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Book Review: Realizing Reason: A Narrative of Truth and Knowing by Danielle Macbeth

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Synopsis

This review examines Danielle Macbeth's novel and compelling account of the formal languages of mathematics, from Euclid's geometrical diagrams to the algebraic equations of Descartes and the differential equations of Newton and Leibniz, to the much more abstract language of Galois, Bolzano, and Riemann. She argues that the practice of those 19th century mathematicians, reasoning deductively from abstract concepts like *group* and *manifold*, inspired the philosophical logician Gottlob Frege, whose *Begriffsschrift* captures the procedures of those who reasoned in concepts. However, his way of formalizing mathematical reasoning was obscured by the success of Bertrand Russell and Alfred North Whitehead's *Principia Mathematica*, which gave rise to modern predicate logic. Macbeth argues that the *Begriffsschrift* should be re-examined and revived.

Realizing Reason: A Narrative of Truth and Knowing.
By Danielle Macbeth, Oxford University Press, New York, 2014.
(Hardcover US$99.00, ISBN: 978-0-19-870475-1. 512 pages.)

A Brief Chapter by Chapter Account of the Book

In this book, Danielle Macbeth offers a novel and compelling account of the formal languages of mathematics, from Euclid's geometrical diagrams to the algebraic equations of Descartes and the differential equations of Newton and Leibniz, to the much more abstract language of Galois, Bolzano, and Riemann. The practice of those 19th century mathematicians, reasoning deductively from abstract concepts like group' and manifold', she argues, inspired the philosophical logician Gottlob Frege, whose *Begriffsschrift* captures the procedures of those who reasoned in concepts. She notes that his way of formalizing mathematical reasoning was left aside, in the wake of the success of Bertrand Russell and Alfred North Whitehead's *Principia Mathematica*, which gave rise to modern predicate logic, and argues that it should be revived and re-employed. Thus the first part of her book traces the development (she calls it a Hegelian dialectic, after the pattern made famous by the great 19th century German philosopher Hegel: thesis, antithesis, synthesis) of mathematical languages, and the second part is a strong defense of the usefulness and philosophical interest of Frege's *Begriffsschrift*.

Chapter 1, "Where We Begin," moves from a philosophical reflection on how human cognition emerged from the natural world, to end with a brief account of aspects of Aristotle's epistemology. Aristotle's theory of knowledge starts from sense perception (and takes it as giving reliable access to truth) and so leads to a metaphysics of substance and reliance on natural language; it is oriented towards objects. Natural language, Macbeth argues, "is at once inferentially articulated and essentially referential. It is also essentially sensory..." (page 52). It is also historical, changing with changes in culture and the natural environment, and it is vague and often ambiguous, features that serve human diplomacy, imagination and ruse. Moreover, as she asserts a bit later, "the symbolic languages of mathematics are quite unlike natural languages. Neither narrative nor sensory they are special purpose instruments designed for particular purposes and useless for others. They are not constitutively social and historical, and they have no inherent tendency to change with use. Unlike natural languages, they also can (at least in some cases) be used merely mechanically, without understanding, used, that is, not as languages at all but as useful calculating devices" (page 108).

To illustrate the contrast between natural language and the specialized idioms of mathematics, Macbeth points out in Chapter 2, "Ancient Greek Diagrammatic Practice," that while you can describe a procedure for finding (say) a complicated long division problem in English, you cannot calculate the result in English: you have to use Indo-Arabic numerals with the base ten, positional notation and the algorithemic procedures we all learned in primary school. Your calculation, written on paper with a pencil, does not describe or report the solution to the problem, but rather performs it—or one might say displays it or embodies it. You are reasoning in a system of signs, in a specially formulated artificial language. Whereas natural languages change with use and historical-cultural conditions, are first and foremost oral, and are amazingly versatile, mathematical languages have no inherent tendency to change with time, are inherently written (we perform them by writing with them, correctly), and are designed for very specific uses. These special idioms make possible reasoning that has something like the rigor of deduction but is also ampliative, extending and adding to what we know.

