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Cover Page Footnote
This essay originated as an assignment for Professor Thomas Martin's Plutarch seminar at Holy Cross in Fall 2016. I want to thank him and the referees for a number of helpful comments and suggestions.

A Mathematician Reads Plutarch:
Plato's Criticism of Geometers of His Time[1]

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Synopsis

This essay describes the author's recent encounter with two well-known passages in Plutarch that touch on a crucial episode in the history of the Greek mathematics of the fourth century BCE involving various approaches to the problem of the duplication of the cube. One theme will be the way key sources for understanding the history of our subject sometimes come from texts that have much wider cultural contexts and resonances. Sensitivity to the history, to the mathematics, and to the language is necessary to tease out the meaning of such texts. However, in the past, historians of mathematics often interpreted these sources using the mathematics of their own times. Their sometimes anachronistic accounts have often been presented in the mainstream histories of mathematics to which mathematicians who do not read Greek must turn to learn about that history. With the original sources, the tidy and inevitable picture of the development of mathematics disappears and we are left with a much more interesting, if ultimately somewhat inconclusive, story.

1. Introduction

Textbook presentations of mathematics itself and histories of the subject unfortunately tend to suffer from some similar defects. In most mathematics textbooks, everything is seemingly inevitably and tidily organized. The

[1]This essay originated as an assignment for Professor Thomas Martin's Plutarch seminar at Holy Cross in Fall 2016. I want to thank him and the referees for a number of helpful comments and suggestions.

actual process by which humans discover or invent new mathematics (even something new only to a student) rarely, if ever, comes through; the focus is often on acquiring cut-and-dried techniques and mastering well-known algorithms. Analogously, until quite recently, histories have often presented aspects of the mathematical past as leading inevitably and tidily to our current understanding of the subject. However, the mathematics of the past often starts from completely different assumptions and uses different methods that may not be captured in modern expositions or reconstructions.

Moreover, in many cases, meeting the past on its own terms can be extremely difficult because of the paucity of the surviving materials. In addition, key evidence can come from texts that have much wider cultural contexts and resonances. Sensitivity to underlying history, to the language of the original sources, and to the mathematics involved may be necessary to tease out the meanings of those texts. Yet, historians of mathematics in the past often interpreted these sources using the mathematics of their own times and produced conclusions that are open to question. Unfortunately it is those sometimes anachronistic accounts that have been presented in mainstream histories of the subject to which mathematicians who do not read the original languages must turn to learn about that history. At the same time, some historians and experts in the original languages lack the mathematical background knowledge to appreciate that aspect of the content of these sources. Once one returns to actual source material, questions with no easy answers often abound and it is amazing to see how much we still do not understand about critical junctures in the history of our subject. The tidy and inevitable picture of the development of mathematics disappears and we are left with a much more interesting, if ultimately somewhat inconclusive, story.

In this essay I will present an extended example illustrating all of the points touched on in the previous paragraph. This stems from an encounter I have recently had with two well-known passages from Plutarch. The first comes from the so-called *Moralia*, the compendium of occasional essays and miscellaneous writings that accompanies the series of parallel *Lives* of illustrious Greeks and Romans in Plutarch's immense output, and the other occurs in his *Life of Marcellus*.[2] These passages touch on a crucial episode in the

[2]The occasion for this study was that, motivated by a desire to read works such as Euclid's *Elements* and the *Conics* of Apollonius (or at least the portions of Apollonius that survive in Greek) in their original forms, I have been studying ancient Greek for the

history of the Greek mathematics of the fourth century BCE that stimulated mathematical research into the 19th century CE and whose influence is still felt at one point in the undergraduate pure mathematics curriculum–the story of various approaches to the duplication of the cube. This problem is usually grouped together with two others, the quadrature of the circle (i.e. the problem of constructing a square or rectangle with the same area as a given circle), and the trisection of a general angle. Wilbur Knorr examined this tradition of geometric problems in detail in [4]. These passages from Plutarch have been seen by some[3] as a major contributing factor to the later notion that in Greek geometric constructions, *only* the compass and straightedge were acceptable tools. In fact constructions using auxiliary curves of various sorts were developed for all of these problems. Whether these constructions could be accomplished using only the Euclidean tools remained an open question until the work of P. Wantzel and others in the 19th century CE, completing a line of thought initiated by Descartes in *La Géometrie*. It is now known that *none* of them is possible under those restrictive conditions and many undergraduate mathematics majors learn proofs of these facts in abstract algebra courses.

Because this period and the associated questions have been intensively studied since the 19th century CE, I cannot claim that this essay presents any new historical scholarship. However, I hope that it may prove useful to instructors and other readers who are interested in finding more nuanced accounts of some of the Greek work on the duplication of the cube than are available in some of the standard histories.

2. The Plutarch Texts

Plutarch of Chaeronea (ca. 45–ca. 120 CE) was a Greek writing for a mixed Greek and Roman audience during the early empire. He himself records that he studied philosophy and mathematics in Athens and his writings reveal a strong connection with Platonic traditions. We would call him an essayist and biographer, although most of his writing is more devoted to ethical lessons than to history, *per se*. The first passage I will discuss comes from

past four years in courses taught by several very kind colleagues in my home institution's well-regarded Classics department.

[3] See [15] for a discussion.

a section of his *Moralia* known as the *Quaestiones Convivales*, or "Table Talk." Each section of this work is presented as a record of conversation at a *sumposion*, or drinking party, arranged by Plutarch for a group of guests. Philosophical questions are always debated and it is amusing to see what Plutarch says about the rationale for this: in "... our entertainments we should use learned and philosophical discourse ..." so that even if the guests become drunk, "... every thing that is brutish and outrageous in it [i.e. the drunkenness] is concealed ..." [12, 716g, Book 8, Chapter 0, Section 2]. In other words, to keep your next party from degenerating into a drunken brawl, have your guests converse about Plato!

