

On French Pudding and a German Mathematician

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On French Pudding and a German Mathematician

Cover Page Footnote

A special thanks to Professor Gizem Karaali for her guidance and instruction in my first foray into the world of Analysis back in 2011, and for all of her support and encouragement since then.

On French Pudding and a German Mathematician

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Synopsis

At the turn of the 19th century, mathematics developed the rigor that is now considered essential to the field. Mathematicians began going back and proving theorems and statements that had been taken to be true on face value, ensuring the underpinnings of mathematics were solid. This era of mathematics was characterized not only by setting foundations, but also by pushing the boundaries of new ideas. In 1830, Bolzano found an example of a function that was nowhere differentiable, despite being continuous. Thirty years later, Cellier and Riemann each discovered another example of such a pathological function. The first everywhere continuous nowhere differentiable function to be published appeared in *Borchardt's Journal* in 1875. This function was proposed by Karl Weierstrass. His function, in addition to similar examples that followed it, revolutionized the ideas of continuity, differentiability, and limits. Due to these pathological functions, the traditional definitions were rethought and revised. Here we explore some of these pathological functions.

1. A Historical Introduction

“Mathematics” has etymological roots in the Greek word “μαθαινω”, meaning “learn”. Prior to the 19th century, the field was characterized by an intuitive investigation of numbers, quantity, and space. The new pieces of knowledge being discovered and learned were far from rigorous and occasionally even faulty. At the turn of the 19th century, though, Mathematics took a more critical approach to its namesake, to learning. It was Analysis that brought about this rigor by going back and proving theorems and statements that had been taken to be true on face value.

One such assumed truth was Ampère's Theorem, which stated that every continuous function is differentiable except at a few isolated points. Indeed, functions like $y = |x|$, with problematic points of non-differentiability (in this case the corner at $x = 0$) were known to exist. The prospect of a function's being everywhere continuous yet nowhere differentiable, however, seemed absurd. At the close of the 19th century, in 1899, Poincaré said, "A hundred years ago, such a function would have been considered an outrage on common sense." Indeed, it would have been. [5]

The 19th century, however, was an opportune environment for mathematicians to push the limits of their field (pun intended). In 1821, Cauchy published his famous textbook that contained requirements for the rigorization of mathematics, *Cours d'Analyse* [3]. The text begins with a list of questions (when we read between the lines):

- What is a derivative really? Answer: a limit.
- What is an integral really? Answer: a limit.
- What is an infinite series $a_1 + a_2 + a_3 + \cdots$ really? Answer: a limit.
- This leads to: What is a limit? Answer: a number.
- And, finally, the last question: What is a number?

And thus mathematicians like Weierstrass, Heine, Cantor, and others began to rigorously define the foundations of analysis, and thereby, of mathematics [5].

This era of mathematics was not only characterized by a foundational focus, but also by a pushing of the boundaries of new ideas. In 1830, Bolzano found an example of a function that was nowhere differentiable, despite being continuous. Thirty years later, Cellier and Riemann each discovered an example of such a pathological function. The first everywhere continuous nowhere differentiable function to be published appeared in *Borchardt's Journal* in 1875. This function was proposed by Karl Weierstrass. His function, in addition to similar examples that superseded it, revolutionized the ideas of continuity, differentiability, and limits. Due to these pathological functions, the traditional definitions were rethought and revised. [5, 11]

Prior to the revision, the accepted definitions of limits and continuity were those of Cauchy: "... $f(x)$ will be called a *continuous* function, if ... the numerical values of the difference $f(x + \alpha) - f(x)$ decrease indefinitely with those of α ..." [3, page 43]. In 1874, Weierstrass and Bolzano brought the definition of limits to the precise definition we still currently use: "Here we call a quantity y a continuous function of x , if after choosing a quantity ε the existence of δ can be proved, such that for any value between $x_0 - \delta \dots x_0 + \delta$ the corresponding value of y lies between $y_0 - \varepsilon \dots y_0 + \varepsilon$." [5]

Here we have very briefly summarized the history of what is today known as the birth of rigorous analysis. Readers interested in this history will find careful scholarship and much detail in [4]; a more recent exploration of Cauchy's introduction to [3] can be found in [1]. Our focus here is merely mathematical. So without further ado, let us delve into the seemingly mind-bending, logic-defying functions that in fact are truth-disclosing, logical certitudes.

2. Defining Continuity and Differentiability

Not too soon, though! Before delving into specific examples proving the existence of everywhere continuous nowhere differentiable functions, it would be beneficial to review the definitions of continuity and differentiability. Therefore we begin with the basic definition below:

Continuity A function $f : D \rightarrow \mathbb{R}$ is **continuous at** c if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in D$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

In the following we will also need the notion of uniform continuity: For a function to be **uniformly continuous** on some interval D , it must, for a single δ , satisfy the above definition of continuity at c for every point $c \in D$.

