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Victor J. Katz
University of the District of Columbia

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Victor J. Katz
Professor of Mathematics
University of the District of Columbia

As a college teacher of mathematics I receive many new texts each year in precalculus and calculus, each one trumpeting its virtues and its new ideas. But a study of the actual material presented shows that not only are there few new ideas but that chapter for chapter and almost section for section each such book is a repeat of every other one. In fact, I amazed my daughter one day by telling her the titles of the first ten chapters in the calculus text she was using, without ever having seen the book itself. Since all such texts are the same, one could assume that there is a general agreement in the mathematical community that there is one correct way to teach precalculus and calculus. With all the conferences and position papers of the past several years, first on the desirability of discrete mathematics and more recently on the lean and lively calculus, it appears, however, that large numbers of faculty members are dissatisfied with the way these courses are presented. As a matter of fact, the high failure rate in calculus seems to indicate that students too are dissatisfied.

As a possible answer to this dissatisfaction, and as a new way of organizing these two courses, I have experimented over the past several years with an historical approach to both precalculus and calculus — considered together as a four or five term sequence. Not only does this approach help integrate the discrete algorithmic material with the continuous analytic mathematics, since in fact much of the former was developed alongside of the latter, but it also helps to introduce our science and engineering students to the relationship between mathematics and the rest of our culture. As it stands now, many students are sadly lacking in an awareness of the place which mathematics occupies in our culture. They are interested in the mastery of technique to the exclusion of learning the reasons that the ideas were developed and the use of mathematics in the world. And without the intellectual content behind the mathematical techniques, it appears that in large measure the students fail to grasp even the techniques. An historical approach to these courses helps to provide a solid motivation for the learning of mathematics as it ties together much of the students' backgrounds in history and literature with their scientific studies. It also encourages the student at every stage of his/her studies to explore the ramifications of scientific work as it relates to the world around them. And it seems to me that the prospective scientists and engineers I am teaching will more than ever need a sense of how their own highly technical work fits in with the needs of society as they make decisions which will affect the fate of the world.

By an historical approach to mathematics teaching, I do not mean simply giving the historical background for each separate topic or giving a biographical sketch of the developers of various ideas. I do mean the organizing of the desired topics in essentially their historical order of development. I do mean discussing the historical motivations for the development of each of these topics, both those within mathematics and those from other fields. I also mean connecting the development of each of the mathematical topics with the development of the other sciences and with the other things which were going on in the world. Naturally, I cannot always stick precisely to the historical record. Many seemingly good ideas led to dead ends or to methods which are too difficult for the level of these particular courses. And one should make use of today's technology of calculators and computers to perform tedious computations rather than have the students repeat their ancestors' hand calculations. Nevertheless, using history as a general guide does provide an organizing tool for both precalculus and calculus which helps to motivate the students and shows them that mathematics has always been an important part of the overall culture, not only in the West but also in other parts of the world.

A reasonable question at this point is how we can do all this when we can barely cover all the material in the syllabus as it is. I do not have a simple answer to that question. My experience is that to a large extent, this historical approach is more time-efficient than a more standard approach and that the historical connections drawn help to motivate and excite the students, enabling

them to do more work independently. Of course, we cannot neglect the development of technical proficiency or of problem solving skills. But again, both of these aspects can be set into the historical context. And if, in fact, this approach forces us to drop a few topics in the current syllabus, this will only be in line with current recommendations in any case.

Let me now describe the mathematics course I have in mind. In the context of the school at which I teach, it is necessary to begin with precalculus material. But it would not be difficult to begin this course at a later point. As will be clear, virtually all of the standard precalculus and calculus topics will be dealt with, but often in an order and a context differing from the standard ones. The course description will emphasize the novel aspects of my approach and how it better serves the students than the usual method.

The course begins with a description of the common mathematical knowledge of various ancient civilizations, namely, basic algebra through quadratic equations, basic geometric formulas, the Pythagorean theorem, and the calculation of square roots. We discuss the nature of these ancient societies, the Babylonian, the Egyptian, the Chinese, and what it means for a certain society to "know" mathematical ideas. Who in the society knew these ideas? Why did they know them? What kinds of problems did they need to solve? In particular, we discuss the value of pi. What does it mean to approximate the ratio of the circumference of a circle to its diameter and how good an approximation is necessary? In the same mode, we also discuss the square root algorithm and its geometric origins. In this context, an introductory discussion of the nature of an algorithm is warranted along with a discussion of accuracy. A review of the quadratic formula is also useful along with the Chinese numerical method of solving quadratic equations.