In the second half of Chapter 2, she goes on to treat the great revolution that Euclidean geometry represents, the emergence of a formal system in which one can reason in diagrams, aside from natural language, with greater power though with strictly constrained scope. Like Aristotle, Euclid works in a mode that is object-oriented. Macbeth writes, "One reasons in the diagram rather than merely on it in Euclidean diagrammatic practice... Because the diagram reformulates the contents of concepts in the way that it does, namely, by combining primitive parts into wholes that are themselves parts of the diagram as a whole, various parts of the diagram can be conceived now this way and now that in an ordered series of steps. One does not in this case merely shift one's gaze in order to see explicitly something that was already implicit in the diagram as drawn... Because what is displayed are the contents of concepts the parts of which can be recombined with parts of other concepts, something new can emerge that was not there even implicitly in that with which one began" (page 105). In her detailed discussion of Proposition I.1 of Euclid's *Elements*, which demonstrates how to construct an equilateral triangle on a given line segment as its base, she shows how the diagram allows for ampliative and yet deductively justified reasoning.

We draw the circle DCB with radius AB and the circle ACE with AB as radius; and this gives us the point C, the opposite vertex of the equilateral triangle with base AB; see Figure 1. Macbeth notes that if we were reasoning in natural language, this problem might seem intractable: "No mere diagram of some drawn circles, however iconic, can justify an inference from a claim about the radii of circles to a claim about a triangle" (pages 92–93). However, working within the diagram, we can read one and the same line segment as belonging to a circle (in the role of one of its radii), indeed to two distinct circles, and as also belonging to a triangle (in the role of one of its sides), and how the proof hinges on this triple import. She concludes, "The demonstration is fruitful, a real extension of our knowledge, for just

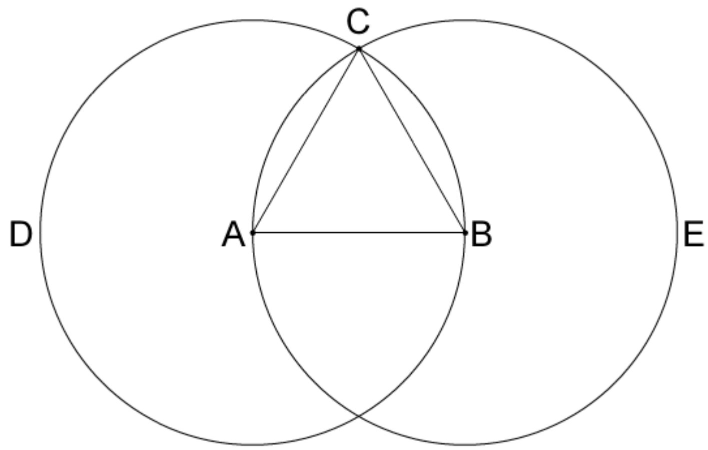

Figure 1: Proposition I.1 of Euclid's Elements, which demonstrates how to construct an equilateral triangle on a given line segment as its base.

this reason: because we are able perceptually to take a part of one whole and combine it with a part of another whole to form an utterly new and hitherto unavailable whole, we are able to discover something that was simply not there, even implicitly, in the materials with which we began" (page 93).

The dialectic shifts with the advent of Descartes' *Geometry*. Euclidean geometry is a geometry of constructions; thus it is fixed on constructible objects like line segments, circles, and triangles. The next step in the dialectic, according to Macbeth, is explored in Chapter 3, "A New World Order," the development of analytic geometry in the work of Descartes, using polynomial equations, born of the new algebra, to tackle geometrical problems. While a Greek geometer reasons in the diagram, Descartes reasons in the symbolic language of elementary algebra. Here, Macbeth argues, the mathematician shifts focus from objects to relations, moving up a level of abstractness. She notes that the language of algebra, for Viète, is an uninterpreted calculus: "Because the *logistice speciosa* has no meaning or content of its own, the results that are derivable in it can be interpreted either arithmetically or geometrically" (pages 126–127). By contrast, she argues that for Descartes algebra is a fully meaningful language. "In his mathematical practice... Descartes is not interested in any traditional subject matter but instead in order and measure, that is, in the patterns the objects forming the subject matters of various disciplines exhibit. One such pattern is that displayed by the length of the hypotenuse of a right triangle relative to the lengths of the other two sides, the pattern exhibited in the equation $x = \sqrt{(a^2 + b^2)}$... the whole discussion is effectively moved up a level, from talk about, or better formulation of, relata, that is, objects such as numbers and figures, to talk about, the formulation of, relations, the patterns such objects can be seen to exhibit. The function of the letters 'a' and 'b' in the equation '$x = \sqrt{(a^2 + b^2)}$' is to enable one to exhibit the precise mathematical relation that holds between the length of the hypotenuse of a right triangle and the lengths of the other two sides. The Cartesian geometer is in this way directed not on independently existing objects, whether sensible or intelligible, but instead on the relations, proportions, and patterns that such objects can display" (page 132).

