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A Generalised Song

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Synopsis

We consider parallels between words and music. We call a triple of structures, one verbal, one musical, and one mathematical, in which the mathematical structure is related to the verbal and musical structures, a *generalised song*. With the intention of exhibiting the potential of this form, we describe a generalised song called 'Cube.'

1. Generalised songs.

Traditional songs have a verbal element and a musical element. There is a formal element, the metre, which is a structural feature of both the verbal element and the musical element. For example, in the folk song 'Blacksmith' [4] the verbal element begins 'A blacksmith courted me Nine months and better. He fairly won my heart Wrote me a letter.' The song has a tune, the length of whose notes we approximately record in a metre that begins LSSSSL, SSSSL, LSSSSL, SSSSL (here L stands for long syllable, and S for short syllable). The words of the verbal component are in one-to-one correspondence with the letters of the metre, and these letters correspond to notes, or pairs of notes in the tune. We represent the situation with a diagram:

$$\begin{tikzcd} & Me \arrow[dl] \arrow[dr] & \\ Mu & & V \end{tikzcd}$$

where Me stands for metre, Mu stands for the musical element, V stands for the verbal element, and the arrows represent structural relations between them.

Inspired by the above, we define a generalised song to be given by a diagram

$$\begin{tikzcd} & Ma \arrow[dl] \arrow[dr] & \\ Mu & & V \end{tikzcd}$$

where Mu is a musical structure, V is a verbal structure, where Ma is a mathematical structure, and the arrows describe formal relations between them. Given that it involves vague terms like 'mathematical structure' and 'musical structure' this is not a mathematically rigorous definition.

Here we present a generalised song called 'Cube.' In 'Cube', the relevant mathematical structure consists of a cube, the sequence $\{a\}, \{b\}, \{a,b\}$, and six three-point motifs. In writing 'Cube' we looked to design a triple simple enough to be described in a few pages, but complex enough to demonstrate the artistic potential of the form. In the main we disregarded conventional song techniques.

The musical component of 'Cube' is recorded on a stave in 'Cube score' [8]. An audio file for this music is available [7].

Before outlining 'Cube' in detail, we describe in general terms the techniques we use to relate mathematical structures to verbal and musical structures.

2. Some techniques for relating mathematical structures to verbal structures.

There are numerous ways of relating words and mathematics. Mathematical structures can be used in the composition of verbal structures, like sonnets, or rotationally symmetric crosswords, or G. Perec's novel *La Vie mode d'emploi* [5]. Here we describe a number of techniques for relating mathematical structures to verbal structures in generalised songs. This is intended to be a guide to the techniques we use in 'Cube'.

A narrative concerning a collection of characters will commonly consist of a number of scenes. The characters in those scenes determine a sequence of subsets of the set of characters appearing in the narrative. This sequence is a formal mathematical structure. For example, 'The Adventure of the Blue Carbuncle' by A. Conan Doyle [3] has set of all characters indexed by the set

$$\{H, W, P, Ba, L, Br, R\}$$

(Holmes, Watson, Peterson, Baker, Landlord, Breckinridge, Ryder). The scenes determine a sequence of subsets

$$\{H, W, P\}, \{H, W, Ba\}, \{H, W, L\}, \{H, W, Br, R\}, \{H, W, R\}.$$

Note that the definition of scene is somewhat subjective, so this sequence is not canonically defined. Analogously, in a generalised song, our mathematical structure Ma may naturally give rise to a set of subsets, which may be used to thus construct the verbal structure V.

A given scene in a narrative may have a number of different narrators. These narrators may be different characters in the scene, or even different components of a character's psyche. We have a set consisting of all narrators, and each scene involves some subset of narrators. In 'Cube' we have eight narrators, each of which is a single component of the protaganist Daniel's psyche. These are in one-to-one correspondence with the vertices of a cube, and are indexed by the set

$$\{Si, M, Q, O, De, F, So, Di\}$$

(Sight, Metaphor, Query, Other, Desire, Fantasy, Sound, Distinction). The scenes correspond to subsets

$$\{Q, Si, M, O\}, \{M, De, So, O\}, \{So, Di, Q, O\},$$

$$\{De, M, Si, F\}, \{Si, Q, Di, F\}, \{Di, So, De, F\}.$$

These four-element subsets correspond to faces of our cube, as depicted in Figure 1. The faces labelled T, W and N correspond to scenes featuring transactions, walking, and narratives, respectively.

