of (closed) sets of supermodular maximal l-ideals.*

For a in A and M in $\mathscr{M}(A)$, we shall write $M(a)$ for the image of a in the l-simple f-ring A/M. Note

6.5. *If A is an f-ring, $M_0 \in \mathscr{M}(A)$, $\mathscr{U}$ is a neighborhood of M_0, and t is an element of A/M_0, then there exists a in A such that $M_0(a) = t$ but $M(a) = 0$ for all M in $\mathscr{M}(A) - \mathscr{U}$.*

The proof is straightforward.

We prove next

6.6. LEMMA. *If A is an S-semisimple f-ring, then $\mathscr{M}(A)$ is compact if and only if A has a superunit.*

Proof. By 6.1, we need only prove the necessity. Assume that A does not have a superunit, and let $\mathscr{I}$ denote the family of all supermodular l-ideals of A. We will show that the family $\mathscr{F} = \{h(I^\perp) : I \in \mathscr{I})\}$ of closed sets has the finite intersection property, and has empty intersection.

For any $I \in \mathscr{I}$, $I \cap I^\perp = 0$; so every $M \in \mathscr{M}(A)$ contains I or $I^\perp$ by 5.2. Since A is S-semisimple, and has no superunit, at least one such M contains $I^\perp$; so $h(I^\perp)$ is nonempty for all $I \in \mathscr{I}$. Since $(I \cap J)^\perp \supset I^\perp + J^\perp$ for all $I, J, \mathscr{F}$ has the finite intersection property. Finally, let $M \in \mathscr{M}(A)$. By 6.1, then is an $I \in \mathscr{I}$ such that $h(I)$ is a neighborhood of M, so, by 6.5, $I^\perp$ is not contained in M. Hence the intersection of all the elements of $\mathscr{E}$ is empty.

6.7. THEOREM. *Let A be an S-semisimple f-ring.*

(i) *An l-ideal of A is a supermodular direct summand if and only if it is the kernel of a compact open subset of $\mathscr{M}(A)$.*

(ii) *An l-ideal of A is a polar ideal if and only if it is the kernel of the closure of an open subset of $\mathscr{M}(A)$.*

Proof. In (i), necessity is clear.

Conversely, if $\mathscr{U}$ and $\mathscr{M}(A) - \mathscr{U}$ are open and closed sets with kernels H and K, then, since A is S-semisimple, $H \cap K = 0$. If $M \in \mathscr{M}(A)$ contains both H and K, then M must be in $h(H) \cap h(K) = \mathscr{U} \cap (\mathscr{M}(A) - \mathscr{U})$, which is empty. Hence, if $H + K$ is a proper ideal, any maximal l-ideal L containing it fails to be supermodular. It follows that A/H has no superunit, whence by 6.6, $\mathscr{U}$ is not compact. This proves (i).

Similarly, if H and $H^\perp$ are polar ideals, then $y \in H^\perp$ if and only if the open set of all M for which $|M(y)| > 0$ is contained in the hull of H; and symmetrically. Hence $H^\perp$ and H are kernels of their hulls. Moreover, the hull of $H^\perp$ is the complement of the union of the open sets just described, which by 6.5 is the complement of the interior of the hull of H. The complement of the interior of a closed set is the closure of its own interior, and the argument applies to H as well. Conversely, if $\mathscr{V} \subset \mathscr{M}(A)$ is the closure of an open set $\mathscr{U}$, and H is the kernel of $\mathscr{V}$, then 6.5 shows that there are elements of $H^\perp$ taking nonzero values at each point of $\mathscr{U}$, so that $H^{\perp\perp}$ can only be H.

REMARK. The supermodular polar ideals, of course, correspond to compact subsets of the structure space that are closures of open sets. There is no such simple description of the unrestricted direct summands.

A ring isomorphism i of an f-ring A onto an f-ring B is called a *reordering*. For the balance of the paper, we discuss properties of certain classes of f-rings that remain invariant under reordering. If A and B are S-semisimple, we call i an *S-semisimple reordering*.