The major change from these ancient civilizations to the classical Greek in terms of mathematics was in the introduction of deductive proof. Thus we deal briefly with the nature of Greek civilization and its differences from those of Egypt and Babylonia. Then, even though the students have generally had some course in geometry, a discussion of some salient points of Euclid's *Elements* is warranted. Among the topics included are the nature of logical argument and the idea of an axiomatic system, the basic triangle and parallelogram theorems of Book I, the circle theorems of Book III, the construction of the pentagon in Book IV (for its later use in trigonometry), the similarity results of Book VI, and finally, Euclid's result on the area of a circle from Book XII.

Curiously, there is no biographical data available on the author of the world's most famous mathematics text except a few stories dating from some 700 years later than the time of its writing. Since the text is a compilation of several different strands of Greek mathematics and since it was written in Alexandria, a city whose inhabitants came from many different backgrounds, some historians have speculated that it was written by a committee. Among the possibilities for members of this committee would be native Africans, Jews, and even women. Though this particular idea on the authorship of the *Elements* is pure speculation, it should be made clear to students that though certain groups have traditionally been excluded from mathematical knowledge, there were always individuals who somehow managed to buck the trend and make their own contributions. Unfortunately, sometimes the history books ignore these contributions or possibilities, denying a sense of "ownership" to large parts of our population.

Euclid's work is followed in the course by an introduction to conic sections in terms of sections of a cone and in the context of the problem of doubling the cube. I generally stretch the historical record a bit here and interpret Apollonius' work in terms of coordinate geometry as I develop the equations and some of the elementary properties of these important curves. I also introduce the beginnings of mathematical physics in the work of Archimedes, emphasizing in particular the idea of a mathematical model.

The idea of a model is further developed with the study of trigonometry, since that subject originated as a mathematical tool for astronomy, which in turn, in the Greek world, was based on the two-sphere model of the universe. Thus I introduce the students to some ideas in astronomy, using the model having the earth fixed in the center of the heavens. It is interesting, in fact, to ask the students for any evidence that the earth is not stationary. I introduce trigonometry itself in essence as Ptolemy treated it, though rather than deal with his chords I give the modern ratio definitions of the sine, cosine, and tangent. But the sum, difference, and half-angle formulas are done following Ptolemy's geometric proofs and these are used to begin the actual calculation of values of the trigonometric functions. The values for 30, 45, and 60 degrees come from simple right triangle geometry, while those for 36 and 72 degrees come from Euclid's pentagon construction. I think it is important for the students actually to perform some of these calculations themselves, so that they learn that the sine function on their calculator is not magically generated nor does it come from actually measuring triangles. Among the benefits of these calculations is an appreciation of interpolation and

approximation as well as the realization that the sine function is nearly linear for small angles. In fact, it was that realization which enabled Ptolemy to complete the calculation of his table. I even recalculate the sine using radian measure, since, in effect, radian measure was used both in Greece and in India early on. It was realized that for small angles, using this measure, sine x is very nearly equal to x itself.

The students naturally use their calculators rather than the calculated table in applying trigonometry to solve problems. These problems include not only plane problems but also spherical ones, since it was the latter which were most important to the solving of astronomical problems. With a brief discussion of some astronomical concepts and the introduction of the important spherical trigonometry formulas, the students can easily solve problems such as determining the length of daylight on a given day at a given location or the exact direction in which the sun rises or sets.

The next major topic, as we move out of the Greek period, is that of equation solving. I begin this section with a geometric justification of the quadratic formula taken from the work of the Arab algebraist al-Khwarizmi. (That is, completing the square means exactly what it says.) This justification needs a geometric version of the binomial theorem $(a+b)^2 = a^2 + 2ab + b^2$. Continuing in this vein, I ask the natural question of how to solve a cubic equation. There are several medieval Arab works which seek to answer this question, although the answers they give are probably not what students expect. We therefore discuss what it means to "solve" a cubic (or any) equation. The Chinese in the thirteenth century certainly could solve such an equation numerically, while the Arabs of the same time period knew how to do it geometrically. One interesting method to discuss is that of al-Tusi, who used techniques related to the calculus to decide what types of solutions to a cubic were possible.

To find an algebraic solution, however, one must turn to sixteenth century Italy and the story of Tartaglia and Cardano, a story that should certainly be shared with the students. The students should also be made aware of the Renaissance background here and learn why there was a renewed urge in Europe to do mathematics. In particular, it was often the merchants who brought mathematics back to Europe through their travels to Africa and Asia. In any case, the algebraic solution of the cubic begins with the binomial theorem in degree three and, at least in the simple cases, is not difficult to understand. I do not give a complete treatment of cubics, but only enough for the students to get the general idea and see why, in the irreducible case, complex numbers are necessary. I then can introduce these numbers, as Bombelli did late in the sixteenth century, and use them in solving not only cubic but also quadratic equations.