She illustrates her point with an example from the *Geometry*, which Descartes uses to define multiplication as well as division on line lengths; one of his great conceptual innovations was to give up the Greek habit of interpreting multiplication of two line lengths as an area and of three line lengths as a volume; see Figure 2.

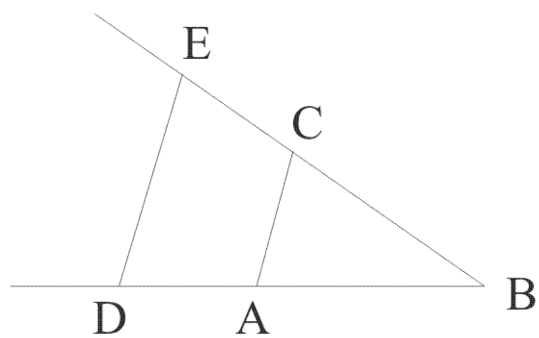

Figure 2: Descartes uses this figure to explain how the multiplication of line segments may result in a line segment.

This new convention allowed him to establish arithmetic for line segments. Macbeth notes, "Descartes claims... that the geometrical relationship between the line lengths that is presented in the above diagram is not merely analogous to but an alternative expression of that which is expressed symbolically [here, proportions and elsewhere algebraic equations]. It is the relationship itself, which is common to the two means of expression, that is displayed both in the diagram and in symbols. .. What is exhibited both in the diagram and in the symbolic language is not an object (that is, a number or geometrical figure) but a relation that objects can stand in" (page 128). Aristotle's focus on the unity of substances shifts to a focus on the unity of the law-governed function.

In Chapter 4, Macbeth traces "Kant's Critical Turn," reminding us that while Kant asserted that knowledge is objective because it is knowledge of objects, the unity of the objects of mathematics is not given, but constructed by the mind, the expression of a transcendental synthesis. In mathematics, what are given to sensibility are only the pure forms of space and time, with respect to which we employ the pure concepts of the understanding to construct the objects of mathematics. But this means that Kant cannot really account for the application of mathematics in the exact sciences, and in particular in physics. How do we know that things as they appear to us, can be the key to the way things are in themselves? (In Kantian terms, we cannot.) And how can we understand mathematics as self-correcting?

The title of Chapter 5 is "Mathematics Transformed, Again." Taking as her examples the development of non-Euclidean geometry, projective geometry, the development of abstract algebra, and Riemann's work on manifolds, Macbeth notes that mathematics itself led away from and belied Kant's reliance on intuition, the pure forms of space and time which he identified with three-dimensional Euclidean (flat) geometry and the number line, supposed to be one-dimensional, directed and continuous. The mathematicians of 19$^{\text{th}}$ century Germany and France, she argues, began to work at a higher level of abstraction, eschewing calculation to work rather with concepts. She writes, "Developments within the discipline … reveal new concepts, and indeed new kinds of concepts, which once they are available can be refined and their consequences explored." The concept of a group or of a topological space is a kind of hypothesis that allows for the exploration of new territory; thus they are in a sense provisional and drive a novel mathematics. Reasoning from concepts, she argues, is ampliative (increases knowledge) and may be viewed as, like physics, self-correcting. She gives as an example Galois's explanation of why there is no general solution (no algorithm) for quintic (fifth degree) polynomial equations over the rationals $\mathbb{Q}$: the Galois group associated with each quintic equation is in general not solvable, that it, it is either not Abelian or does not have an Abelian subgroup. And this result generalizes to higher degree polynomial equations.

