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Hamilton's Icosian Calculus and His Icosian Game

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In 1856 the Irish mathematician, Sir William Rowan Hamilton, discovered a new system of mathematics which he called Icosian Calculus because it was based on the relationship of the sides of an icosahedron. This system is an early development in what is now called graph theory.

Eventually, however, Hamilton used the dual of the icosahedron, the dodecahedron, with its twenty vertices and thirty edges, to serve as a model for this calculus (Fig. 1). The formula for the Icosian Calculus gave the only possible route which would yield a complete circuit of the vertices of the dodecahedron and would return to the vertex where the path began. Such a circuit which traverses each vertex of a graph exactly once and returns to its starting point is called a Hamiltonian circuit. A path through all the vertices that does not return to the starting point is a Hamiltonian path.

John Jaques and Sons, a London toymaker, paid Hamilton 25 pounds for all the rights to produce his Icosian Game. Hamilton did insist that a description of the mathematics of the game be included with the instruction sheet, and it was—at the end of the directions. Although Hamilton found the game more difficult than did many children, the game was not enthusiastically received because it was "too easy." It is reported that the toymaker never recovered his investment. Hamilton was delighted, however, that he had finally received money, rather than "just medals," for some of his mathematical discoveries.

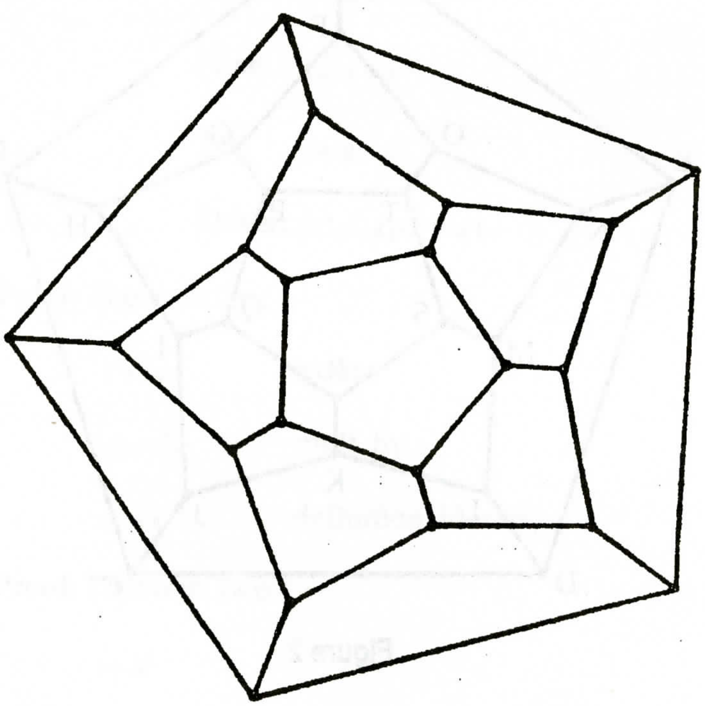

Figure 1

His Icosian Calculus is considered to be a significant contribution to graph theory and to the theory of abstract groups.

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Francis Edgeworth went to Ireland from England about 1583 from Edgeware, Middlesex. His great-great-great-grandson was Richard Lovell Edgeworth. RLE was married to four different wives, serially, and in 48 years fathered 21 children. The oldest of these, Richard, born in 1764, came to Charleston, SC, in the 1780's and became my late husband's great-great-great-grandfather. The second oldest of these, Maria, became a popular novelist in her day and influenced Sir Walter Scott—he tried to depict the Scots in the way that Maria had depicted the Irish. One of the younger children, Francis, who hated mathematics but enjoyed philosophical debate, was a friend of Hamilton. Maria was 56 years old when Hamilton met her in 1824; he was 18, and Francis was 14.

RLE was a Fellow of the London Royal Society (1781) and was one of the founders of the Royal Irish Academy in 1785. Hamilton was elected President of the RIA in 1837. While he was President, Maria was invited to join the RIA—with the understanding that she would "excuse herself" during business meetings when the men would drink port and smoke cigars. She was the first female invited to join this august organization.

During a recent trip to Ireland (1987) I had the oppor-

tunity to visit the old homeplace in Edgeworthstown and learned that Hamilton had visited with the Edgeworths many times. The next day on a trip to Trinity College in Dublin, I met Professor Sterling—an expert on Hamilton. I was interested, especially, in visiting the bridge where Hamilton had determined how quaternions were to be multiplied. He gave me directions to the bridge, and when I told him of my connection with the Edgeworths he suggested that I visit the RIA; he is a current member. I needed no encouragement.