Next, we provide the definition of differentiability:

Differentiability Let I be an interval in the domain, and let $c \in I$. The function $f : I \rightarrow \mathbb{R}$ is **differentiable at** c if the following limit (denoted by $f'(c)$) exists:

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

For a function to be **nowhere differentiable**, the above limit must not exist at every point c .

3. Weierstrass' Function (1875)

3.1. Historical Background

In one of his lectures at the Royal Academy in Berlin in 1872, Weierstrass proposed the existence of everywhere continuous nowhere differentiable functions. He said,

“As I know from some pupils of Riemann, he as the first one (around 1861 or earlier) suggested as a counterexample to Ampère’s Theorem [which perhaps could be interpreted as: every continuous function is differentiable except at a few isolated points]; for example, the function R does not satisfy this theorem. Unfortunately, Riemann’s proof was unpublished and, as I think, it is neither in his notes nor in oral transfers...”

Weierstrass then went forth to disprove Ampère’s Theorem with a proof of the existence of continuous, nowhere differentiable functions. The function he used as an example was an infinite sum of cosine functions that formed a fractal. His lecture notes were given to *Borchardt’s Journal* by du Bois-Reymond in 1873, but Weierstrass spent two years attempting to make more progress and remarks. In 1875, the proof was published in almost identical form. [11]

3.2. The Function

The original Weierstrass function was:

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n x), \quad (1)$$

for $0 < a < 1$, $b > 1$ is an odd integer, and $ab > 1 + \frac{3\pi}{2}$.

The many restrictions placed on the variables were simplified by Hardy [6] in 1916, but the function retained all of its properties and general form (see Figure 1 on the following page).

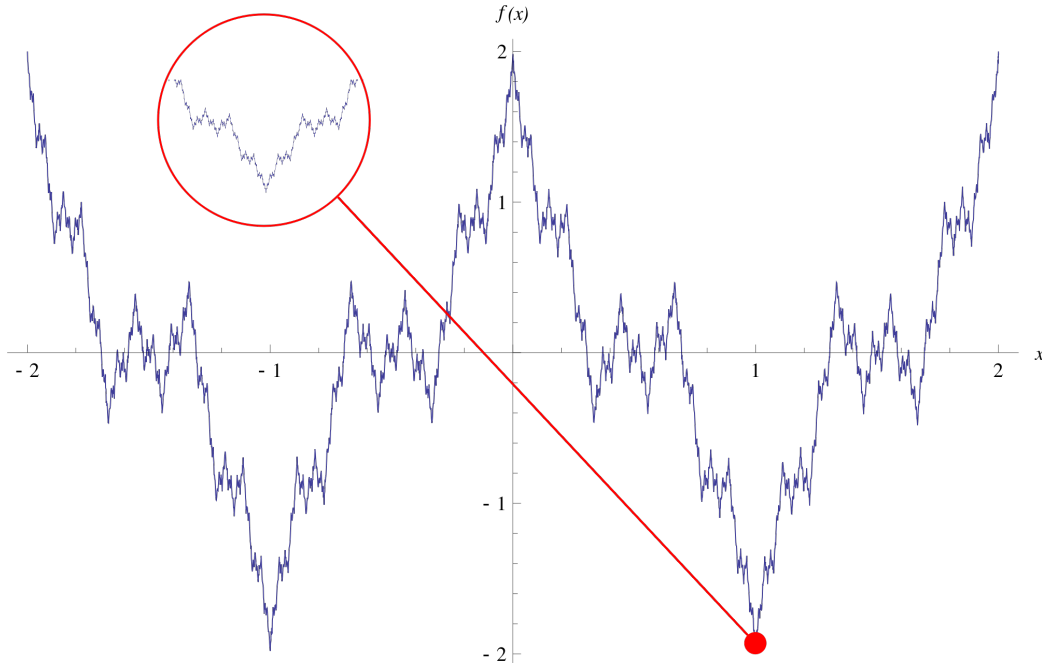


Figure 1: Plot of the Weierstrass function. A section of the plot is zoomed in on to illustrate the fractal nature of the function. Public domain image created by Eeyore22 on 26 October 2008, available at https://en.wikipedia.org/wiki/Weierstrass_function, accessed last on June 17, 2017.

3.3. Proof of Uniform Continuity and Nowhere Differentiability

Before proving the continuity of the function f , we must delve into some of Weierstrass's other findings. In particular, we will describe the Weierstrass M-test. An accessible reference for this material is [2].

We begin with a definition.