I think that Macbeth is right to argue, along with a variety of other contemporary philosophers, that late 19th century mathematics is accurately characterized "as rejecting a computational, algebraic approach in favor of a more conceptual one." And this leads to the very interesting observation, in Chapter 6, "Mathematics and Language," and Chapter 7, "Reasoning in Frege's *Begriffsschrift*," that Frege's project is best understood as an attempt to formalize this novel kind of reasoning, reasoning in concepts. Her exposition of the centrally important Theorem 133 is a model of clarity and recommends further study of Frege's way of formalizing reasoning. It proves that "if f is a single-valued function and y and m both follow x in the f-sequence, then either m follows y or y belongs to the f-sequence beginning with m" (pages 348–351). This means that it may be worth our while, as philosophical logicians interested in mathematical reasoning, and as mathematics educators, to take up the *Begriffsschrift* itself, which differs markedly from Russell and Whitehead's predicate logic, and re-investigate its expressive powers.

She makes a good case for this proposal in the last chapters, Chapters 8 and 9. In Chapter 8, "Truth and Knowledge in Mathematics," she argues that Frege gives us a good way to understand ampliative proof, which surpasses merely explicative proof by playing a positive role in research and education. In ampliative proof, we see that concepts are not merely a sum of their parts, but have an important intrinsic unity; and that the most fruitful concepts arrive with "inference licenses," which Frege's system highlights, for further philosophical scrutiny. Good examples are the concepts of *being hereditary in a sequence* and *prime*. Thus in her view, what is essential to the notation is the fact that primitive signs have no designation independent of a context of use, enabling one to exhibit the contents of concepts. I would add that another interesting way of looking at the *Begriffsschrift* is in terms of its close relation to the theory of trees, graph theory, and of reasoning in terms of the tree notation offered by Dale Jacquette in his textbook *Symbolic Logic* [3]. Emiliano Ippoliti too has written many interesting essays about reasoning in graph notation, which could be developed with reference to a re-investigation and re-implementation of the *Begriffsschrift*; see for instance [2].

In Chapter 9, "The View from Here," she argues that the same kind of reasoning from concepts that we see in the work of Klein and Riemann can also be seen in the work of Einstein. On her view, the dynamical space-time that emerges from Special and General Relativity constitute a physics of mathematically defined structures liberated from the conventional objects of classical physics, disclosing a fundamental aspect of reality to pure thought. Using Frege's distinction between *Sinn* and *Bedeutung*, she notes that Fregean sense (*Sinn*) is the way we achieve direct cognitive contact with reality. When we employ natural language in this endeavor, what we encounter is everyday life, the world of living things with their natures and powers. When we employ the language of mathematics, what we encounter is truth, the same for everyone. The main thesis of her book is that Frege's *Begriffsschrift* is the best vehicle for the kind of reasoning that arrives at truth.

A Brief Critique of the Project

Though I find Macbeth's dialectical account of the history of Western mathematics and her defense of the philosophical and mathematical interest of Frege's *Begriffsschrift* compelling, I also find her claims too strong. A moderation of her claims and recommendations would make them more plausible, and also perhaps better understood and more widely implemented. For example, in Chapter 5, Macbeth endorses a very strong version of the claim about the distinctive nature of reasoning in concepts in 19$^{\text{th}}$ century mathematics. She writes, "The whole of mathematics was to be set on a new, purely conceptual foundation. Not only are objects of no concern to mathematics, even particular relations are to be set aside. Nothing intuitive is to be allowed in this new, purified mathematics; and insofar as it succeeds in banishing all intuition, not only our intuitive understanding of space and time (whether ancient or early modern) but also our constructive paper-and-pencil models of mathematical reasoning..."