While visiting the RIA they brought out Hamilton's Icosian Game for me to see.

Of course, I knew that I must research this topic when I returned home. One of the original games was given to the Edgeworths, but, of course, I have no idea what has happened to it.

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Hamilton, on a visit to John Graves while attending a meeting of the British Association in 1856, got the idea for his Icosian Calculus and executed its development in two months. Thomas P. Kirkman contributed a paper to the London Royal Society on August 6, 1855, in which he stated conditions for a complete circuit of the vertices of any polyhedron, but it contained errors. It did, however, correctly identify a general class of graphs that does not possess a complete circuit. This was a significant contribution, and some believe that he should be credited with priority. However, perhaps because Kirkman was a rector and an amateur, though enthusiastic mathematician, Hamilton is credited with this area of mathematics. Certainly Hamilton, alone, developed the algebra for his Icosian Calculus and did his work independently.

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The directions for the game may be varied for the number of players. Solitaire play consists of finding the Hamiltonian circuit starting at any point. If two people play, the first begins a path through five consecutive vertices and challenges his/her opponent to complete the circuit. Other variations can be described for multiple players.

I have noted, in my readings, two problems presented on the playing board. I have adapted them from numbers four and five in the bibliography, respectively.

1) Two delivery-persons start out on their routes from town A and reunite at town C at the end of the day (Fig. 2). Their journeys do not cross and no highway is used more than once.

   Mr. First travels from A to E to N to M to L to D and then to C. Ms. Second makes deliveries at each of the remaining towns before meeting her colleague at C. Can you find Ms. Second's route?

2) Al and his son Bob start out on their collection routes on the same morning, intending to spend 19 nights on the road, one in each of the towns they visit before returning home. Al will begin and end his trip in town A and Bob will begin and end his trip in town B (Fig. 2).

   The fifth night of their tours they met each other in town L and the following night they again found themselves together in another town.

   At what other towns (if any) did they spend the night together?

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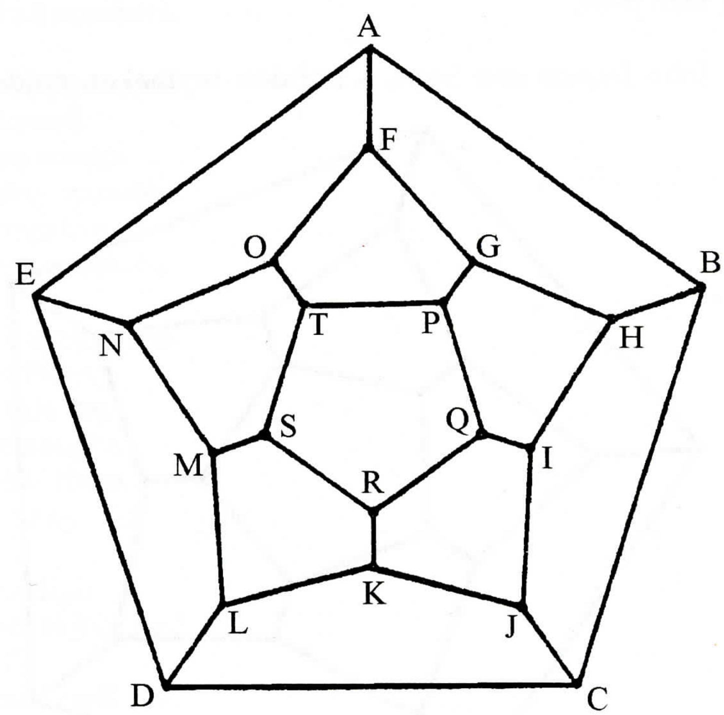

Figure 2

ICOSIAN CALCULUS

Definitions:

$\iota^2 = 1$

$\kappa^3 = 1$

$\lambda^5 = 1$

Axioms: Multiplication is associative, but not commutative.

$\lambda = \iota\kappa$

1 is the multiplicative identity

Theorems:

One: $\kappa = \iota\lambda$

Two: $1 = (\iota\kappa^2)^5$ let $\mu = \iota\kappa^2$

Three: $\mu = \lambda\iota\lambda$

Four: $\lambda = \mu\iota\mu$

Five: $\iota = \lambda\iota\mu$

Six: $\iota = \mu\iota\lambda$

Seven: $\lambda\mu^2\lambda = \mu\lambda\mu$

Eight: $\mu\lambda^2\mu = \lambda\mu\lambda$

Nine: $\lambda\mu^3\lambda = \mu^2$

Ten: $\mu\lambda^3\mu = \lambda^2$

Eleven: $[\lambda^3\mu^3(\lambda\mu)^2]^2 = 1$

Proof: Theorem One

$\lambda = \iota\kappa$	axiom
$\iota\lambda = \iota^2\kappa$	mult. by ι
$\iota\lambda = \kappa$	definition, identity