Uniform Convergence Consider a sequence f_1, f_2, \dots of functions of the form $f_n : D \rightarrow \mathbb{R}$ such that, for each $x \in D$, the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges. This series is *uniformly convergent* to f on D if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in D$,

$$|(f_1(x) + f_2(x) + \dots + f_n(x)) - f(x)| < \varepsilon.$$

We are now ready for:

Lemma 1 (Weierstrass M-Test). *Let f_1, f_2, f_3, \dots be a sequence of functions $f_n : D \rightarrow \mathbb{R}$ and let M be a constant. If $\sum a_n$ is a convergent series of nonnegative terms such that for all $n \in \mathbb{N}$ and $x \in D$, $|f_n(x)| \leq Ma_n$, then the series $\sum_n f_n$ is uniformly convergent on D .*

Proof. Given $\varepsilon > 0$, choose an $N \in \mathbb{N}$ such that $M \sum_{n=N+1}^{\infty} a_n = M(a_{N+1} + a_{N+2} + \dots) < \varepsilon$. Then for $n > N$,

$$\sum_n f_n = |(f_1(x) + f_2(x) + \dots + f_n(x)) - f(x)| < M(a_{N+1} + a_{N+2} + \dots) < \varepsilon$$

for all $x \in D$. That is, $\sum_n f_n$ is uniformly convergent. \square

We will also need:

Lemma 2. *Let f_1, f_2, f_3, \dots be a sequence of functions $f_n : D \rightarrow \mathbb{R}$ such that each $f_n(x)$ is continuous at every point $c \in D$. If $\sum_n f_n$ is uniformly convergent to f on D , then f is uniformly continuous on D .*

For a proof of this lemma, see Beardon [2, page 129] or Hairer [5, page 215]. Finally we can prove:

Theorem 1. *The Weierstrass function f from Equation (1) is everywhere continuous.*

Proof. First note that for $0 < a < 1$, our a^n form a geometric series that converges to a finite number: $\sum_{n=1}^{\infty} a^n = \frac{1}{1-a} < \infty$. Also note that by definition of cosine functions, $|\cos(b^k \pi x)| \leq 1$. So for all x ,

$$|f_n(x)| = |a^n \cos(b^n \pi x)| \leq a_n.$$

Thus by the Weierstrass M-test (where $M = 1$), $\sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$ converges uniformly to $f(x)$ on \mathbb{R} . Our infinite sum is uniformly convergent, and each partial summand, $f_n(x)$ is uniformly continuous (since cosine functions are uniformly continuous and retain said continuity when elementary operations are applied to them). Hence, we can now apply Lemma 2 and conclude that our Weierstrass function f is indeed everywhere continuous. \square

We state the second half of our main result as:

Theorem 2. *The Weierstrass function f from Equation (1) is nowhere differentiable.*

Multiple proofs of the Weierstrass function's non-differentiability at all points have been published; several of these use Fourier Series, which goes beyond the space constraints of this paper (and the time constraints of its author). For a complete proof of Theorem 2, reference [6, 7].

Intuitively, it makes sense that the Weierstrass function is not differentiable at any point due to its fractal form. No matter how closely you zoom in on a single point of the function, the curve will never be smooth. That is, a tangent line will never be able to be found; the limit of the slopes will be undefined; the derivative will not exist. This is true for every point. (Reference Figure 1 again.)

4. McCarthy's Function (1953)

4.1. Historical Background

After Weierstrass's proof was published in the late 19th century, subsequent mathematicians sought out unique and novel examples of such functions. By the mid-20th century, the goal became to find examples with shorter proofs, more intriguing forms, and more unexpected approaches. John McCarthy's thirteen-line proof is an example of success in the first category. [9]

4.2. Building the Function

Before defining our function $f(x)$, consider the function

$$g(x) = \begin{cases} 1 + x & \text{for } -2 \leq x \leq 0 \\ 1 - x & \text{for } 0 \leq x \leq 2 \end{cases}$$

extended periodically to the rest of the real line. That is, $g(x)$ has period 4. (See Figure 2 on the following page.)

Let $g_n(x) = g(2^{2^n} x)$ for $n \in \mathbb{N}$. Note that $g_n(x)$ has period $4 \cdot 2^{-2^n}$. We then define our function as

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g_n(x). \quad (2)$$

See Figure 3 for a depiction of f .

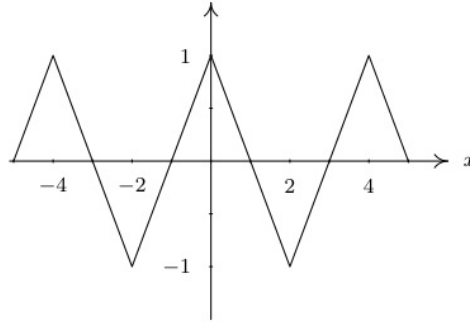


Figure 2: McCarthy's $g(x)$ on $[-5, 5]$. Image from [11], used with author's permission.