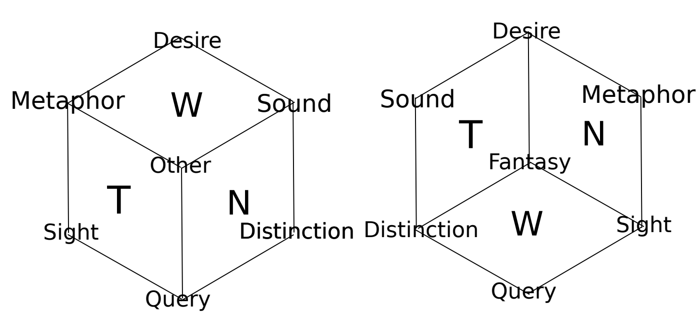

Figure 1: The components of Daniel's psyche.

In ‘Cube’ we have placed the narratives of a given scene on a single double page so they can be read together (the pages are intended be read two to a screen, or printed off two to a sheet). Since we are discussing relations between words and music, it seems worth mentioning that this technique of placing parallel narratives together on a page is motivated by music in which a number of instruments are commonly played simultaneously in harmony.

Amongst the techniques we have described so far, we may choose a symmetrical mathematical object to represent verbally. In this case our narrative will have some symmetry which we may choose to draw out further in our presentation of the narrative. In ‘Cube’ we have presented the narrative in a PDF file [6], in which all four of the orientations of a face of the cube on the screen are drawn up, making the 48 symmetries of the cube less obscure to the reader. We navigate the cube by clicking on small circles at the top, bottom, left, and right of the screen. We think of the screen as displaying a face of the cube, and one click corresponding to a rotation of the cube by 90 degrees in the relevant direction.

Every face of the cube has a symmetrically opposing face. We have given opposing faces similar subject matter (the three pairs concern transactions, walks, and narratives respectively). In one of each opposing pair the protagonist is alone, in the other there are other people present. It is these ‘other’ faces of the cube that contain a vertex corresponding to the ‘other’ narrator. This other narrator superficially appears to be one of the other people present, but is in fact a component of our protagonist’s psyche.

It may not be desirable to have a linear narrative, in which scenes follow one after another. We may instead choose to index scenes by the vertices of a graph whose edges correspond to some relation between scenes (a common character, a common theme, or indeed a closeness in time). This graph is a mathematical structure that may be naturally related to the mathematical structure of our generalised song.

For example, in ‘Cube’ the relevant graph is the dual graph of a cube. In the PDF file associated to ‘Cube’ [6], as we click to navigate our nonlinear narrative we move between scenes with two common narrators. If we click on a circle on one side of the screen we find the narrators on that side of the screen become the new narrators on the opposite side of the screen.

We have discussed how relations between scenes of a narrative can give rise to sequences of sets. Likewise relations between paragraphs can give rise to sequences of sets. For example, the first six paragraphs of 'The Adventure of the Blue Carbuncle' give rise to the following sequence of six sets

$$\{N, H\}, \{W, N\}, \{S, N, H\}, \{N, W, C\}, \{S, C, N\}, \{W, N, C\}$$

where we record an N if we find the voice of the narrator in the relevant paragraph, a W if we find Watson's speech, an S if we find Sherlock Holmes' speech, an H if we find mention of a hat, and a C if we find mention of a crime or criminal. The subsets associated to a sequence of paragraphs depend on the features we choose to associate, which may be motifs, or voices of given characters, or themes, for example.

In a generalised song, our mathematical structure Ma may contain a sequence of subsets which we then use to structure the paragraphs of the verbal structure V. For example, in 'Cube' the sequence $\{a\}, \{b\}, \{a, b\}$ has been used to structure our narratives as follows: each face of our cube corresponds to a scene with four narrators, components of Daniel's psyche, each of which describes the scene in three paragraphs, indexed by $\{a\}$, $\{b\}$, $\{a, b\}$ respectively. In paragraph $\{a\}$ a scene is set by the narrator. In paragraph $\{b\}$ a memory Daniel has of a female is described. In paragraph $\{a, b\}$ the narration of the original scene continues in the light of the memory Daniel has had.