Proof: Theorem Two

$1 = \lambda^5$

$1 = (\iota\kappa)^5$

$1 = \iota\kappa\iota\kappa\iota\kappa\iota\kappa\iota\kappa$

$1 \cdot \kappa^2 = \iota\kappa\iota\kappa\iota\kappa\iota\kappa\iota\kappa \cdot \kappa^2$

$\kappa^2 = \iota\kappa\iota\kappa\iota\kappa\iota\kappa\iota$

$\kappa^2\iota = \iota\kappa\iota\kappa\iota\kappa\iota\kappa$

$\kappa^2\iota\kappa^2 = \iota\kappa\iota\kappa\iota\kappa\iota$

$\kappa^2\iota\kappa^2\iota = \iota\kappa\iota\kappa\iota\kappa$

$\kappa^2\iota\kappa^2\iota\kappa^2 = \iota\kappa\iota\kappa\iota$

$\kappa^2\iota\kappa^2\iota\kappa^2\iota = \iota\kappa\iota\kappa$

$\kappa^2\iota\kappa^2\iota\kappa^2\iota\kappa^2 = \iota\kappa\iota$

$\kappa^2\iota\kappa^2\iota\kappa^2\iota\kappa^2\iota = \iota\kappa$

$(\kappa^2\iota)^4 = \iota\kappa$

$1 = \iota^2\kappa^3$

$1 = \iota(\iota\kappa)\kappa^2$

$1 = \iota(\kappa^2\iota)^4\kappa^2$

$1 = \iota\kappa^2\iota\kappa^2\iota\kappa^2\iota\kappa^2\iota\kappa^2$

$1 = (\iota\kappa^2)^5$

Proof: Theorem Eleven

$$[\lambda^3\mu^3(\lambda\mu)^2]^2 = [\lambda^2(\lambda\mu^3\lambda)\mu\lambda\mu]^2$$

$$= [\lambda^2\mu^2\mu\lambda\mu]^2$$

$$= [\lambda(\lambda\mu^3\lambda)\mu]^2$$

$$= [\lambda\mu^2\mu]^2$$

$$= [\lambda\mu^3]^2$$

$$= (\lambda\mu^3\lambda)\mu^3$$

$$= \mu^2\mu^3$$

$$= \mu^5$$

$$= 1$$

without signs of grouping this says:

$$\lambda^3\mu^3\lambda\mu\lambda\mu\lambda^3\mu^3\lambda\mu\lambda\mu = 1$$

APPLICATION OF THE MODEL

$\lambda(PQ)$ **means**

start at vertex P and move along the edge to vertex Q and *"turn to the right"* to vertex R

so $\lambda(PQ) = QR$; $\lambda^5(PQ) = (PQ)$

$\mu(PQ)$ **means**

start at vertex P and move along the edge to vertex Q and *"turn to the left"* to vertex I

so $\mu(PQ) = QI$; $\mu^5(PQ) = (PQ)$

So the *circuit* may start at any vertex and be completed by following the "directions" of the last theorem:

right, right, right, left, left, left, right, left, right, left
right, right, right, left, left, left, right, left, right, left

Of course, a mirror image is obtained if "right" is interchanged with "left." Another circuit is obtained if transversed in the opposite order—it is not *different*, however.

$\iota(PQ)$ **means**

"reverse direction" $\iota(PQ) = QP$; $\iota^2(PQ) = PQ$

$\kappa(PQ)$ **means**

"rotate counter clockwise about Q"

so $\kappa(PQ) = RQ$; $\kappa^2(PQ) = IQ$; $\kappa^3(PQ) = PQ$

To illustrate the axiom, $\lambda = \iota\kappa$ we use the symmetric property of equality and

$$\begin{aligned}\iota\kappa(PQ) &= \iota(RQ)\\ &= (QR)\\ &= \lambda(PQ)\end{aligned}$$

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Representation

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In honor of the newest standard of the National Council of Teachers of Mathematics' Principles and Standards for School Mathematics (2000).

(may be sung to the tune of "Anticipation" by Carly Simon and Jacob Brackman)

We...can never know a phenomenon
If we...view it only one way.
What if we...supplement our formula
With words or a graph or tabular display?

Chorus:
Representations, representations
Are helping me to see...are helping my learning...

And I tell you...the more ways I model this
And the more...I translate back and forth,
Then the more...I understand conceptu'ly
And that helps me when I report!