4.3. Proof of Uniform Continuity and Nowhere Differentiability

We begin with:

Theorem 3. *McCarthy's function f from Equation (2) is uniformly continuous.*

Proof. Much like in the proof of Theorem 1, we will use the Weierstrass M-test (Lemma 1) and Lemma 2 to prove uniform continuity. Let $a^n = 2^{-n}$. As before, this forms a geometric series that converges to a finite number:

$$\sum_{n=1}^{\infty} a^n = \frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = 2.$$

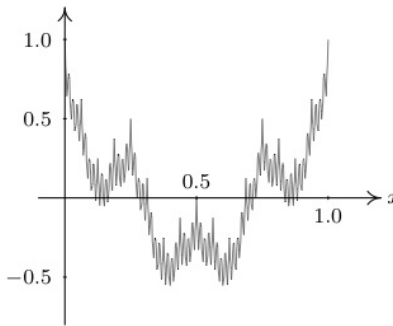


Figure 3: This is the general form of the McCarthy function $f(x)$. Image from [11], used with author's permission.

And, for each $g_n(x)$, we know $|g_n(x)| \leq 1 \leq Ma_n$ for any fixed $M < \frac{1}{2}$. Thus $\sum_{n=1}^{\infty} 2^{-n}g_n(x)$ converges uniformly to $f(x)$ on \mathbb{R} . Our infinite sum is uniformly convergent, and each partial summand, $f_n(x)$ is uniformly continuous (because the product of two continuous functions is continuous). Hence, we can apply Lemma 2 to conclude that McCarthy's function f is indeed everywhere continuous. \square

Here is the second half of the main result:

Theorem 4. *McCarthy's function f from Equation (2) is nowhere differentiable.*

Proof. Let $\Delta x = \pm 2^{-2^k}$, choosing whichever sign makes x and $x + \Delta x$ lie on the same linear segment. Now let us consider $\Delta g_n(x)$ for $n > k$, $n = k$, and $n < k$.

Case 1: For $n > k$, consider $\Delta g_n(x) = |g_n(x) - g_n(x + \Delta x)|$. Given, without loss of generality, $\Delta x = 2^{-2^k}$, we note that the period of $g_n(x)$ divides Δx , as

$$(2^{-2^k}) = (2^2 \cdot 2^{-2^n}) \cdot (2^{2^n - 2^k - 2}).$$

So, Δx can be written as a multiple of the period. By definition of the period of a function, we have that

$$g_n(x) = g_n(x + 4 \cdot 2^{-2^n}) = g_n(x + 2(4 \cdot 2^{-2^n})) = \cdots = g_n(x + \Delta x).$$

Returning to our $\Delta g_n(x)$ we can conclude $\Delta g_n(x) = 0$.

Case 2: For $n = k$, consider

$$\begin{aligned} |\Delta g_n(x)| &= |\Delta g(2^{2^n} x)| = |\Delta g(2^{2^k} x)| \\ &= |g(2^{2^k} x) - g(2^{2^k} x + 2^{2^k} \Delta x)|. \end{aligned}$$

Plugging the two arguments in for the piecewise function $g(x)$ when $\Delta x = 2^{-2^k}$, without loss of generality we discover $|\Delta g_n(x)| = |- (2^{2^k})(2^{-2^k})| = 1$.

Case 3: For $n < k$, consider $|\Delta g_n(x)| = |\Delta g(2^{2^n} x)| = |g(2^{2^n} x) - g(2^{2^n} x + 2^{2^n} \Delta x)|$. Plugging the two arguments in for the piecewise function $g(x)$ when $\Delta x = 2^{-2^k}$, without loss of generality we discover

$$|\Delta g_n(x)| = |- (2^{2^n})(2^{-2^k})| \leq (2^{2^k - 1})(2^{-2^k})$$

for all $n < k$.

Thus, when we consider $\Delta f(x)$ for $n < k$, we have

$$\begin{aligned} \left| \Delta \sum_{n=1}^{k-1} g(2^{2^n} x) \right| &\leq (k-1) \max |\Delta g(2^{2^n} x)| \\ &\leq (k-1) 2^{2^{k-1}} 2^{-2^k} \\ &< 2^k 2^{-2^{k-1}}. \end{aligned}$$

So from our three cases, we know that when $n > k$, $\Delta g_n(x)$ does not contribute to the total sum. When $n = k$, $\Delta g_n(x)$ is, at its least, -1 . And when $n < k$, the partial sum is less than $2^k 2^{-2^{k-1}}$.