In Book 8, Chapter 2, Section 1 (classicists refer to this via the so-called Stephanus page 718ef, from one of the first modern printed versions of the Greek text), Plutarch presents a conversation between the grammarian Diogenianus and the Spartan Tyndares concerning the role of the study of geometry in Plato's thought. Diogenianus begins this phase of the conversation by raising the question why Plato asserted that "God always geometrizes." He also says he is not aware of any specific text where Plato said precisely that, though he thinks it sounds like something Plato would have said. Tyndares replies that there is no great mystery there and asks Diogenianus whether it was not true that Plato had written that geometry is "... taking us away from the sensible and turning us back to the eternal nature we can perceive with our minds, whose contemplation is the goal of philosophy ... ."[4] After some elaboration of these points, Tyndares presents an interesting piece of evidence concerning this aspect of Plato's thought: "Therefore even Plato himself strongly criticized Eudoxus, Archytas, and Menaechmus" (or possibly "those around Eudoxus, Archytas, and Menaechmus") "for attempting to reduce the duplication of the cube to tool-based and mechanical constructions ... ."[5] Tyndares continues in a technical vein, "... just as though they were trying, in an unreasoning way, to take two mean proportionals in con-

[4]In the text, I will present my translations of the Greek. Here: "ἀποσπῶσαν ἡμᾶς προσισχομένους τῇ αἰσθήσει καὶ ἀποστρέφουσαν ἐπὶ τὴν νοητὴν καὶ ἀίδιον φύσιν, ἧς θέα τέλος ἐστὶ φιλοσοφίας ... ." All Greek quotations from the Plutarch passage are from the Loeb Classical Library/Perseus online text of Plutarch's *Quaestiones Convivales*, [12].

[5]"δίο καὶ Πλάτων αὐτὸς ἐμέμψατο τοὺς περὶ Εὔδοξον καὶ Ἀρχύταν καὶ Μέναιχμον εἰς ὀργανικὰς καὶ μηχανικὰς κατασκευὰς τὸν τοῦ στερεοῦ διπλασιασμὸν ἀπαγεῖν ἐπιχειροῦντας ... "

tinued proportion any way that they might ... ."[6] It is quite interesting that Tyndares seems to be assuming that all of his listeners would be familiar with this terminology and the episode in the history of geometry to which he is referring. Tyndares concludes his summary of Plato's criticism by claiming that in this way (i.e. by using mechanical procedures with tools) "... the good of geometry is utterly destroyed and it falls back on things of the senses; it is neither carried above nor apprehends the eternal and immaterial forms, before which God is always God."[7]

It is interesting to note that Plutarch gives a second, partially parallel account of this criticism in Chapter 14, Sections 5 and 6 of his *Life of Marcellus*, in the context of a discussion of the geometrical and mechanical work of Archimedes and the tradition that King Hiero of Syracuse persuaded him to take up mechanics to design engines of war in defence of his native city-state. Marcellus was, of course, the commmander of the Roman forces in the siege of Syracuse in 212 BCE during which Archimedes was killed. In that passage, Plutarch says (in the English translation by Bernadotte Perrin from [13]; we discuss some issues related to word choices in the footnotes):

> "For the art of mechanics, now so celebrated and admired, was first originated by Eudoxus and Archytas, who embellished geometry with its subtleties, and gave to problems incapable of proof by word and diagram, a support derived from mechanical illustrations that were patent to the senses. For instance in solving the problem of finding two mean proportional lines, a necessary requisite for many geometrical figures, both mathematicians had recourse to mechanical arrangements[8] adapting to their purposes certain intermediate portions of curved lines and sections.[9] But Plato was incensed at this, and inveighed against them as corrupters and destroyers of the pure excellence of ge-

[6] "ὥσπερ πειρωμένους δι' ἀλόγου δύο μέσας ἀνάλογον, ᾗ παρείκοι, λαβεῖν ... " The δι' ἀλόγου is hard to translate and may not even be what Plutarch originally wrote. This specific phrase has a rather large number of textual issues as evidenced by the variant readings discussed in the Loeb Classical Library/Perseus version of the Plutarch text. At least one editor has suggested that the whole phrase should be omitted from the text!

[7] "ἀπόλλυσθαι γὰρ οὕτω καί διαφείρεσθαι τὸ γεωμετρίας ἀγαθὸν αὖθις ἐπὶ τὰ αἰσθητὰ παλινδρομούσης καὶ μὴ φερομένης ἄνω μηδ' ἀντιλαμβανομένης τῶν ἀίδιων καὶ ἀσωμάτων εἰκόνων πρὸς αἷσπερ ὢν ὁ θεὸς ἀεὶ θεός ἐστι."

[8] "κατασκευάς" – "constructions" would be another, perhaps better, translation here.

[9] "μεσογράφους τινὰς ἀπὸ καμπύλων καὶ τμημάτων μεθαρμόζοντες" – I think a better translation here is "adapting to their purposes mean proportionals found from curved

> ometry, which thus turned her back upon the incorporeal things of abstract thought and descended to the things of sense, making use, moreover, of objects which required much mean and manual labor. For this reason, mechanics was made entirely distinct from geometry, and being for a long time ignored by philosophers, came to be regarded as one of the military arts."

One can see the dilemma of a non-mathematician translating technical discussions! In the passage as a whole, one can also glimpse some of the less attractive aspects of Plato's thought. These passages in Plutarch provide a fascinating, but also ultimately somewhat cryptic, sidelight on a key episode in the history of Greek mathematics.

3. Just What Was Plato Criticizing?

To understand the import of Plutarch's account, we need to take a small detour into Platonic philosophy to start. Tyndares is evidently thinking of passages like 527b in Book VII of the *Republic*, where, in the midst of a discussion of the subjects in which men should be educated, Plato has Socrates say in reference to geometry that "... it is the knowledge of that which always is, and not of a something which at some time comes into being and passes away. ... [I]t would tend to draw the soul to truth, and would be productive of a philosophical attitude of mind, directing upward the faculties that are now wrongly turned downward."[10]

I would argue that Plato's supposed objection in our passages refers specifically to the use of tools within the realm of pure geometry, conceived of as an exercise of pure thought apprehending properties of an unchanging external

lines and sections." The meaning of μεσογράφους here is the same as in a passage from a commentary of Eutocius quoting Eratosthenes that will be discussed below.