In the composition of a generalised song, elements of the verbal component V can be associated to musical motifs. These motifs form combinatorial data, which are elements of the mathematical component Ma. In 'Cube' we define six motifs in this way, one for each face of our cube.

Let us define our terms. Consider the set of musical passages written on a stave that are a minim long, written without dynamics. We have an equivalence relation on such passages given by $p_1 \sim p_2$ if p_1 can be obtained from p_2 by transposition by some interval. An equivalence class under this equivalence relation we call a *motif*. If a motif features n distinct notes, we call it an *n-point motif*. An element of a motif we call a *motif representative*.

There are various strategies for associating a motif to a narrative. We illustrate some of these by describing the manner in which we chose six motifs for 'Cube score', one for each face of our cube. All motifs in 'Cube score' are 3-point motifs, with the intervals of the motif determined by some choices of generating intervals made by the player.

In ‘Cube score’ the 3-element set of notes for a motif is determined by the three numbers in the corresponding face of the cube depicted in Figure 2. To obtain a set of three notes for a motif representative, take a fixed note and raise it by the number of semitones in the corresponding face.

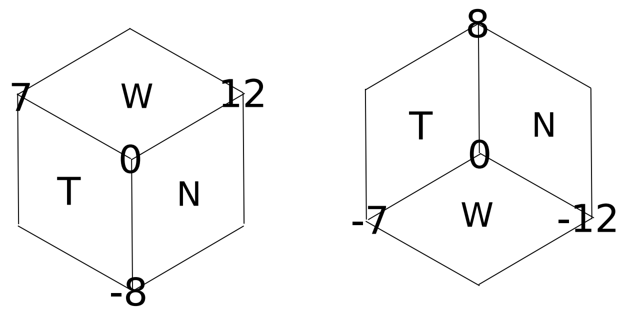

Figure 2: Intervals for motifs in ‘Cube score’.

We label the scenes of ‘Cube’ with a T,W,N, and an O, or F according to whether they concern *T*ransaction, *W*alking, or *N*arrative, and whether they have an *O*ther or *F*antasy narrator. The scene at the beginning of our pdf of ‘Cube’ is thus labelled TO [6].

In writing ‘Cube score,’ rather than associating a motif to the entirety of a scene, it seemed easier to focus attention on a significant feature of the scene and associate a motif to that. The relevant feature of the narrative was then written in the score. The motif ‘Swipe’ in bars 1-8 is thus associated with the swipe of the supermarket cashier in the supermarket scene TO. The motif ‘Exhilaration’ in bars 10-17 is associated with Daniel’s feeling as he walks in the wind in the WO scene. ‘Drops of blood’ in bars 19-26 is associated with the scattering of blood by the Witch’s wand in the NO scene. ‘Rising’ in bars 29-36 is associated with bread rising in the oven as it is baked in the NF scene. ‘Petal’ in bars 38-45 is associated with the child in the doctor’s surgery in the WF scene. ‘Granny’ in bars 47-54 is associated with the lady in Daniel’s memory as he falls asleep in the TF scene.

Certain motifs in ‘Cube score’ were suggested by physical elements of the narratives. For example the ‘Swipe’ motif is a swift descending sound, comparable with the swift monotone motion of a swipe. The ‘Rising’ motif is a relatively slow ascending phrase, comparable with the increase in volume of bread as it rises in the oven. The ‘Drops of blood’ motif is rhythmically complex, as it appeared to me the scattering of blood would be. The ‘Granny’ motif has two sounds, one for each syllable of the word granny. The associations for the other motifs were less direct. The 'Petal' motif was chosen to be simple, to be comparable with the simplicity of Petal's language. The 'Exhilaration' motif was chosen because the combination of the longer notes, the speedy transition of the shorter notes, and the large harmonic leap at the end of the motif, suggest variation, which Daniel experiences in his windy setting.

Note there is a great deal of flexibility and subjectivity involved in the definition of these motifs.

