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Cello Tangents of Quartic Polynomials

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Synopsis

This note uses easy calculus and linear algebra to analyze the situation of a line being tangent to two points of a fourth-degree polynomial curve.

A cello is a sensitive musical instrument quite responsive to its owner’s moods. When I’m too busy doing mathematics to play mine, it tries to join me (or call some attention to itself) by resting on its side in the corner with its bow lying on top in the mathematically elegant configuration of a line tangent to a curve at two points (see Figure 1 below). This note uses easy calculus and linear algebra to analyze such “cello tangents” in the simplest case that the curve is a quartic polynomial.

Figure 1: A cello resting on its side, with its bow lying on top. Photo by Philip Choi, used with his permission.

\[ \text{\textsuperscript{1}} \text{ Dedicated to my fine cello teacher Julie Meyers King.} \]
One may derive first a general condition: Suppose that the curve is a differentiable function \( f(x) \), the two tangent points occur at \( x = a \) and \( x = b \) \((a < b)\), and the line is
\[
y = mx + c = f'(a)x + f(a) - f'(a)a.
\]
Then the “cello tangent” condition is
\[
f'(a) = f'(b) = \frac{f(b) - f(a)}{b - a}.
\]
If \( f \) is a polynomial, since it has two peaks it must have degree greater than or equal to 4. So suppose it is a quartic, with degree exactly 4. By rescaling the axes one may take its leading coefficient as \(-1\), giving it the form
\[
f(x) = -x^4 + Ax^3 + Bx^2 + Cx + D.
\]
One thus obtains a set of four equations:
\[
\begin{align*}
Aa^3 + Ba^2 + Ca + D &= a^4 + f(a) \\
3Aa^2 + 2Ba + C &= 4a^3 + f'(a) \\
Ab^3 + Bb^2 + Cb + D &= b^4 + f(b) \\
3Ab^2 + 2Bb + C &= 4b^3 + f'(a).
\end{align*}
\]
stating, respectively, the definitions of \( f(a) \), \( f'(a) \), \( f(b) \), and \( f'(b) \) equal to \( f'(a) \). To obtain specified values, one simply solves these equations for \( A \), \( B \), \( C \), and \( D \).

For example, setting \( a = -1 \), \( b = 1 \), \( f(a) = .4 \), \( f(b) = .8 \), yielding \( f'(a) = f'(b) = .2 \), gives the matrix equation:
\[
\begin{bmatrix}
-1 & 1 & -1 & 1 \\
3 & -2 & 1 & 0 \\
1 & 1 & 1 & 1 \\
3 & 2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix} = 
\begin{bmatrix}
1.4 \\
-3.8 \\
1.8 \\
4.2
\end{bmatrix}
\]
which, inverting the matrix, yields:
\[
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & -1 & 0 & 1 \\
-3 & -1 & 3 & -1 \\
2 & 1 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
1.4 \\
-3.8 \\
1.8 \\
4.2
\end{bmatrix} = 
\begin{bmatrix}
0 \\
2 \\
.2 \\
-.4
\end{bmatrix}
\]
so that \( f(x) = -x^4 + 2x^2 + .2x - .4 \), see Figure 2. A little more generally, \( f(a) = r \) and \( f(b) = s \) comes from

\[
f(x) = -x^4 + 2x^2 + \frac{s-r}{2}x + \frac{s+r-2}{2}.
\]

Figure 2: The figure shows the graphs of the quartic polynomial curve \( y = -x^4 + 2x^2 + .2x - .4 \) and the line \( y = .2x + .6 \). The line is tangent to the curve at two points \((-1, .4)\) and \((1, .8)\) that lie slightly to the left of the curve’s maxima of \((- .9742, .4026)\) and \((1.0241, .8024)\).

Since the tangent line tilts upward, with a slope of .2, the “cello points” (at \( x = -1 \) and 1) occur slightly to the left of the curve’s maxima (at \( x = -.9742 \ldots \) and \( 1.0241 \ldots \)).

One might wonder whether the same ideas work for polynomials with three, four, etc. peaks (degree 6, 8, \ldots), or even infinitely many (e.g. an undamped sinusoid). Let me go off to play my cello and think about it!