[10]English translation from [10, page 758]. I am indebted to one of the referees for the observation that Xenophon's Socrates has a rather different viewpoint on geometry from Plato's Socrates. In the *Memorabilia*, 4.7.2-3, Xenophon has Socrates say that practical geometry of measurement and apportionment is important and that men should be able to demonstrate the correctness of their work but that higher geometry is something whose usefulness he cannot see; see [16]. In *Republic*, 526de, Plato's Socrates also acknowledges the practical uses of geometry, then considers the question of whether "... the greater and more advanced part of it tends to facilitate the apprehension of the idea of the good." He clearly believes that it does do so [10, page 759].

reality. For the adjective “tool-based”[11] to apply, I believe some physical device manipulated by the geometer and requiring input from the senses of the geometer must be involved in a construction. Input from the senses might involve some sort of approximation of a length or an angle or manipulation of elements of a figure to put them into a special configuration (e.g. moving elements to make a collection of points collinear). We will see some concrete examples of this later. While such a process might be sufficient for a practical application, I think Plato would have objected on principle to any such construction that claimed to be a solution of a geometric problem because the solution it yields must be an approximation to an exact theoretical solution.

There is a point here that may be subtle for some modern readers of these works. The Greeks, even though they used physical straightedges to draw lines and physical compasses to draw circles while constructing diagrams, also considered those tools in *idealized versions* that were constructs of the mind and thus not dependent on the senses. The first three postulates in Book I of the *Elements* of Euclid (ca. 300 BCE) describe their uses and properties in abstract terms. In particular, the idealized straightedge can be used to draw lines, but not to measure distances; it has no distance scale like a modern ruler. Similarly, the idealized Euclidean compass can be used to draw circles, but not to measure or transfer distances. So we should certainly not take criticisms such as the one ascribed to Plato here to refer to constructions that involve the Euclidean tools.

Moreover, it is certainly not the case that Plato was opposed in all cases to the use of tools or mechanical devices to understand the cosmos. For instance in the *Timaeus* at 40d he says that trying to explain the motions of the planets and other astronomical phenomena “... without an inspection of models of these movements would be labor in vain.”[12]

I think the statement in the quotation from the *Life of Marcellus* gives several additional clues about Plutarch’s understanding of Plato’s supposed objections. For Plutarch, from the evidence of this statement, it seems that the ad-

[11] ὀργανικός

[12] [10, page 1169]. I thank one of the referees for pointing out this facet of the Platonic tradition. The word translated as “models” is μιμήματα – literally “imitations.” I would take this as possibly meaning some sort of orrery or other device representing the motions.

jective "mechanical"[13] referred to the use of mathematics to emulate and/or design real-world machines, and perhaps machines of war in particular. It is possible that for him the adjectives "tool-based" and "mechanical" overlap in meaning to some degree. However, I would suggest that another aspect that might make a geometrical construction "mechanical," apart from the use of actual tools, is that it has some element of *motion* (real or imagined) possibly as in a real-world mechanical device. Plutarch says that Archytas in particular "gave to problems incapable of proof by word and diagram, a support derived from mechanical illustrations that were patent to the senses." Note that any sort of change over time in a figure would by itself seem to violate Plato's vision of the eternal and unchanging nature of the world of the forms.

Even the language used in many Greek mathematical arguments reinforces this point. The third person perfect imperative verb forms typically used in Greek to express the steps of geometric constructions (e.g. γεγράφθω – "let it have been drawn") seem to emphasize that the figure or diagram *has been constructed* as a whole, and thus connote something static rather than something dynamic.[14] This convention seems in fact to agree perfectly with the Platonic conception of geometry that I would argue forms the basis of the supposed criticism in our Plutarch passages.

Tyndares is saying that Plato criticized the nature of the solutions proposed by Eudoxus, Archytas, and Menaechmus because they in effect subverted what he (i.e. Plato) saw as the true purpose of geometry: its *raison d'être* was not merely to solve problems "by any means necessary,"[15] but rather to take us away from the realm of the senses and prepare us for the study of dialectic.[16]

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[13] μηχανικός

[14] See, for instance, the discussion in [7, page 197].

[15] that is, the ᾗ παρείκοι in Plutarch's formulation

[16] As one of the referees has pointed out, it is important to acknowledge that while Plato gave geometry a *special* place in his epistemology, he certainly did not think of it as the *highest* form of knowledge, because of its essentially hypothetical nature. At 533c in the *Republic*, for instance, he says that "... geometry and the studies that accompany it are, as we see, dreaming about being, but the clear waking vision of it is impossible for them as long as they leave the assumptions which they employ undisturbed and cannot give any account of them." Dialectic, in which we give an account of our basic assumptions and reason from them, is even more important to master [10, page 765].

In his thought-provoking study [4] of the Greek work on construction problems, Wilbur Knorr has argued in effect that by this period Greek mathematics had distanced itself from the sort of philosophical or religious underpinning that Plutarch says Plato claimed for it and was already very close to a modern research program, in which the goal *often is indeed* to solve problems by whatever means are necessary, and then to understand what methodological restrictions might still allow a solution.[17] Needless to say, this view is not universal among historians of Greek mathematics.

4. The Historical Background

Eudoxus of Cnidus (409–356 BCE), Archytas of Tarentum (428–347 BCE), and Menaechmus of Alopeconnesus (380–320 BCE) were three of the most accomplished Greek mathematicians active in the 4th century BCE. Archytas is often identified as a Pythagorean and there are traditions that Eudoxus was a pupil of his and Menaechmus was a pupil of Eudoxus. All three were associated with Plato and his Academy in Athens in some way. We have much of this from sources such as the commentary on Book I of Euclid's *Elements* by Proclus,[18] though the fact that Proclus is writing roughly 800 years after this period raises the question of how reliable his information is.

As almost all mathematicians know, the *duplication of the cube* was a geometrical problem asking for the construction of the side of a cube whose volume would be twice the volume of a given cube. Various traditions deal with the genesis of this problem. One says that seeking direction in order to stem the progress of a plague on their island (or perhaps political conflicts; different versions of the story differ on this point), the people of Delos consulted the oracle at Delphi, whereupon the Pythia replied that they must find a way to double the size of an cubical altar of Apollo.[19] When they were unable to do this themselves, the Delians supposedly consulted Plato and the geometers at his Academy to find the required geometric construction; for this reason the problem of the duplication of the cube is often called the "Delian problem."

[17] See the discussions at [4, pages 39–41 and 88] in particular.

[18] [14, pages 54–56].

[19] A somewhat parallel story about King Minos seeking how to double the size of a tomb also appears in a letter of Eratosthenes to King Ptolemy III Euergetes of Egypt that will be discussed below. See [3, page 245].