6.8. *Polar ideals and direct summands are preserved by S-semisimple reorderings.*

Proof. Recall that in an l-simple f-ring, $xy = 0$ implies $x = 0$ or $y = 0$. Hence, in any S-semisimple f-ring, $|x| \wedge |y| = 0$ if and only if $xy = 0$. This proves the invariance of polar ideals, and a direct summand is just a polar ideal H such that $H + H^\perp$ is the whole ring.

If i is a reordering of A onto B, a point M of $\mathscr{M}(B)$ is called a *distinguishing point of i* if there is an $a \in A^+$ such that $M(i(a)) < 0$.

6.9. *An S-semisimple reordering is an isomorphism if and only if it has no distinguishing points.*

Proof. If i is an S-semisimple reordering of A onto B without distinguishing points that is not an isomorphism, then $i(a) \in B^+$ for all $a \in A^+$, and $i(c) \in B^+$ for some c not in A^+. But $i(|c|)$ is a different positive element of B which has the same square as $i(c)$, and this is a contradiction. The converse is trivial.

6.10. THEOREM. *The set of distinguishing points of an S-semisimple* *reordering i of A onto B is open. If A and B have superunits, the distinguishing points of i also form a totally disconnected set.*

Proof. The set $\mathscr{D}$ of distinguishing points of i is the union of $\{M \in \mathscr{M}(B) : M(i(a)) < 0\}$ as a ranges over A^+. Thus, $\mathscr{D}$ is open.

Now, let e be a superunit for A, let $M \in \mathscr{D}$, and let N be any other point of $\mathscr{M}(B)$. Then, for some $a \in A^+$, $M(i(a)) < 0$. We will show next that there is a $z \in A^+$ which is in no proper l-ideal of A, such that $M(i(z)) < 0$ and $N(i(z)) > 0$.

Note that $M(i(e^2)) \geqq 0 > M(i(a))$. Since B/M is l-simple, some multiple t of $-M(i(a))$ exceeds $M(i(e^2))$. Since B/M has a superunit, the multiplier can be a square element of B. Applying i^{-1}, we conclude there is an element y of A^+ of the form $x^2a + e^2$. By adding to it a suitable square that lies in M, we obtain a z satisfying all the desired conditions.

Then $i(z)$ has nonzero positive part u, and nonzero negative part v, since $N(u) > 0$ and $M(v) > 0$. Let $g = i^{-1}(u)$, and $h = i^{-1}(v)$. Since $z = g - h$, $gh = 0$, and z lies in no proper l-ideal of A, the l-ideals G, H generated by g, h respectively, are a pair of direct summands. Therefore, by 6.8, so are $i(G)$ and $i(H)$, and the hulls of these ideals are open and closed sets separating M and N.

It is well known [3, p. 174] that every compact totally disconnected space is zero-dimensional (i.e., has a base of open and closed sets.) Hence the same is true for locally compact spaces. It follows that a union of open totally disconnected subspaces of a locally compact space is totally disconnected. This will be used in establishing

6.11. COROLLARY. *The set of distinguishing points $\mathscr{D}$ for any S-semisimple reordering i of A onto B is open and has a dense open subset that is totally disconnected.*

Proof. By 6.10, $\mathscr{D}$ is open. Since, by 6.1, $\mathscr{M}(B)$ is locally compact, as remarked above, the union $\mathscr{E}$ of all the open totally disconnected subsets of $\mathscr{D}$ is open and totally disconnected. To prove that $\mathscr{E}$ is dense in $\mathscr{D}$, it suffices to show that there is no open set in $\mathscr{D}$ whose closure $\mathscr{Q}$ is contained in $\mathscr{D} - \mathscr{E}$.