It is now worthwhile to continue the study of solutions of equations from other points of view, once the complexity of the cubic and, perhaps, the quartic formulas are understood. For example, I use the work of Descartes to develop the factor and remainder theorems as well as the methods of finding rational solutions to polynomial equations. If no rational solutions exist, the Chinese method already discussed, as well as Newton's method, itself an adaptation of earlier work, can be used to approximate solutions. So we return to the notion of an algorithm and by using it can explore the idea of convergence. Again, even though modern computer software will enable any polynomial equation to be solved numerically with the touch of a button, it is always important for the students to understand how the algorithms built into the software were developed.

Having already considered the binomial theorem in the cases $n = 2$ and $n = 3$, it is now time to give a more detailed treatment of combinatorics concentrating on the work of Pascal and his medieval Chinese, Arabic, and European predecessors. First, the binomial theorem for any positive integral exponent needs to be developed. Naturally, this is the place to introduce and study the notion of mathematical induction. A consideration of the background and motivation for Pascal's triangle is useful as an introduction to the ideas of probability and the calculations of permutations and combinations. A second major result is the formula for the sums of powers of integers. This is often associated with Bernoulli, but the basic ideas date from somewhat earlier. A third important idea treated here is the general idea of arithmetic and geometric sequences. These latter provide a gentle introduction to the idea of an infinite series and its sum, an idea already understood in some sense by Archimedes.

Once we have geometric and arithmetic sequences, it is time to investigate logarithms. I discuss the scientific need for logarithms as an aid in the tedious computations necessary for the preparation of astronomical tables. So I also need to discuss the need for such tables in the age of European exploration and discovery. In any event, the relationship between geometric and arithmetic sequences and the desire to convert multiplication to addition by using this relationship leads to the necessity for the choice of a good base. Adapting the ideas of Napier and Briggs, I show the students why "natural" logarithms as well as common logarithms were both developed. In fact, the question of the "naturalness" of natural logarithms leads via the work of Napier himself to some important

ideas of differential calculus as well as to the idea of the exponential function.

Another idea being developed in the 17th century, through the work of Fermat and Descartes, was that of analytic geometry. We therefore return to Apollonius' conic sections and show algebraically that any quadratic equation in two variables leads to a conic section. The important tangent and focal properties of these curves are then treated as a further introduction to ideas of calculus. Kepler's laws are then discussed as a continuation of our earlier notion of a mathematical model. Of course, whenever one discusses Kepler, one also must discuss Galileo and the idea that the scientific success of a particular mathematical model does not always mean its acceptance. But we also deal with Galileo as a mathematical physicist as we treat the motion of projectiles and the general idea of a function. Though functions have been discussed earlier on an ad hoc basis, it is here that I make the first attempt to develop the idea in detail, first in terms of physical phenomena and then as a purely mathematical idea. In particular, we consider and graph polynomial functions and some easy rational functions, including one we will use often later, $y = 1/(x+1)$. And again the idea of a mathematical model occurs, this time in earthly terms rather than in the heavens. Finally, the graphs of the logarithm, exponential, and trigonometric functions are briefly discussed, leaving more details to the development of the calculus of these functions.

Certain calculus ideas having been introduced earlier, it is now time for a detailed discussion, using the work of Fermat, of the two basic problems of calculus, areas and extrema (or tangents). I first solve these two problems for the curves $y = x^2$ and $y = x^3$ and then proceed to generalize to $y = x^n$. For derivatives, I use the binomial theorem, and for integrals, I use either the material on Bernoulli numbers or the work on geometric sequences. In both cases, we need the basic idea of a limit, so an intuitive discussion of limits is warranted here. The elementary results on power functions can then be extended to a procedure for finding derivatives and integrals of polynomials. Now although Fermat essentially had these results, he never noticed what Newton did, the Fundamental Theorem. In any case, we have now reached the central figure in the scientific revolution. A statement and at least a sketch of a proof of the Fundamental Theorem can now be given, probably in terms of velocity and distance. In fact, with the right preparation, the students can "discover" this theorem for themselves. But I also discuss to some extent the scientific revolution itself and Newton's part in it. In particular, throughout the remainder of the course I deal with some of the problems the calculus enabled Newton and others to solve.