However, this version of the story is almost certainly fanciful (at least as the origin of the problem–the chronology seems to be wrong, for one thing, since as we will see presently there was work on the question somewhat before the time of Plato (428 – 348 BCE)). Plutarch himself, in another section of the *Moralia* called *The E at Delphi*, says that the underlying point of the story was that the god was commanding the Greeks to apply themselves more assiduously to geometry.[20] There is no doubt that the duplication of the cube was one of a series of geometric construction problems that stimulated the development of Greek mathematics throughout the Classical period.

5. Two Mean Proportionals In Continued Proportion

For a full understanding of our Plutarch passages, and of Plato's supposed objection to the work of Eudoxus, Archytas, and Menaechmus, we need to introduce an important piece of progress that had been made earlier and definitely before the time of Plato by Hippocrates of Chios (ca. 470–ca. 410 BCE). None of Hippocrates' own writings have survived and we know about the following only from sources such as fragments of a history of pre-Euclidean mathematics by Eudemus of Rhodes (ca. 370–ca. 300 BCE) preserved in other sources. Given two line segments AB and GH, we say line segments CD and EF are *two mean proportionals in continued proportion*[21] between AB and GH if their lengths satisfy:

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{GH}. \tag{1}$$

Hippocrates' contribution was the realization that if we start with

$$GH = 2AB,$$

then the construction of two mean proportionals as in (1) would *solve the problem of the duplication of the cube.* The idea is straightforward: If

$$\frac{AB}{CD} = \frac{CD}{EF} = \frac{EF}{2AB},$$

[20][11, Chapter 6, 386e].

[21]In the first Plutarch passage above, this appears in the accusative as δύο μέσας ἀνάλογον. The ἀνάλογον seems to be essentially equivalent to the ἀνὰ λόγον from the second passage, and that is conventionally translated as "in continued proportion."

then some simple algebra (which the Greeks would have emulated with parallel manipulation of proportions) shows

$$CD^3 = 2AB^3.$$

In other words, if AB is the side of the original cube, then CD is the side of the cube with twice the volume.

It is important to realize that Hippocrates in effect only reduced one problem to another one. Finding a geometric construction of the two mean proportionals in continued proportion was still an open question but this approach did provide a definite way to attack the duplication of the cube and essentially all later work took this reduction as a starting point.

6. Eutocius' Catalog of Constructions

Plutarch does not include any discussion of what Eudoxus, Archytas, or Menaechmus actually did in their work on the duplication of the cube. However, quite detailed accounts of the contributions of Archytas and Menaechmus, as well as the contributions of many others on this problem, have survived in ancient sources. The most important is a much later commentary on Archimedes' *On the Sphere and the Cylinder* by Eutocius of Ascalon (ca. 480–ca. 540 CE), in which Eutocius surveys a wide selection of different solutions to the problem of duplicating a cube by finding two mean proportionals.[22] The occasion for the commentary was the fact that Archimedes assumed the construction was possible in some way in the proof of the first proposition in Book II of *On the Sphere and the Cylinder*, but he did not provide any explanation. Because of the roughly 900 years intervening between the time of the Platonic geometers and Eutocius' time, the caveat we made above about accepting evidence from Proclus' writings uncritically also applies to Eutocius' catalog. Eutocius' account includes detailed information about the approaches of Archytas and Menaechmus. Unfortunately, the solution by Eudoxus, "by means of curved lines," is not presented in detail by Eutocius because he believes his sources for it are corrupt. Hence we do not have

[22] The Greek original with a Latin translation is included in Volume III of J. Heiberg's *Archimedis Opera Omnia*, [1]. A near-literal English translation is given by Netz in [9, pages 270–306].

enough information to draw any firm conclusions about how it connects with what Plutarch writes.[23]

Before we turn to what Eutocius says about the work of Archytas and Menaechmus, we will examine his discussion of a purported letter to King Ptolemy III of Egypt by Eratosthenes of Cyrene (276–194 BCE) with a summary of earlier work and Eratosthenes' own solution making use of an instrument he dubbed the *mesolabe*, or "mean-taker."[24] I include the following brief description of Eratosthenes' device to illustrate a construction where the adjectives "tool-based" and "mechanical" clearly do apply in the senses we have suggested above.

Let $AE > DH$ be two unequal line segments and suppose it is required to find two mean proportionals in continued proportion between them. Imagine the configuration in Figure 1 showing the mesolabe in its original position, with three rectangular panels whose horizontal sides are all equal: $EF = FG = GH$. Hence the diagonals AF, JG, KH are parallel.

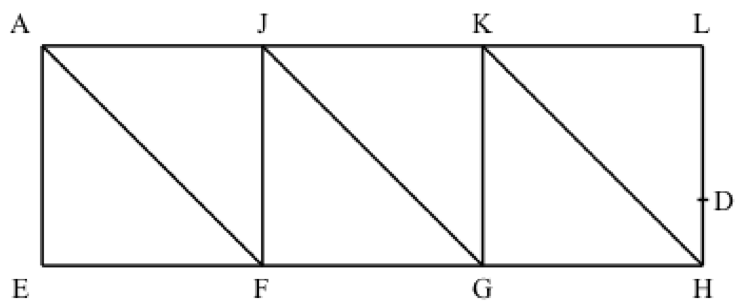

Figure 1: The *mesolabe* in original position.

The three panels are arranged something like a set of sliding screen doors on separate grooves; the left panel $AEFJ$ can slide to the right, over and partially covering the middle panel $JFGK$. Meanwhile, the right panel $KGHL$ can slide to the left behind the middle panel.

[23]A number of conjectural attempts to reconstruct Eudoxus' solution have been made, however. See the discussion in [4, pages 52–57].

[24]The purpose of the letter is essentially to claim the superiority of Eratosthenes' tool-based mechanical method for practical use. It was dismissed as a later forgery by some 19th and early 20th century historians, but more recently, the tide of opinion has seemingly changed and sources such as [4] and [9] argue that it should be accepted as authentic.

Using his or her senses with some trial and error, the geometer using the mesolabe maneuvers the left and right panels until a configuration like the one in Figure 2 is reached. The labels J, F, K, G, L, H now show the final positions of the points on the right-hand edges of the three panels.

