Cello Tangents of Quartic Polynomials

Robert Haas Ph.D.

Follow this and additional works at: <https://scholarship.claremont.edu/jhm>

Part of the [Arts and Humanities Commons](#), and the [Mathematics Commons](#)

Recommended Citation

Robert Haas Ph.D., "Cello Tangents of Quartic Polynomials," *Journal of Humanistic Mathematics*, Volume 8 Issue 1 (January 2018), pages 329-331. DOI: 10.5642/jhummath.201801.15. Available at: <https://scholarship.claremont.edu/jhm/vol8/iss1/15>

©2018 by the authors. This work is licensed under a Creative Commons License.

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | <http://scholarship.claremont.edu/jhm/>

The editorial staff of JHM works hard to make sure the scholarship disseminated in JHM is accurate and upholds professional ethical guidelines. However the views and opinions expressed in each published manuscript belong exclusively to the individual contributor(s). The publisher and the editors do not endorse or accept responsibility for them. See <https://scholarship.claremont.edu/jhm/policies.html> for more information.

Cello Tangents of Quartic Polynomials¹

Robert Haas

Cleveland Heights, Ohio, 44112, USA
rhaas3141@yahoo.com

Synopsis

This note uses easy calculus and linear algebra to analyze the situation of a line being tangent to two points of a fourth-degree polynomial curve.

A cello is a sensitive musical instrument quite responsive to its owner's moods. When I'm too busy doing mathematics to play mine, it tries to join me (or call some attention to itself) by resting on its side in the corner with its bow lying on top in the mathematically elegant configuration of a line tangent to a curve at two points (see Figure 1 below). This note uses easy calculus and linear algebra to analyze such “cello tangents” in the simplest case that the curve is a quartic polynomial.

Figure 1: A cello resting on its side, with its bow lying on top. Photo by Philip Choi, used with his permission.

---

¹ Dedicated to my fine cello teacher Julie Meyers King.

One may derive first a general condition: Suppose that the curve is a differentiable function $f(x)$, the two tangent points occur at $x = a$ and $x = b$ ($a < b$), and the line is

$$y = mx + c = f'(a)x + f(a) - f'(a)a.$$

Then the “cello tangent” condition is

$$f'(a) = f'(b) = \frac{f(b) - f(a)}{b - a}.$$

If f is a polynomial, since it has two peaks it must have degree greater than or equal to 4. So suppose it is a quartic, with degree exactly 4. By rescaling the axes one may take its leading coefficient as -1 , giving it the form

$$f(x) = -x^4 + Ax^3 + Bx^2 + Cx + D.$$

One thus obtains a set of four equations:

$$\begin{aligned} Aa^3 + Ba^2 + Ca + D &= a^4 + f(a) \\ 3Aa^2 + 2Ba + C &= 4a^3 + f'(a) \\ Ab^3 + Bb^2 + Cb + D &= b^4 + f(b) \\ 3Ab^2 + 2Bb + C &= 4b^3 + f'(b). \end{aligned}$$

stating, respectively, the definitions of $f(a)$, $f'(a)$, $f(b)$, and $f'(b)$ equal to $f'(a)$. To obtain specified values, one simply solves these equations for A , B , C , and D .

For example, setting $a = -1$, $b = 1$, $f(a) = .4$, $f(b) = .8$, yielding $f'(a) = f'(b) = .2$, gives the matrix equation:

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1.4 \\ -3.8 \\ 1.8 \\ 4.2 \end{bmatrix}$$

which, inverting the matrix, yields:

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -3 & -1 & 3 & -1 \\ 2 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1.4 \\ -3.8 \\ 1.8 \\ 4.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ .2 \\ -.4 \end{bmatrix}$$

so that $f(x) = -x^4 + 2x^2 + .2x - .4$, see Figure 2. A little more generally, $f(a) = r$ and $f(b) = s$ comes from

$$f(x) = -x^4 + 2x^2 + \frac{s-r}{2}x + \frac{s+r-2}{2}.$$

Figure 2: The figure shows the graphs of the quartic polynomial curve $y = -x^4 + 2x^2 + .2x - .4$ and the line $y = .2x + .6$. The line is tangent to the curve at two points $(-1, .4)$ and $(1, .8)$ that lie slightly to the left of the curve's maxima of $(-.9742, .4026)$ and $(1.0241, .8024)$.

Since the tangent line tilts upward, with a slope of .2, the “cello points” (at $x = -1$ and 1) occur slightly to the left of the curve's maxima (at $x = -.9742\dots$ and $1.0241\dots$).

One might wonder whether the same ideas work for polynomials with three, four, etc. peaks (degree 6, 8, $\dots$), or even infinitely many (e.g. an undamped sinusoid). Let me go off to play my cello and think about it!