She adds, "Although the science of mathematics begins with the visual appearance of things, it then moves on, first to the symbolic formulae that underlie and explain those appearances, and then finally to the fundamental and essential properties of functions that underlie and explain the formulae in which they are first expressed." Thus for example, "functions are not to be displayed, exhibited in formulae, but instead described, thought through concepts, and the science of mathematics is in this way to achieve an understanding of its subject matter that is intrinsic to that subject matter, purified of all the contingencies that attach to the ways that subject matter manifests or reveals itself to us, whether in drawn diagrams or in the symbolic language of arithmetic and algebra. That at least is the aspiration" (pages 220–221). What follows from this extreme version is that Frege's *Begriffsschrift*, the purest expression of thinking logically in concepts, should be widely adopted, by philosophers, mathematicians, and educators. Indeed, she seems sometimes to be arguing that it should supplant other ways of expressing mathematical ideas, in research as well as in mathematical education.

For example, in Chapter 8, having reminded us in very interesting detail (in Chapter 7) how the *Begriffsschrift* works, she observes, "Because everything is made maximally explicit in reasoning in Frege's system—the starting points, the rules governing inferences, and each individual inference from the starting points to the desired conclusion—errors are more easily detected, and where no errors are found, although this is no guarantee that problems will not later come to light, one has very reasonable assurance that the theorem is true and good reason to think one has discovered the nature of its grounds, whether in logic alone or in the basic laws of some special science. It is in just this that the rationality of the endeavor consists. Because it makes everything maximally explicit and hence available to critically reflective scrutiny and criticism, the method gives one cognitive control over the domain of inquiry" (page 381). Such reasoning, she adds, is both ampliative and non-foundationalist.

My objections to these formulations can be explained most clearly by beginning with the example of Descartes, before moving to the 19$^{\text{th}}$ century. On my reading of Descartes' *Geometry*, the algebra does not replace the geometry, but rather is superimposed upon it [1, Chapters 1–2]. Most of Descartes' reasonings require us to read his diagrams both as about relations among line lengths and as about Euclidean objects like triangles and circles. In the diagram from the Geometry given above, for example, Descartes makes use of Euclidean results about similar triangles to arrive at his conclusions. (Note that whereas polynomial equations are very good at expressing circles ($x^2 + y^2 = 1$, for example), they do not express triangles.) The ampliative thrust of Cartesian geometry is to lead from the investigation of the conic sections to algebraic curves of higher degree, including some of the cubic curves. When Leibniz and the Bernoullis extend Descartes' notation and methods, the investigation extends to transcendental curves. But the tractrix and the catenary are no more and no less mathematical objects (and indeed constructible objects) than a cubic curve or an ellipse or a circle. It may be helpful to look at them as relations; but it is also helpful to recall their intrinsic unity as shapes. They can be expressed by a diagram, a souped-up polynomial equation and a differential equation: but all of these expressions work together in the mathematicians' research. As Henk Bos points out, Descartes checks all the conclusions he arrives at via equations back against the diagrams. So in terms of Descartes' practice as a geometer, it is inaccurate to say that circles for him are objects whose essences are given by equations that are grasped by the pure intellect independent of any images.

And, I would argue, when we come to the 19^{th} century, an extra level is added to the reasoning. The use of concepts of a manifold, a topological space, a group, ring or field, is added to the use of earlier geometrical and algebraic representations. Such reasoning in concepts is a powerful addition to the repertoire of research mathematicians, but it does not supplant or replace reasoning at other levels. Rather, it reorganizes earlier reasoning, and enriches it by that extra layer. One of my favorite developments in the 19^{th} century is Gauss's reformulation of the complex numbers in terms of the Euclidean plane; this identification got complex analysis off the ground, and it flew. In this novel conceptual context, which combines Euclidean geometry and analysis, the circle comes to harbor a surprising collection of previously unsuspected things, the nth roots of unity, that is, the complex numbers satisfying the equation $z^n = 1$. For this discovery to take place, the mathematical community had to embed the study of the integers and the reals in the study of the complex numbers (algebraic number theory on the one hand and complex analysis on the other do this in quite distinct ways), and further recast that study in terms of the complex plane, so that the resources of Euclidean geometry, projective geometry, and eventually non-Euclidean geometry could be used to frame, unify, and solve clusters of problems. On the complex plane, the nth roots of unity are the vertices of a regular n-polygon inscribed in the unit circle $|z| = 1$, where $\sqrt{-1} = i$, $z = x + iy$ and $|z| = x^2 + y^2$. We can also 'read off' the unit circle, understood this way, as $x = \cos\theta$ and $y = \sin\theta$. Then Euler's famous formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is valid for all real θ, along with the representation of the nth roots of unity ζ_n as $e^{2\pi i\theta k/n}$.

