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SUMMARY
The author gives a personal history of experiences in finding the square root of a number by the "do it thus" method—from algorithm to table to calculator. Why each procedure works is elucidated, making liberal use of the history of mathematics.

ODE TO THE SQUARE ROOT: A HISTORICAL JOURNEY

Just as the scribe Ahmes in 1650 B.C. would direct the reader of the Rhind Papyrus to "Do it thus"[1] in solving a problem, so would my teachers instruct me to find the square root of a number in the secondary schools of the 1940's. It was an elaborate, laborious procedure, performed by rote, one mysterious step after the other.

In college we abandoned that square root algorithm and turned to tables. I still own my copy of "Mathematical Tables from the Handbook of Chemistry and Physics,"[2] which also contained trigonometric and logarithmic tables, tables of squares, cubes, cube roots, reciprocals and factorials, interest tables and pages of all kinds of mathematical formulas.

Fresh out of college in the late 40's, and wanting to work in the "real world" (as opposed to the academic world), I became a junior mathematician for a company that manufactured an early analogue computer. I was assigned to calculate the numerical solution of a differential equation describing the motion of a guided missile. To find the value of a trigonometric function correct to ten places, I used the giant books of tables prepared by mathematicians hired by the Works Progress Administration (WPA) during the depression. But to find the square root of a number correct to ten places I was directed to use Newton's Method. The directions given were in the style of the Rind Papyrus: "Do it thus." No reference was given to Newton's iterative formula. Only the algorithm, sometimes called the divide-and-average method,[3] was prescribed. Fortunately, I had at my disposal large electromechanical desk calculators (Frieden, Marchant, Monroe) capable of performing division, as well as multiplication, addition and subtraction.

What a relief it was in the 60's to have access to the electronic handheld scientific calculator to perform these arithmetic operations and soon after to just press a key to get the square root of any positive real number.

Now I am old and gray and have access to the graphing calculator, to the computer, and I can surf the Internet. To find the square root of a number, or its cube root or any root, is a trivial procedure—and I'm happy about it.

How and Why the Square Root Algorithm Works

The square root algorithm taught in the 40's was taught in Victorian times.[4] More than two thousand years ago the Greeks used a similar method. Basic to both methods is Proposition 4 in Book II of Euclid's Elements: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments: (See Fig. 1).[5] Since this proposition, like all fourteen propositions in Book II, can be interpreted algebraically, Euclid's diagram has been given an algebraic interpretation, the identity $(a+x)^2 = a^2 + 2ax + x^2$.

To find the square root of n we use a trial and error process. Let a represent the first digit in the square root of n, where a is in the place held by the highest power of ten in the square root. Now we use the identity to find x, by dividing n-a^2 by $2a$, yielding x as a quotient, and at the same time ascertaining that $2ax+x^2$ be less than n-a^2. Suppose the highest possible value of x satisfying the condition is b, then $2ab+b^2$ would be subtracted from the first remainder n-a^2 and from the second remainder left a third digit in the square root would be found in the same way.[6]

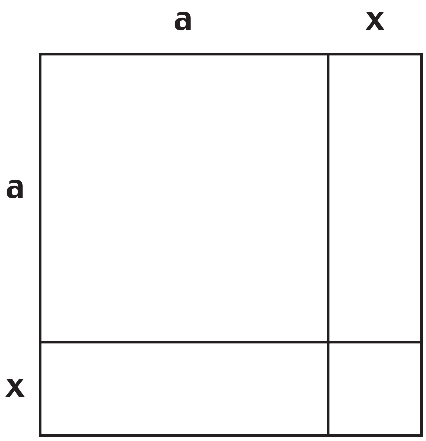

Figure 1

Suppose n = 1225. Guess a = 3, so 3•10 is our first guess of the square root of 1225. If $(3\bullet10)^2(30)^2$ is subtracted from 1225 we get 325, which must contain not only twice the product of 30 and the next digit in the square root, but also the square of the next digit. Now twice 30 is 60, and dividing 325 by 60 suggests 5 as the next digit in the square root. This happens to be exactly what we need, since $(2\bullet30\bullet5)+5^2 = 325$. See Fig. 2.

In a typical Victorian text[4], the algorithm is given without a geometric explanation:

1. Designate in the given number n "periods" of two digits each, counting from the decimal point toward the left and the right.
2. Find the greatest square number in the most left-hand period, and write its square root for the first digit in the square root of n. Subtract the square number from the left-hand period, and to the remainder bring down the next period providing a dividend.
3. At the left of the dividend write twice the first digit in the square root of n, for a trial divisor. Divide the dividend, exclusive of its right-hand digit, by the trial divisor, and write the quotient for the next trial digit in the square root of n.
4. Annex the trial digit of the square root of n to the trial divisor for a complete divisor. Multiply the complete divisor by the trial digit in the square root of n, subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.
5. So far there are two digits in the square root of n. Double this number and use as the next trial divisor, and proceed as before.