Putting together all parts, we can say

$$\begin{aligned} \left| \frac{\Delta f}{\Delta x} \right| &\geq \frac{2^k 2^{-2^{k-1}} - 1}{2^{-2^k}} \\ &\geq 2^{-k} 2^{2^k} - 2^k 2^{2^{k-1}}, \end{aligned}$$

which goes to infinity with k . Because the quotient is unbounded, its limit cannot be defined at any point x .

Therefore, McCarthy's function f is nowhere differentiable. \square

As seen above, it took us a bit longer than thirteen lines to prove this result! McCarthy's construction is interesting but perhaps a bit opaque to the beginner.

5. The Blancmange Function (1982)

5.1. Historical Background

In 1903, Teiji Takagi developed another function with our desired, pathological property. David Tall revised the function in 1982 [10] and dubbed it the “blancmange” function, due to its resemblance to the medieval European flan-like gelatinous dessert in both general shape and in “wobbliness”, see Figure 4. In some ways, the construction involved here is a construction that may feel more intuitive to a beginner, though again, the resultant function, as we will see, is obviously quite wonky!



Figure 4: A serving of blancmange, “made according to a rather purist recipe, containing only almonds, water, milk, sugar and gelatin”. Image by SKopp - Own work, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=33502092>

5.2. Building the Function

In establishing our function, we must start with its building blocks. First consider the function

$$f_1(x) = \begin{cases} x & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 1 - x & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

where values repeat over each succeeding unit interval (i.e. $f_1(x+1) = f_1(x)$). (See Figure 5.)

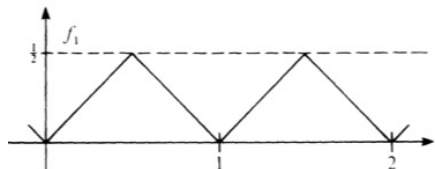


Figure 5: The first blancmange sawtooth function: $f_1(x)$. Image from [10] used with author's permission.

For subsequent functions we let

$$f_n(x) = \frac{1}{2^{n-1}} f_1(2^{n-1}x).$$

This gives us $f_2(x) = \frac{1}{2}f_1(2x)$, $f_3(x) = \frac{1}{4}f_1(4x)$, and so on.

Building off of this function, we form sequences $b_n(x)$ of their sums:

$$\begin{aligned} b_1(x) &= f_1(x), \\ b_2(x) &= f_1(x) + f_2(x), \\ &\dots \\ b_n(x) &= b_{n-1}(x) + f_n(x) = \sum_{k=1}^n f_k(x). \end{aligned}$$

(See Figure 6 on the next page.)

We are now ready to define our blancmange function as $b(x) = \lim_{n \rightarrow \infty} b_n(x)$ for all $x \in \mathbb{R}$. Or, equivalently:

$$b(x) = \sum_{n=1}^{\infty} f_n(x). \quad (3)$$

At this juncture, do note some algebraic and geometric properties of our function h . Algebraically, the $b_n(x)$ sequences as n approaches infinity rapidly approximate $b(x)$ very precisely. Take, for example, $n = 20$. Then,

$$b_{20}(x) \leq b(x) \leq b_{20}(x) + 0.000001.$$

That is, $b_{20}(x)$ approximates $b(x)$ to within 0.000001. Geometrically, though, no matter how large n is, $b_n(x)$ will never be a good approximation of $b(x)$.