We can go even further. The field $\mathbb{Q}[i]$ is an algebraic extension of the field of rational numbers $\mathbb{Q}$, obtained by adjoining the square root of -1 ($\sqrt{-1} = i$) to the rationals. Its elements are of the form $a + bi$ where a and b are rational numbers, which can be added, subtracted, and multiplied according to the usual rules of arithmetic, augmented by the equation $i^2 = -1$; and a bit of computation shows that every element $a + bi$ is invertible. Within this field $\mathbb{Q}[i]$ we can locate the analogue of the integers $\mathbb{Z}$ within $\mathbb{Q}$: it is $\mathbb{Z}[i]$, the Gaussian integers, whose elements are of the form $a + bi$ where a and b are integers. Like $\mathbb{Z}$, $\mathbb{Z}[i]$ enjoys unique factorization (that is, each nonzero element of $\mathbb{Z}[i]$ can be expressed as a unique (up to units) product of primes), but its primes are different from those of $\mathbb{Z}$; and instead of two units it has four: 1, i, -1, and $-i$. To be a prime in $\mathbb{Z}[i]$, $a + bi$ must satisfy these conditions: either a is 0 and b is a prime in $\mathbb{Z}$ and is congruent to 3(mod 4); or b is 0 and a is a prime in $\mathbb{Z}$ and is congruent to 3(mod 4); or neither a nor b are 0 and $a^2 + b^2$ is a prime in $\mathbb{Z}$ and is not congruent to 3(mod 4). When we use the Euclidean plane as a model for $\mathbb{C}$, the units are then modeled by the square with endpoints 1, i, -1 and $-i$. This suggests the generalization that models the set of nth roots of unity as vertices of regular n-polygons centered at 0 on the complex plane, with one vertex at 1. This nesting of all such polygons within the circle on the complex plane provides a kind of visual index for the generalization from $\mathbb{Q}[i]$ to other algebraic fields called cyclotomic fields, where an nth root of unity (ζ_n) is adjoined to $\mathbb{Q}$. ($Q[i]$ is $Q[\zeta_4]$). Those roots of unity, regarded as vertices, suggest the notion of symmetry, and in that light may be studied in terms of groups of symmetries. For each cyclotomic field $\mathbb{Q}[\zeta_n]$ there is a group of automorphisms that permute the roots of unity while mapping $\mathbb{Q}$ to itself; this is the Galois group, $\mathrm{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q})$. In this interesting development, we find the circle and the regular polygons, straight out of Euclid, and quite a bit of algebra, and the use of high-level concepts like automorphism, Galois group, cyclotomic field, and algebraic number field. The abstract concepts play an essential role here; but the older objects and concepts persist alongside them.

Finally, Frege's *Begriffsschrift* may, when revived, turn out to have important features as a language designed to exhibit inference, which predicate logic does not have. And we may decide to start using it and teaching it more often, and that use may lead to interesting research and improved mathematics instruction. But every logical system has its strengths and limitations; and no system that is designed to formalize inference, I would argue, can all by itself and at the same time formalize the study of shapes and numbers and spaces and structures. Mathematics, and logic as well, need a variety of ways of expressing and investigating the venerable constructed objects and the novel conceptual unities that drive mathematical research. I prefer to talk about the fruitfulness of the conceptual tension between reference to problematic objects, and analysis, the search for conditions of intelligibility of problematic objects, or the search for conditions of solvability of the problems in which they occur. The vocabulary is Leibniz's, and my case studies are also drawn from mathematics and cosmology; so it seems that there may be another dialectic here, but it will have to be spelled out in another context.

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