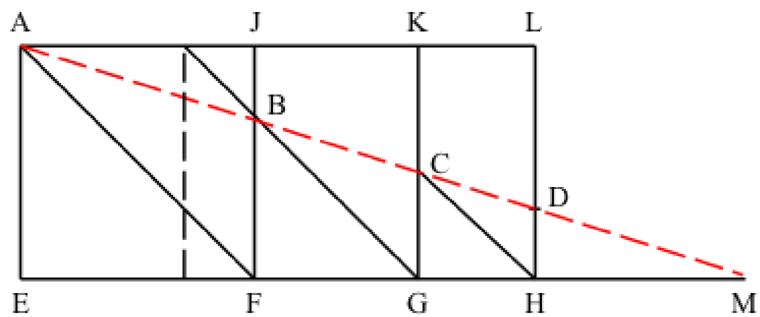

Figure 2: The *mesolabe* in final position.

The left-hand edge of the middle panel is shown with the dashed black line; it lies behind the final position of $AEJF$. The left-hand edge of the right panel, which lies behind the middle panel, has been omitted here for legibility. Here point B is the intersection of JF and the diagonal of the middle panel, which does not move. Similarly, point C is the intersection of KG, the right-hand edge of the middle panel and the diagonal of the right panel. *The final configuration is arranged to make the four points A, B, C, D collinear.* We claim that when such a configuration is reached, we have the two required mean proportionals in continued proportion. This follows easily because ΔAEF, ΔBFG, and ΔCGH are similar triangles, and the same is true for ΔABF, ΔBCG and ΔCDH. Hence

$$\frac{AE}{BF} = \frac{AF}{BG} = \frac{BF}{CG}.$$

Similarly, we can see that

$$\frac{BF}{CG} = \frac{BG}{CH} = \frac{CG}{DH},$$

and hence

$$\frac{AE}{BF} = \frac{BF}{CG} = \frac{CG}{DH}.$$

I will next discuss the main ideas behind what Eutocius says about the approaches of Archytas and Menaechmus and the ways this evidence has been interpreted. My goals here are to shed some additional light on the content of the Plutarch passages and to show how different interpretations of what Eutocius says have led to quite different understandings of this part of Greek geometry. Some of these seem more faithful to context of this work and some seem more anachronistic.

7. The Work of Archytas

In Eutocius' presentation, the account of the approach by Archytas is specifically attributed to Eudemus' history, which means what we have may actually be a commentary on a commentary.[25] The solution is essentially based on a geometric configuration in which it can be seen that two mean proportionals in continued proportion have been found. Borrowing from [7], we will call these *Archytas configurations.* One of these is shown in Figure 3.

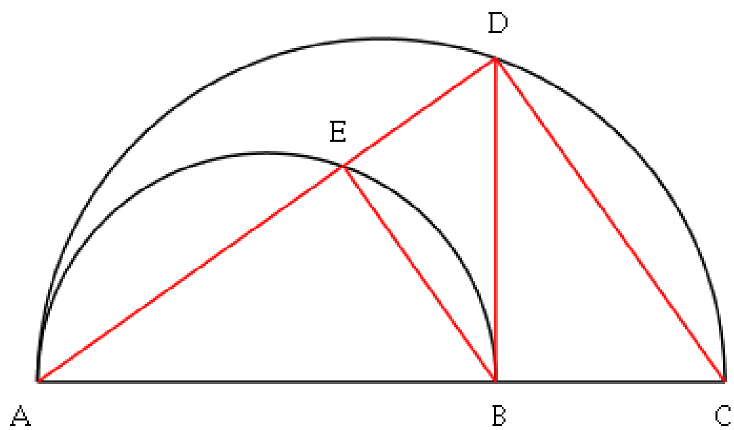

Figure 3: An *Archytas configuration.*

Here AEB and ADC are two semicircles tangent at A, and BD is tangent to the smaller semicircle at B.

[25] The other sections of Eutocius' commentary are not labeled in this way; they give the name of the author, and sometimes the title of the work from which Eutocius is quoting.

It follows from some standard geometric facts that ΔBAE, ΔCAD, ΔDBE, ΔCDB and ΔDAB are all similar. This follows because the angles $\angle AEB$ and $\angle ADC$ are inscribed in semicircles, hence right angles, and hence the lines $\overleftrightarrow{EB}$ and $\overleftrightarrow{DC}$ are parallel. From this we can see immediately that taking ratios of longer sides to hypotenuses in three of these triangles,

$$\frac{AE}{AB} = \frac{AB}{AD} = \frac{AD}{AC}.$$

In other words, AB and AD are two mean proportionals in continued proportion between AE and AC.

But now, we must address the question of how such a configuration would be constructed given the lengths $AE < AC$. A modern explanation might run as follows. The issue is that although we can always take the segment AC as the diameter of the larger semicircle, there is no direct way to find the smaller semicircle, the perpendicular BD to AC and the point E without some sort of continuity argument or approximation process. Consider the situation in Figure 4.

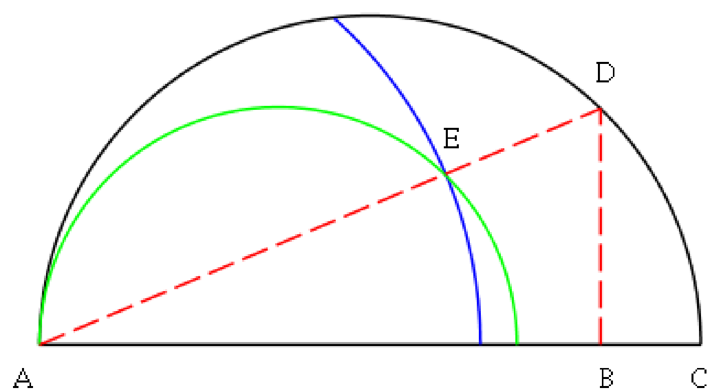

Figure 4: A failed attempted construction of an *Archytas configuration*.