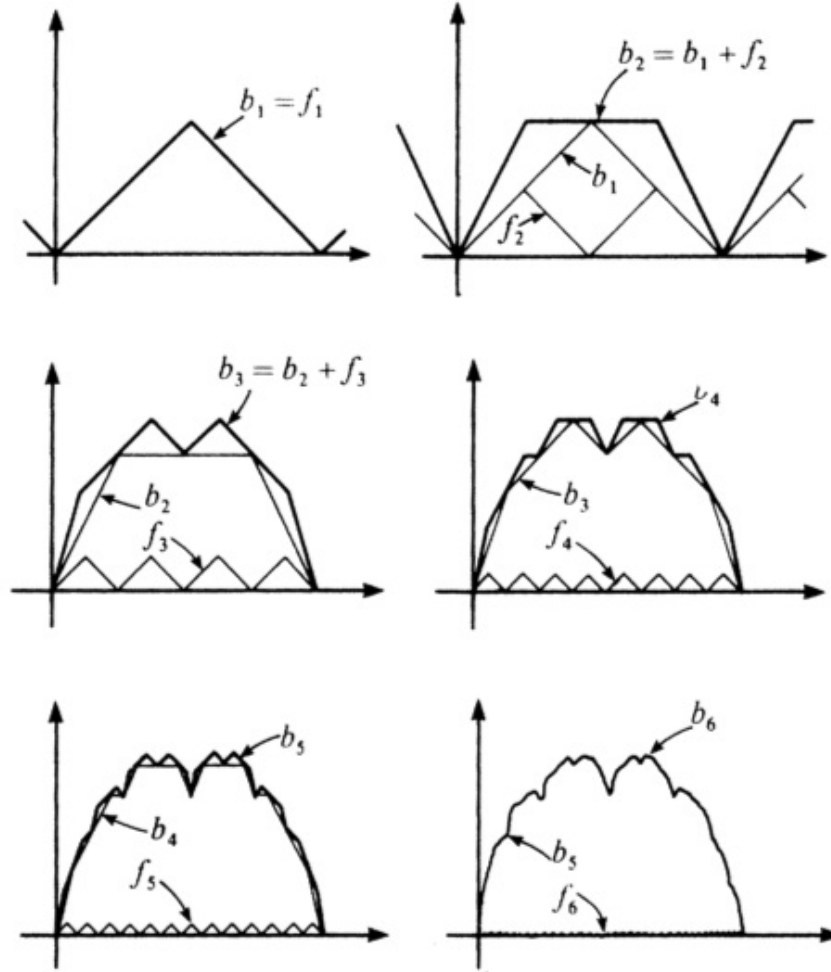


Figure 6: Here are the first few steps in the construction of $b_n(x)$. Notice how the function *already* begins to stabilize slightly at $b_6(x)$, due to the rapid scaling-down of the sharktooth $f_n(x)$. Image from [10] used with author's permission.

Considering the difference between our function and one of its partial sums for an arbitrarily large value of n , we get that

$$b(x) - b_n(x) = \frac{1}{2^n} b(2^n x).$$

That is, $b(x)$ is a fractal.

5.3. Proof of Uniform Continuity and Nowhere Differentiability

We begin with two lemmas:

Lemma 3. *The difference $b(x) - b_n(x)$ can be made arbitrarily small.*

Proof. For any fixed value of x , the sequence $b_1(x), b_2(x), \dots, b_k(x), \dots$ is increasing and bounded above by 1. For any $k > n$ we know

$$\begin{aligned} 0 \leq b_k(x) - b_n(x) &= f_{n+1}(x) + \dots + f_k(x) \leq \left(\frac{1}{2}\right)^{n+1} + \dots + \left(\frac{1}{2}\right)^k \\ &= \frac{\left(\frac{1}{2}\right)^{n+1} \left(1 - \left(\frac{1}{2}\right)^{k-n}\right)}{1 - \frac{1}{2}} \\ &< \left(\frac{1}{2}\right)^{n+1}. \end{aligned}$$

So, $b_n(x) \leq b_k(x) \leq b_n(x) + \left(\frac{1}{2}\right)^{n+1}$ for $k > n$. Allowing n to increase so that $b_k(x)$ tends to $b(x)$, the above statement becomes

$$b_n(x) \leq b(x) \leq b_n(x) + \left(\frac{1}{2}\right)^{n+1}$$

That is,

$$b(x) - b_n(x) \leq \left(\frac{1}{2}\right)^{n+1}$$

As n increases, $\left(\frac{1}{2}\right)^{n+1}$ decreases exponentially. Thus, a large n can be chosen to make the difference $b(x) - b_n(x)$ arbitrarily small. \square

Lemma 4. *If $|s - c| < \delta$, then $|b_n(s) - b_n(c)| < \delta n$.*

Proof. While $b(x)$ has a wobbly, fractal form, $b_n(x)$ is made up of straight line segments produced in summing together n saw teeth, each of which with a slope of ± 1 . Thus the gradients of the line segments of $b_n(x)$ lie between $-n$ and $+n$. That is, $\left|\frac{b_n(s) - b_n(c)}{s - c}\right| \leq n$. Or, equivalently,

$$|b_n(s) - b_n(c)| \leq n|s - c|.$$

So, if $|s - c| < \delta$, then we can substitute into the above equation to obtain

$$|b_n(s) - b_n(c)| < \delta n. \quad \square$$

Now we are ready to prove the first part of our main result:

Theorem 5. *The blancmange function $b(x)$ of Equation (3) is everywhere continuous.*

Proof. Given any fixed $\varepsilon > 0$ and $s \in \mathbb{R}$, let $\delta = \frac{\varepsilon}{2^n}$, and assume $|s - c| < \delta$. Consider

$$|b(s) - b(c)| = |(b(s) - b_n(s) + b_n(c) - b(c)) + (b_n(s) - b_n(c))|.$$

By applying the triangle inequality to the right hand side of the above equation, we obtain

$$|b(s) - b(c)| \leq |(b(s) - b_n(s)) - (b(c) - b_n(c))| + |(b_n(s) - b_n(c))|.$$