If there were, the kernel of $\mathscr{Q}$ would be a proper polar ideal, which, by 5.7, would be contained in a supermodular polar ideal K. By 6.8, $i^{-1}(K)$ is a polar ideal I in A, which also is contained in a supermodular polar ideal J. Then, by 6.8, $i(J) = L$ is still polar, and since it contains K, still supermodular. Thus i induces a reordering j of A/J onto B/L. (Note that both of these are S-semisimple and have superunits.) Since $h(L) \subset \mathscr{Q} \subset \mathscr{D}$, every point of $h(L)$ is distinguishing for j. So, by 6.10, $h(L)$ is totally disconnected. By 6.7, the interior of $h(L)$ is nonempty; a contradiction.

Since $\mathscr{E}$ is open in the locally compact space $\mathscr{M}(B)$, it is locally compact, and hence is zero dimensional. Thus, every point of E has a compact open neighborhood. Hence, we have established

6.12. COROLLARY. *An S-semisimple f-ring which is not totally ordered and has no direct sum decomposition admits no S-semisimple reorderings except isomorphisms.*

6.13. THEOREM. *An S-semisimple reordering i of A onto B, where A and B have superunits, induces a homeomorphism of $\mathscr{M}(B)$ onto $\mathscr{M}(A)$.*

Proof. Note first that if M is a nondistinguishing point of i, then $i^{-1}(M)$ is an l-ideal of A. For otherwise, there would be a y in $i^{-1}(M)$ and an x not in $i^{-1}(M)$ with $0 \leqq x \leqq y$. If $M(i(x)) > 0$, then $M(i(y-x)) = M(i(y)) - M(i(x)) = -M(i(x)) < 0$, although $y - x \geqq 0$; so M is distinguishing. The same conclusion holds if $M(i(x)) < 0$, since $x \geqq 0$.

Moreover, i maps $A/i^{-1}(M)$ isomorphically onto B/M. Hence $i^{-1}(M)$ is a point of $\mathscr{M}(A)$.

We begin by defining $\phi(M) = i^{-1}(M)$ for all non-distinguishing $M \in \mathscr{M}(B)$.

A distinguishing point M of $\mathscr{M}(B)$ is determined by the compact open sets containing it. Since, by 6.6, $\mathscr{M}(B)$ is compact, these correspond exactly to direct summands contained in the l-ideal M. Evidently the corresponding direct summands in A determine a corresponding point $\phi(M)$ in $\mathscr{M}(A)$.

Omitting the details, the correspondence we have described is one-to-one onto because its inverse can be recovered from the reordering i^{-1}; the correspondence between nondistinguishing points is a homeomorphism because it preserves kernels and hulls; the sets of distinguishing points are homeomorphic because they are totally disconnected locally compact and have the same compact open sets. Finally, a set of distinguishing points M_α has a nondistinguishing point M as a limit point if and only if for any direct summands I_α each contained in M_α, their kernel is contained in M. This condition is invariant under i.

REMARK. We can give a more convenient description of the homeomorphism ϕ of 6.13. We shall show that $\phi(M)$ contains a polar ideal J of A if and only if M contains $i(J)$. Since $\mathscr{M}(A)$ is a Hausdorff space, this will characterize ϕ (by 6.7 (ii)).

Since $\phi(M) = i^{-1}(M)$ if M is nondistinguishing, the assertion is obvious in this case. We note next that M is distinguishing if and only if $\phi(M)$ is distinguishing. Let J be a polar ideal of A such that $i(J)$ is not contained in the distinguishing point M. This means that M is not in $h(i(J))$. By 6.10, M has a compact open neighborhood disjoint from $h(i(J))$. By 6.7 (i), this means there is a direct summand H contained in M such that $H + i(J) = B$. Then $i^{-1}(H) + J = A$. From the construction of ϕ, since M contains the direct summand H, $\phi(M)$ contains $i^{-1}(H)$. Hence $\phi(M) + J = A$. So $\phi(M)$ does not contain J. With the symmetric argument, the proof is complete.