Given the lengths $AE < AC$, the possible locations of the point E lie on an arc of the circle with center at A and radius equal to a specified length. One such arc is shown in blue in Figure 4. Through each point E on that arc, there is exactly one semicircle tangent at A to the semicircle with diameter AC, shown in green in the figure. The line through A, E meets the outer semicircle at D and B is the foot of the perpendicular from D to AC. However, note that with this choice of E, DB does not meet the smaller semicircle at all. However, by rotating the segment AE about A and increasing the angle $\angle CAE$, we would eventually find that the corresponding BD cut through the corresponding smaller semicircle. Hence there must be some point E on the blue arc that yields an Archytas configuration as in Figure 3, by continuity. As we have described it, a naive process of finding that point might involve motion and exactly the sort of resort to "eyeballing" or use of the senses and approximation that Tyndares says Plato criticized in our passages from Plutarch! We can also easily locate such a point using modern coordinate geometry, trigonometry, and numerical root finding. But needless to say, all of that is well beyond the scope of Greek mathematics.

What Eutocius said that Archytas actually did here has been interpreted in a number of different ways by different modern scholars. One tradition known from influential sources such as Heath's history of Greek mathematics, [3], interprets Archytas' solution as a bold foray into solid geometry whereby a suitable point like our E in Figure 3 is found by the intersection of three different surfaces in three dimensions (a cylinder, a cone and a degenerate semi-torus–the surface of revolution generated by rotating the semicircle with diameter AC about its tangent line at A). Heath characterizes this solution as "the most remarkable of all" those discussed by Eutocius because of the sophisticated use of three-dimensional geometry he sees in it [3, pages 246–249]. Similarly, Knorr calls it a "stunning *tour de force* of stereometric insight" [4, page 50].

Very recently, however, a new interpretation based on a close reading of the Greek text of Eutocius has appeared in the historical literature in [7]. As the author Masià points out, while surfaces in three-dimensional space are indeed mentioned in Eutocius' (or perhaps Eudemus') presentation of Archytas' work, it is not easy to see all of the aspects of Heath's description in the actual text. While a (semi-)cylinder and a cone are explicitly mentioned, the semi-torus surface of revolution is not. Moreover, even there, the cone and its properties are not really used in the proof; it seems to be included more for the purposes of visualization and to show how an exact solution could be specified without recourse to approximation [7, page 203]. He also points out that Heath's characterization of this solution as the "most remarkable" does not seem to match the way the solution is presented by Eutocius. It is not given first or last, or singled out in any other way. Hence Heath's interpretation, while certainly a correct way to describe the geometry, seems to be a somewhat *anachronistic* reading because it uses the intersection of three surfaces in three-dimensional space in such a crucial way.

Instead, Masià suggests that the argument can be understood in a fashion that seems much closer to the actual text in Eutocius and to what we know about the state of mathematics at the time of Archytas, in which we think geometry in three dimensions was essentially still in its infancy.[26] Consider first the configuration in Figure 5, in which we have a semicircle and an inscribed triangle two of whose sides have the given lengths AE and AC. The goal is to find two mean proportionals in continued proportion between these.

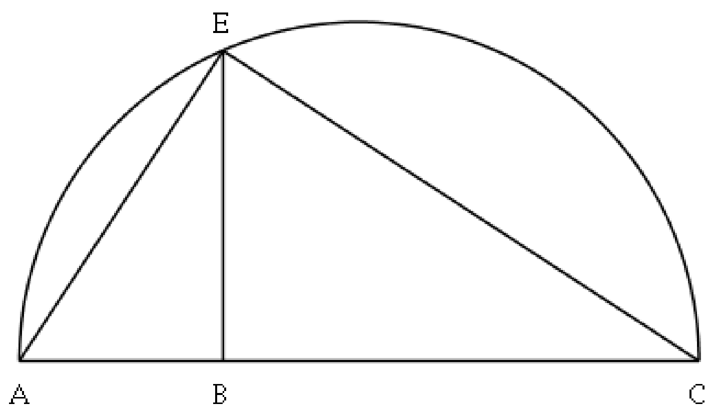

Figure 5: Masià's suggestion, initial configuration.

In Figure 6, imagine a second copy of the semicircle rotating about A (*in the plane*). As this rotation occurs, the line AE is extended to meet the rotated semicircle at E' and a perpendicular is dropped from E' to B' on the diameter AC' of the rotated semicircle. The rotation is continued until B' lies on CE. When this configuration is reached, we have an Archytas configuration because it is easy to see that the points A, B', and E lie on a semicircle tangent to $B'E'$ as in Figure 3.

[26]One piece of evidence for this is the comments about the state of solid geometry in Plato, *Republic*, 528bc, [10, page 760]. How this might relate to Plato's supposed criticism of Archytas for this work is yet another mysterious aspect of this episode.

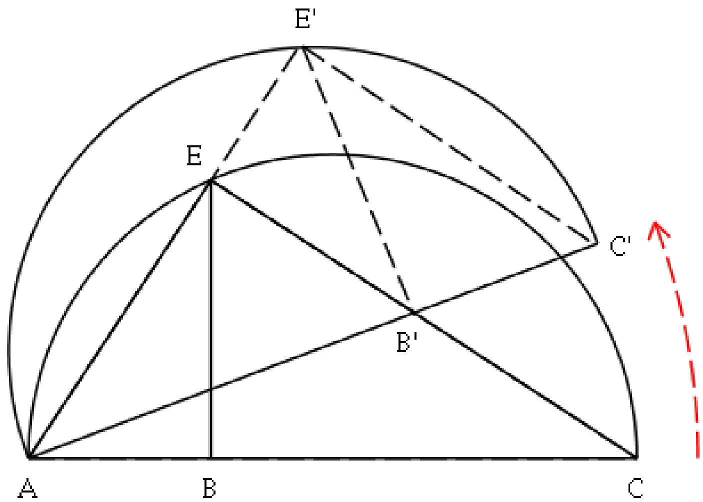

Figure 6: Masià's suggestion, final configuration.

Masià suggests that this motion is then emulated in three dimensions by triangles in two perpendicular planes to produce a construction matching the Greek text of Eutocius very closely.[27] In either our simple presentation, or the kinematic description of the three surfaces in three dimensions, or the new reading of Archytas' construction from [7], there is definitely an aspect of motion that seems to agree with Plato's reported characterization of the construction as "mechanical." How the adjective "instrument-" or "tool-based" might apply is not as clear, although one could easily imagine a device (requiring input from the geometer's senses and hence yielding an approximate solution) to carry out the rotation shown in Figure 6.