By Lemma 3, $(b(s) - b_n(s))$ and $(b(c) - b_n(c))$ can each be made arbitrarily small. Thus their difference can, for our arbitrarily large n , be less than $\frac{\varepsilon}{2}$. By Lemma 4, we have that $|s - c| < \delta = \frac{\varepsilon}{2^n}$, then $|b_n(s) - b_n(c)| < \delta n = \frac{\varepsilon}{2}$. We are thus left with the statement

$$|b(s) - b(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus if $|s - c| < \frac{\varepsilon}{2^n}$, then $|b(s) - b(c)| < \varepsilon$. Because $\varepsilon > 0$ and $s \in \mathbb{R}$ were chosen arbitrarily, the inequality holds for any $\varepsilon > 0$ and $s \in \mathbb{R}$. As such, we can conclude that $b(x)$ is uniformly continuous. \square

Finally we wish to convince the reader that the following holds:

Theorem 6. *The blancmange function $b(x)$ of Equation (3) is nowhere differentiable.*

Proof. Earlier we mentioned the fractal property of the blancmange function. That is,

$$b(x) - b_n(x) = \frac{1}{2^n} b(2^n x).$$

Examining the above equation for various n , we see that when $n = 1$, $b(x) - b_1(x) = \frac{1}{2} b(2x)$ (We display this in Figure 7 below).

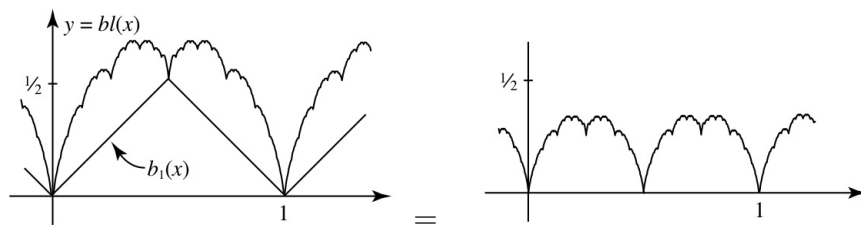


Figure 7: The left image is $b(x) - b_1(x)$, which equals the right image, $\frac{1}{2}b(2x)$, of half-sized blancmange functions. Image from [10] used with author's permission.

See Figure 8 for when $n = 2$ and $n = 4$, that is, for the graphs

$$b(x) - b_2(x) = \frac{1}{4}b(4x) \quad \text{and} \quad b(x) - b_4(x) = \frac{1}{16}b(16x).$$

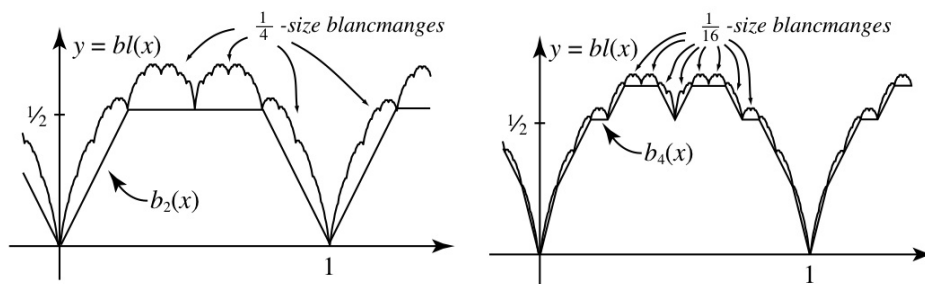


Figure 8: The left image shows the quarter-size blancmange functions that emerge when $n = 2$. The right image shows the sixteenth-sized blancmange functions that emerge when $n = 4$. Image from [10] used with author's permission.

Now, to see the trend as $n \rightarrow \infty$, consider a larger n , say $n = 1000$. Reference Figure 9 to see that the graph $r(x) = b(x) - b_{1000}(x)$ is yet *another* identical blancmange function scaled down.

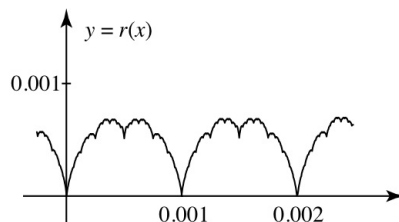


Figure 9: $r(x) = b(x) - b_{1000}(x)$. Image from [10] used with author's permission.

Given these illustrious images of the fractal nature of the blancmange function, we can prove its nowhere differentiability through two thought processes. If we think about the function's behavior in terms of limits, we know that as x tends towards c , $|x - c| \rightarrow 0$, but $b(x)$ and $b(c)$ will always be distinct on yet another $\frac{1}{2^n}$ -sized blancmange function. Thus the quotient of the limit will be undefined; that is, $b'(c)$ will not exist at any point c .