As an example, find the square root of 540577.8576 . See Fig. 3.

Using Tables of Square Roots

The square root table (from the "Handbook of Chemistry and Physics") lists the square roots of a positive integer n from 1 to 1000, correct to seven significant figures. Since the square roots of 10n are also given in the table, values of the square roots of numbers from 1 to 10,000 can be found directly.

For the square roots of numbers above and below this range, a simple adjustment can be made. For example,

```
        3  5
      √1225
         9
  65 | 325
       325
```

Figure 2

```
         7   3   5.  2   4
      √540577.8576
        49
  143 |  505
         429
  1465 |  7677
          7325
  14702 |  35285
           29404
  147044 |  588176
            588176
```

Figure 3

$\sqrt{10.268} = \frac{1}{100}\sqrt{10 \bullet 268}$. The tabular value for the square root of $10n$, for $n = 268$, is 51.76872, so the desired root is .5176872.

How and Why Newton's Method Works

The divide-and-average method, alias Newton's Method, is a common sense algorithm. Let's say we must find the square root of 125. Make a guess; say it's 11.1. Divide 125 by 11.1 and get a quotient 11.26126126. Take the average of 11.1 and 11.26126126, which yields 11.18063063 and let this be the next trial divisor. Now 125 divided by 11.18063063 is 11.18004915. Take the average and let this be the next trial divisor. Continue in this manner until the quotient is equal to the divisor, which is the square root of 125, correct to ten significant figures, 11.18033989.

Newton's Method generally is an iterative procedure used to approximate a solution of an equation $f(x) = 0$. It makes use of a corollary to the Intermediate Value Theorem in differential calculus: "If $f(a)$ denotes a function continuous on a closed interval $[a,b]$ and if $f(a)$ and $f(b)$ have opposite algebraic signs, then there exists some value of x between a and b for which $f(x) = 0$."[7] This means that there is at least one solution of $f(x) = 0$ in the interval (a,b).

Suppose f is differentiable and suppose r represents a solution of $f(x) = 0$. Then the graph of f crosses the x-axis at $x = r$ (See Fig. 4). Examining the graph, we approximate r. Our first guess is x_0. If $f(x_0) = 0$, then usually a better approximation to r can be made by moving along the tangent line to $y = f(x)$ at $x = x_0$, to where the tangent line crosses the x-axis at $x = x_1$.

Slope of line $= f'(x_0) = f(x_0)/(x_0-x_1)$.
Solving for x_1, we get $x_1 = x_0-f(x_0)/f'(x_0)$.

Repeating the procedure at the point $(x_1, f(x_1)$ and observing where the second tangent line crosses the x-axis, yields $f'(x_1) = f(x_1)/(x_1-x_2)$.
Solving for x_2, we get $x_2 = x_1-f(x_1)/f'(x_1)$.

If we continue in this manner, in the usual course of events, we get better and better approximations of r: $x_0, x_1, x_2, \ldots$, where $x_{n+1} = x_n-f(x_n)/f'(x_n)$. Of course, the method is not foolproof. Sometimes $f'(x_t) = 0$ so that x_{t+1} can't be calculated because there is division by 0. Sometimes the approximations $x_0, x_1, x_2, \ldots$ do not converge to the solution r.

Let's see how the divide-and-average method is really Newton's method. We are solving $x^2-125 = 0$, So $f(x) = x^2-125$. $f'(x) = 2x$.

Let $x_0 = 11.1$, then $x_1 =$ 11.1-(f(11.1)/f'(11.1)) = 11.1-((123.21 -125)/22.2) = 11.18063063. Now x_2 =11.18063063-((125.0065013-125)/22.36126126) = 11.18033989. Then x_3 turns out also to be 11.18033989, so we have the square root of 125.

Last Thoughts

I'm not sorry that we no longer must do hideous calculations to find the square root of a number. Looking back at past history makes us more informed and appreciative, too.

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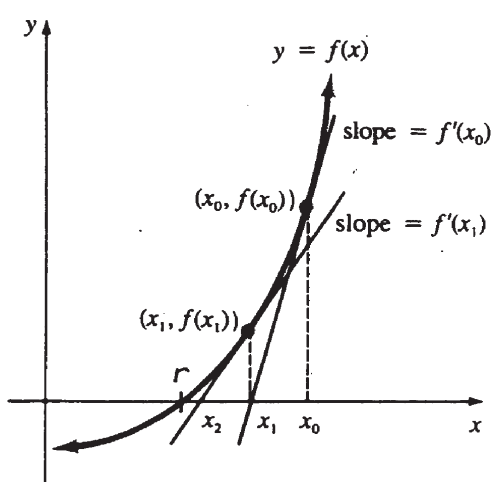

Figure 4