8. The Work of Menaechmus

The approach attributed by Eutocius to Menaechmus is even more problematic although it was evidently extremely influential for the development of a key part of Greek geometry. This approach can be described as follows.[28]

[27] See [7, pages 188–193]. Masià discusses several other possible ways to interpret Archytas' solution in two or three dimensions and discusses other interpretations including the one given in [6, Chapter 5].

[28] This is essentially the presentation given in [3, pages 252–255], although Heath does not use coordinate geometry explicitly in this way.

Given line segments of lengths a, b, finding the two mean proportionals in continued proportion means finding x, y to satisfy:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}. \tag{2}$$

Hence, cross-multiplying and interpreting the resulting equations via coordinate geometry (very anachronistically), we see the solution to the Delian problem will come from the point of intersection of the parabola $ay = x^2$ and the hyperbola $xy = ab$, or one of the points of intersection of the two parabolas $ay = x^2$ and $bx = y^2$.

As beautiful as this is, we must ask whether it is likely that Eutocius has preserved a historically accurate account of Menaechmus' work. In particular, could Menaechmus have recognized that he was dealing with a conic section from an equation like $ay = x^2$, where in his terms, the a, y, x would have represented line segments and each side of the equation would have represented an area? None of Menaechmus' own writings have survived. Suspiciously, the discussion of his work in Eutocius uses the terminology for conic sections introduced much after the time of Menaechmus himself by Apollonius of Perga (262–190 BCE). Apollonius' work does provide exactly the point of view needed to connect sections of a cone with equations such as $ay = x^2$ or $xy = ab$. Since Apollonius' terminology and conceptual framework for conics seems to have been developed by analogy with constructions in the application of areas (a technique that Menaechmus would have known well), there is no doubt that some connection between Menaechmus and the later theory of conics exists. And if Menaechmus already did have a full theory of conics, as traditions preserved by Proclus[29] suggest, there is a favorite candidate for what it may have looked like. Earlier terminology, according to which parabolas are "sections of right-angled cones" (by planes perpendicular to one of the generating lines of the cone) and hyperbolas are "sections of

[29] [14, page 91]. Proclus mentions as evidence the epigram of Eratosthenes on the duplication of the cube that concludes the letter to Ptolemy III mentioned above. This includes the direction "neither seek to cut the cone in the triads of Menaechmus" to obtain a solution (μηδὲ Μεναιχμείους κωνοτόμειν τριάδας διζήαι ...). See [1, page 112]. (The present middle indicative διζήαι should probably be the aorist middle subjunctive διζήσῃ and other sources correct it that way.) Exactly what this phrase means is not at all clear. Some writers have seen in the "triads" the division of conic sections into the three classes of ellipses, parabolas, and hyperbolas. But note that there aren't any ellipses here.

obtuse-angled cones" is preserved in such works as Archimedes' *Quadrature of the Parabola*. This is sometimes used to infer the way Menaechmus may have approached the definition of the conics as well as the way that theory may have been presented in the lost *Conics* of Euclid.[30]

On the other hand, in [4, pages 61–69], Knorr discusses reasons why it is at least questionable that Menaechmus possessed such a theory of conics. He presents an alternative conjectural reconstruction of Menaechmus' work that does not rely on curves derived as sections of cones. The idea is that techniques clearly available to Menaechmus would allow one to construct arbitrarily many pairs x, y of lengths satisfying $ay = x^2$ or $bx = y^2$ and hence approximate a solution of (2) without a construction of the whole curve defined by one of these equations or deeper understanding of the properties of parabolas. The interpretation via the curves obtained by intersecting a cone with a plane in a particular position would have come later. Knorr's version is also at least plausible. In any case, it is virtually certain that either a source Eutocius consulted or Eutocius himself reworked Menaechmus' presentation in the light of later developments. Unfortunately, with our fragmentary knowledge from the surviving ancient sources, we cannot really be sure about any of this. I would venture, though, that attributing a full-blown theory of conic sections (that is, as plane sections of cones) to Menaechmus may be an instance of the sort of *conceptual anachronism* that is unfortunately common in conventional histories of mathematics.[31]

Another slightly mysterious aspect of the Platonic criticism recounted in our Plutarch passages is how the adjectives "mechanical" or "instrument-" or "tool-based" might apply to what is attributed to Menaechmus by Eutocius. It is true that the conic sections apart from the circle cannot be constructed as whole curves using only the Euclidean tools and other sorts of devices would be needed to produce them. Interestingly enough, along these lines, Eutocius' discussion does include a final comment that "the parabola is drawn

[30] See, for instance, the discussions of Menaechmus' work in [3, pages 251–255] and [2, pages 84-87].

[31] This may in fact apply just as much to Eutocius as to modern historians of mathematics. In my opinion it is not always easy for mathematicians to be good historians because of the habits of mind we acquire from the study of mathematics. We don't always preserve distinctions between logically equivalent forms of statements and we find it too easy to attribute our own understanding of those statements to mathematicians of the past.

by the compass invented by our teacher the mechanician Isidore of Miletus … ."[32] Isidore (442–537 CE) was an architect, one of the designers of the *Hagia Sophia* in Constantinople, and thus this note is surely an interpolation from the general period of Eutocius himself, not a part of the older source Eutocius was using to produce this section of his commentary.[33]

9. Plato's Solution?

In a final, decidedly odd, aspect of this story, Eutocius also gives a construction of the two mean proportionals in continued proportion that he ascribes to Plato himself. But that is one of the most mechanical and tool-based of all the solutions he describes.[34]

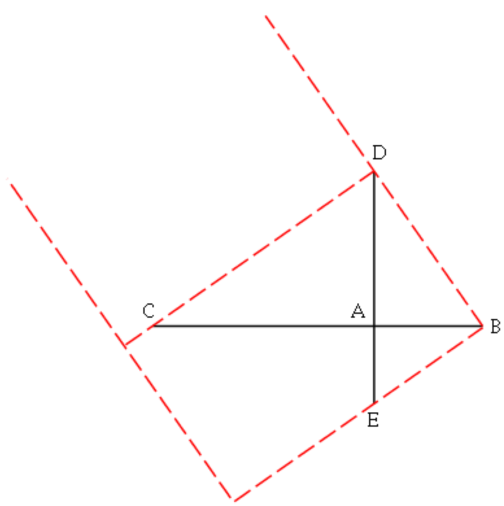

Figure 7: The so-called Plato solution.