6.14. COROLLARY. *An S-semisimple reordering i of A onto B induces a homeomorphism of a dense open subset of $\mathscr{M}(B)$ onto a dense open subset of $\mathscr{M}(A)$.*

Proof. The proof turns on the polar ideals I of B such that both I and $i^{-1}(I)$ are supermodular. Then for this proof, let us call such an I a *useful* ideal. Every proper polar ideal K of B is contained in a useful ideal; for (by 5.8) K is contained in a supermodular polar ideal J, the polar ideal $i^{-1}(J)$ (6.8) is again contained in a supermodular polar ideal I', and $i(I')$ is useful.

Let $\mathscr{H}$ be the set of all $M \in \mathscr{M}(B)$ which are interior to the hull of some useful ideal I. By definition, $\mathscr{H}$ is open. Since the assertion of the last paragraph is equivalent, by 6.7, to the fact that the closure of every open set contains the hull of a useful ideal, $\mathscr{H}$ is dense.

We shall show that for every $M \in \mathscr{H}$, there is a (unique) $\psi(M)$ in $\mathscr{M}(A)$ which contains exactly those polar ideals of A that are inverse images of polar ideals of B contained in M, and ψ is a homeomorphism. To this end, select a useful ideal I having M interior to its hull. The reordering i of A onto B induces a reordering of $A/i^{-1}(I)$ onto B/I. Both of these f-rings are S-semisimple and have superunits; hence by 6.13, i induces a homeomorphism ϕ of their structure spaces. These structure spaces are the hulls of $i^{-1}(I)$ and I respectively.

We will show next that if M contains a polar ideal J, then M contains $(I + J)^{\perp\perp}$. From this it will follow that $\phi(M)$ contains $i^{-1}((I + J)^{\perp\perp})$, which contains $i^{-1}(J)$. From this, and an application of the same argument to the reordering i^{-1} of B onto A, it will follow that $\phi(M)$ is the desired $\psi(M)$. From the continuity of ϕ and ϕ^{-1}, this will complete the proof that ψ is a homeomorphism.

To prove the assertion, note first that $(I + J)^{\perp} = I^{\perp} \cap J^{\perp}$; hence what we must show is that M contains $(I^{\perp} \cap J^{\perp})^{\perp}$, or that if $x > 0$ (mod M) then for some y in $I^{\perp} \cap J^{\perp}$, $x \wedge y \neq 0$. For this it suffices to find a point $N \in \mathscr{M}(B)$ such that $x > 0$ (mod N), $N + I^{\perp} = B$, and $N + J^{\perp} = B$; for then there are p in $I^{\perp}$ and q in $J^{\perp}$ such that $p \wedge q > 0$ (mod N), and we may put $y = p \wedge q$.

From 6.5, the conditions $N + I^{\perp} = N + J^{\perp} = B$ will be satisfied if the hulls of I and of J are neighborhoods of N. Now the set of all $P \in \mathscr{M}(B)$ such that $x > 0$ (mod P) is a neighborhood $\mathscr{U}$ of M. By hypothesis the hull of I is also a neighborhood of M, and its intersection with $\mathscr{U}$ contains an open neighborhood $\mathscr{V}$ of M. Since the hull of J contains M by hypothesis, and is the closure of an open set by 6.7, $\mathscr{V}$ contains an interior point N of the hull of J. Then the proof is complete.

Our final example shows that S-semisimple reorderings need not preserve compactness of the structure space. Consider the subfield F of the ordered real field consisting of all $a + b\sqrt{2}$ with a, b rational numbers. Let A be the f-ring of all sequences of elements of F converging to 0, ordered termwise. Then A has no superunit, whence $\mathscr{M}(A)$ is not compact by 6.6. But one can reorder A by applying the automorphism $a + b\sqrt{2} \to a - b\sqrt{2}$ of F to get an f-ring B. Let $\{s_k\}$, $\{t_k\}$ be sequences of integers such that $\{s_k - t_k\sqrt{2}\}$ converges to 0. It is easily verified that the sequence $\{s_k + t_k\sqrt{2}\}$ is a superunit for B, whence by 6.6 $\mathscr{M}(B)$ is compact.

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