If we think of the function's behavior in terms of tangent lines, it is also evident that the blancmange function is nowhere differentiable. Zooming in on a single point will never cause the curve to flatten, due to the fractal or "wobbly" nature of $b(x)$. Thus, for every c , $\lim_{x \rightarrow c} \frac{b(x) - b(c)}{x - c}$ does not exist, so $b(x)$ is differentiable at *no* point c of our domain. \square

6. Functions That Followed

One of the modern mathematicians to tackle the subject of such functions, Mark Lynch began publishing papers in 1986. As mentioned before, mathematicians of the 20th century began to search for ways to make their examples of nowhere differentiable functions unique. We have been through an explanation of McCarthy's supposedly simple thirteen-line proof. We have been through the more intuitive and visual blancmange function. Lynch's function [8] is unique in that it was the first example of such a function that does not involve uniform convergence in its proof.

Prior to Lynch, everywhere continuous nowhere differentiable functions were proven to be everywhere continuous via the Weierstrass M-test for uniform convergence. (Note that the traditional proof showing that the blancmange function is uniformly continuous uses the Weierstrass M-test and uniform convergence; I chose to do the more laborious proof in Section 5 for the sake of visualization.) Lynch experimented instead with the idea of mapping, using compactness of a function's graph to prove continuity of a function. His article gave another construction of a everywhere continuous, nowhere differentiable function that did not use uniform convergence.¹

¹Incidentally, most of Lynch's papers thereafter took a unique approach to math, especially Topology, focusing on much of the field's quirks and pathological elements (for example, the topologies where differentiable, nowhere continuous function exist) and attempting to find new and different ways to define pre-existing ideas like limits, continuity, and differentiability. His 1992 function is worth examining as another example, for its

7. Prevalence of $ND[0,1]$

As pathological and, at first thought, as counterintuitive these continuous, nowhere differentiable functions may seem, the size of the set of all such functions is even more surprising. Before discussing how it could be that “most functions” are nowhere differentiable, let us define a few terms.

ND For $a < b$, let $ND[a, b]$ be the set of all continuous nowhere differentiable functions $f : [a, b] \rightarrow \mathbb{R}$.

Nowhere dense A set is considered *nowhere dense* if the interior of its closure is non-empty.

Baire’s Category I Sets that can be written as the union of nowhere dense sets are of Baire’s first category. We call these sets *meager*.

Baire’s Category II Sets that are not of the first category, that is, complements of meager sets, are of Baire’s second category. We call these sets *residual*.

We are now ready to state the theorem:

Theorem 7 (Banach-Mazurkiewicz Theorem). *The set $ND[a, b]$ of all continuous, nowhere differentiable functions on $[a, b]$ is of the second category.*

The proof of this theorem (see [11]) uses $[a, b] = [0, 1]$, but a generalization to the universal case follows easily. Defining E_n as the set made up of functions that are differentiable at some point $n \in \mathbb{N}$, it can be shown that E_n is nowhere dense in the space $C[0, 1]$. Therefore, the set of all functions that are differentiable somewhere can be written as the union of nowhere dense sets. That is, $\bigcup E_n$ is meager. The complement of this set, the set of nowhere differentiable functions, is therefore a residual set of Baire’s Category II.

Using other techniques (which will be stated but the complete proof along with certain definitions will be left to reference [11]), we can prove the following:

unique construction and approach. It is strongly recommended by the author that the reader consider reading [8].

Theorem 8. *Almost every function in $C[0, 1]$ is nowhere differentiable. That is, $ND[0, 1]$ is a prevalent subset of $C[0, 1]$.*

Before discussing this fact, we define:

Lipschitz A function $f \in C[a, b]$ is said to be *Lipschitz* at a point $x \in [a, b]$ if there exists an $M > 0$ such that for every $y \in [a, b]$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Analyzing the set of all nowhere Lipschitz functions, we find that its complement is shy. Thus the set itself is a prevalent set. $ND[a, b]$ is contained completely within the set of all nowhere Lipschitz functions, meaning the set of all nowhere differentiable functions is prevalent.

8. Parting Shots

Thank you, generous reader, for taking the time to read my paper. Studying this topic was quite the “mathematikel” feat for me. That is to say, I learned a whole lot. I hope you enjoyed learning about some of the history and mathematics of this subject. Moreover, I hope I was able to somewhat convey how truly beautiful and fascinating the field of Mathematics can be.

Postscript: This article was originally written for a course I took during my first year at Pomona College: Introduction to Analysis, taught by Professor Gizem Karaali. The course, along with this dive I took into everywhere continuous nowhere differentiable functions, were a few of the experiences that made me first fall in love with mathematics. I hope that this article can bring to other students—even the ambitious calculus student!—the joy of discovering some of the beauties and pathologies of mathematics.

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