In Figure 7, let CA and AE be the given lengths, which are laid off along perpendicular lines to start. The red dashed lines in the figure represent a frame (like two "t-squares" joined along an edge with a movable slider

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[32] "γράφεται δὲ ἡ παραβολὴ διὰ τοῦ εὑρεθέντος διαβήτου τῷ Μιλησίῳ μηχανικῷ Ἰσιδώρῳ τῷ ἡμετέρῳ διδασκάλῳ ... ", [1, page 98].

[33] See [9, page 290, note 130].

[34] The text discussion is accompanied by a rare perspective drawing of the device. A possible explanation for the oddness of this attribution is discussed in [8].

connecting the two parallel edges and forming right angles with them). The side through C and D in the figure represents the final position of the slider, parallel to the side through B and E. The device would be used as follows. Keeping one point fixed at E, the frame would be rotated. For each position, the slider would be moved until it passed through the point D where the side BD meets the extension of AE. If the slider passes through the given point C (as in the final position shown), then the triangles $\Delta AEB, \Delta ABD, \Delta ADC$ are similar and it is easy to show that AB and AD are the two required mean proportionals in continued proportion. Once again, some input from the senses of the geometer and some trial and error would be needed to find the required position. The result is an approximation to the ideal configuration desired. As Knorr says, "one is astounded at the flexibility of the traditions which on the one hand attribute such a mechanism to Plato, yet on the other hand portray him as the defender of the purity of geometry and the sharp critic of his colleagues for their use of mechanical procedures in geometric studies" [4, page 59].

10. Concluding Remarks

Our study of the passages in question shows that Plutarch has seemingly preserved a largely accurate picture of Plato's thinking, certainly more accurate than some of the traditions preserved in Eutocius' commentary. But from what we know of the work of Archytas and Menaechmus and from the later work of Archimedes, Apollonius and others, I would argue that if something like Plato's criticism of the geometers in his circle actually happened at this point in history, then its effect on Greek mathematics was rather minimal.

An openness to mechanical techniques can already be seen for instance in the description of the *quadratrix* curve ascribed to Hippias of Elis (late 5^{th} century CE) and used in the period we have considered in solutions of the angle trisection and circle quadrature problems. We often find scholars of the Hellenistic and later periods pursuing both mathematical and mechanical work, sometimes even in combination. Celebrated examples of this trend include Archimedes' work on spirals, a portion of his *Quadrature of the Parabola*, and most strikingly his *Method of Mechanical Theorems*, which presents a somewhat systematic procedure, based on mechanics, to discover geometric area and volume mensuration results.[35] Even later, Heron of Alexandria (ca. 10–ca. 70 CE) gives another different solution of the problem of the two mean proportionals in his Βελοποιϊκά, a treatise on the design of siege engines and artillery(!)

While it drew on philosophy for its norms of logical rigor, I would agree with Knorr that mathematics had in essence emerged as an independent subject in its own right by the time of Eudoxus, Archytas, and Menaechmus. Plutarch was, by training and inclination, a Platonist and this by itself sufficiently explains his interest in preserving traditions about Plato's thinking about mathematics and his criticism of the geometers in his circle. The following passage in *Republic*, 527ab seems to go along with this. There Plato's Socrates pokes fun at geometers in these words: "Their language is most ludicrous, though they cannot help it,[36] for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge" [10, page 759]. But thinking about the implications of this in connection with what Greek mathematicians were doing by this time, it seems doubtful that Plato's ideas about the proper methods or goals of mathematics carried much real weight for many of the actual practitioners of the subject.

References

[1] Archimedes, *Archimedis Opera Omnia* (ed. J. Heiberg), vol. III, Teubner, Leipzig, 1881.

[2] Carl Boyer and Uta Merzbach, *A History of Mathematics*, 3rd ed., Wiley, NY, 2011.

[3] Thomas Little Heath, *A History of Greek Mathematics*, vol. I, Dover, NY 1981; reprint of the original ed., Oxford University Press, 1921.

[35] Archimedes uses dissection procedures akin to the subdivisions used in modern integral calculus, combined with an idealized balance beam. Most modern mathematicians would probably even be happy to consider his arguments as complete proofs, although Archimedes himself had scruples about that point.

[36] Plato uses the words λέγουσι γελοίως τε καὶ ἀναγκαίως, which literally means something like "they speak laughably by necessity."

[4] Wilbur Knorr, *The Ancient Tradition of Geometric Problems*, Dover, NY, 1993; corrected reprint of original ed., Birkhäuser, Boston, 1986.

[5] Wilbur Knorr, "Observations on the Early History of Conics," *Centaurus*, Volume **26** (1982), pages 1–24.

[6] Wilbur Knorr, *Textual Studies in Ancient and Medieval Geometry*, Birkhäuser, Boston, 1989.

[7] Ramon Masià, "A new reading of Archytas' doubling of the cube and its implications," *Archive for History of Exact Sciences*, Volume **70** (2016), pages 175–204.

[8] Reviel Netz, "Plato's Mathematical Construction," *The Classical Quarterly*, Volume **53** (2003), page 500–509.

[9] Reviel Netz, *The Works of Archimedes, Volume 1: The Two Books On the Sphere and the Cylinder*, Cambridge University Press NY, 2004.

[10] Plato, *The Collected Dialogues*, (eds. Edith Hamilton and Huntington Cairns), Bollingen Series LXXI, Princeton University Press, Princeton, 1961.

[11] Plutarch, *De E apud Delphos*, Greek text edited by Gregorius N. Bernardakis, available at http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:2008.01.0242, accessed on June 9, 2017.

[12] Plutarch, *Quaestiones Convivales*, Greek text edited by Gregorius N. Bernardakis, available at http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:2008.01.0311, accessed on June 9, 2017.

[13] Plutarch, *Marcellus*, Greek text edited by Bernadotte Perrin, available at http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:2008.01.0118, accessed on June 9, 2017.

[14] Proclus, *A Commentary on the First Book of Euclid's Elements*, (trans. G. Morrow), Princeton University Press, Princeton, 1970.

[15] A. Steele, "Über die Rolle von Zirkel und Lineal in der Griechischen Mathematik," *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik, Abteilung B–Studien*, Volume **3** (1936), pages 287–369.

[16] Xenophon, *Memorabilia*, Greek text available at http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.01.0207, accessed on June 9, 2017.