It is at this point in the syllabus that I believe the historical approach makes its most important contribution to the study of calculus, namely the early introduction of the notion of a power series. Power series were one of Newton's earliest discoveries in calculus and one which he used constantly. In fact, I also note that for the sine and cosine, power series had been developed even earlier in India. It is quite effective pedagogically to introduce series at the earliest possible moment, namely, as soon as the basic derivative and integral algorithms are known and the fundamental theorem is proved. Power series can then be used as a theme throughout the rest of the course. They provide examples of algorithms, explicit calculations of certain integrals, ideas on the relationships among various functions, a further introduction to the fundamental idea of convergence, and a prime example of the method of discovery through analogy.

As a beginning to the study of power series, we simply deal with them as generalized polynomials which we can add, subtract, multiply, and divide. We then discuss convergence by trying to use power series to represent functions. In particular, we can think of power series as generalizing infinite decimal expansions of numbers. As a first nice example of a power series, I show the generalization of the binomial theorem to negative and fractional exponents. I then use this to calculate square and cube roots, for example. I also note that, since the power series can be considered as generalized polynomial functions, we can take the derivative and anti-derivative term by term.

We do need other techniques in calculus. So we proceed to the basic approximation theorem, $f(x+h) \approx f(x) + f'(x)h$ and then the ideas of Leibniz and his followers, in particular the idea of a differential. The approximation theorem gives us straightforward proofs of the product rule, quotient rule, and chain rule. We can then deal easily with integration by parts and by substitution. On the other hand, we can integrate $\sqrt{1 - x^2}$ and $\sqrt{1 + x^2}$ by using power series since the other methods do not apply. Of course, physical problems, particularly those centered around Newton's laws of motion, must be dealt with here now that the basic tools are available. Maximum-minimum problems are among the more important of these types of problems.

As in any calculus course, it is now necessary to deal with the transcendental functions. The natural logarithm

can be developed as is standard, but is also historically justified, via the integral of $1/(1+x)$. But by considering the power series for that function, we can also get the series for the logarithm. And this, of course, enables us to calculate values. The series for the exponential function can then be developed and discussed by inverting the series for the logarithm, by Euler's method of using the binomial theorem, or, perhaps, by solving the differential equation $y' = ky$, essentially introduced in the earlier treatment of these functions. There are many interesting problems involving the exponential and logarithm functions in early calculus textbooks. It is interesting for today's students to see what kinds of problems earlier students solved, so I show them some from the calculus text of Maria Agnesi, probably the best of those before the works of Euler and evidence that women could and did study mathematics.

The calculus of the trigonometric functions is developed via Newton's idea of relating them to arcs of circles. This leads again to the "naturalness" of radian measure. Thus I begin with arc length and the definition of the inverse functions. Again, these are done in terms of power series. The series for sine, cosine, and tangent can easily be developed by various methods as can the basic rules for derivatives. In fact, the latter is best done by a geometric argument using differentials rather than via the limit argument commonly used. The power series representations of e^x, sin x, and cos x then provide a natural question of how these functions are related. The answer leads into a renewed discussion of complex numbers and also into a treatment of the hyperbolic functions. Finally, the notion of simple harmonic motion and its associated differential equation $y'' = -ky$ is dealt with and shown to result in the trigonometric functions. Thus the basic periodicity of these functions can be understood in terms of an important physical idea.

Other types of physical problems are treated in the process of dealing with various techniques of integration. Not too much time is spent on the techniques themselves, however, since power series methods are available to give results and since new computer algebra systems will generate these results in any case. But we do need to deal with such ideas as arc length, volume, and center of gravity and see what integrals are necessary to solve these problems. At the same time, some of the elementary ideas of differential equations, including the separation of variables and the integration of exact equations, are covered in the context of the physical and mathematical problems which led to their study in the first place.

It is at the end of the one-variable section of the calculus course that I give a detailed treatment of the notions of limit and convergence. It is only after the experience of dealing with these ideas on an intuitive basis for many months that I can expect the students to understand their theoretical underpinnings and the use of epsilons and deltas. After all, the work of Cauchy and Bolzano occurred fully 150 years after that of Newton and Leibniz.

An historical approach to the study of precalculus and calculus can provide valuable insights to the students. With appropriate examples, it can also serve to show that women and minorities have been involved in mathematics in the past and can certainly expect to make further contributions in the future. I am convinced that this approach to this important segment of mathematics is better than the current method which, in calculus at least, begins quite unhistorically, and quite unsuccessfully, with the abstract notion of a limit. Unfortunately, the publishers of mathematics texts hesitate to publish a text which deviates in any major respect from the current model. It is therefore not very easy to get new curriculum ideas out into the mathematics community. If the readers of this essay try out some of the ideas expressed in it, however, and exchange their findings, the mathematical community as a whole may eventually see the benefits of this approach to the teaching of the most popular courses in the mathematics curriculum.