In a generalised song, we can refer explicitly to the mathematical structures in Ma in the verbal component. For example in 'Cube', we have sugar cubes in one of the scenes.

3. Some techniques for relating mathematical structures to musical structures.

There are numerous mathematical techniques to be found in music, for example in the work of J. S. Bach. Here we describe a few techniques for relating mathematical structures to musical structures in generalised songs. These are the techniques we use in 'Cube.'

Given a rooted tree, we define the level of a vertex v to be the number of edges along the unique path between it and the root. Ordered trees (or rooted plane trees) are finite trees with a single root vertex, and a linear ordering of the vertices of level i connected by an edge to v, for every vertex v of level $i-1$, for $i = 1, 2, ...$ Such trees are called rooted plane trees because they can naturally be embedded in the plane, with edges extending upwards from the root vertex, and linear orderings increasing from left to right.

Musical structures have ordered trees associated with their hierarchical structure. For example, a song cycle can be associated with such a tree whose vertices correspond to the cycle, the songs, the verses, the phrases, the bars, and the chords of the cycle. Its root is labelled by the song cycle; it has vertices of level 1 (respectively $2, 3, 4, 5$) labelled by songs (respectively verses, phrases, bars, chords) in the cycle. We have an edge connecting a vertex of level $i-1$ and a vertex of level i for $1 \leq i \leq 5$ if there is containment of the corresponding musical structures. We linearly order vertices by the order the corresponding musical structures come in the song cycle.

In this paper, the linear orderings on our tree vertices are a crucial feature of our musical hierarchies. If we were to drop them we would lose track of the order in which the various musical structures are played.

In a generalised song, we may have a natural construction of an ordered tree from our mathematical structure Ma, with this ordered tree corresponding to such a hierarchy in our musical structure Mu.

Commonly in a piece of music there are formal similarities between adjacent musical substructures of a given musical structure. To give an example from J. S. Bach's Prelude No. 1 from The Well Tempered Clavier Book 1 [1], the minims of the successive bars of this piece differ by 0,-1,1,0,0,-1,0,-2,-7,5,0,-2,0,-1,0,-2,-7,5,0,-7,1,2,-1,0,0,0,0,0,0,0,-7,0,0 semitones respectively. These are small numbers of semitones, corresponding to similar successive sounds.

In a generalised song organised with reference to an ordered tree as above, we have a linear ordering on the set of vertices of level i, for fixed i, inherited from the orderings that define the ordered tree. We can look for formal similarities between the mathematical structures in Ma corresponding to vertices that are adjacent in this ordering.

We have constructed 'Cube' to have sequences of adjacent passages that are similar in that they sound the same motif. Furthermore we have sequences of adjacent passages that are similar in that they sound a common set of notes. These passages are indexed by geometrical features of a cube with common elements. For example, Movement $\{a, b\}$ has six passages indexed by the faces of our cube. Each of these consists of nine notes, combined in a certain way. Faces with a common edge correspond to passages with three common notes. To give another example, Movement $\{a\}$ has passages indexed by so called edges of subfaces of our cube. Each of these consists of a motif, sounded in two parts simultaneously. Edges of a given subface with an element in common correspond to a sounding of the motif at a common pitch.

We have alluded to natural constructions of ordered trees from mathematical structures. To give an example of such a construction, we can associate plane trees to nested families of subsets of a given set S. The root of the tree corresponds to S, the vertices of level 1 correspond to subsets of S, the vertices of level 2 correspond to subsets of those subsets, etc. For example,

$$\begin{tikzcd}[arrows=dash]
\{1,2\} \arrow[d] & \{1,3\} \arrow[dl] & \{2,4\} \arrow[d] & \{3,4\} \arrow[dl] \\
\{1,2,3\} \arrow[d] & & \{2,3,4\} \arrow[dll] & \\
\{1,2,3,4\} & & &
\end{tikzcd}$$

Note subsets on our ordered tree that are adjacent with respect to the defining linear orderings, have common elements. In 'Cube' we construct a more elaborate hierarchy of sets whose elements correspond to the motifs sounding in the piece. We choose adjacent subsets of a given set to have common elements, to create similarities between adjacent musical substructures of a given musical structure.

We can create similarities between the subsets of S alluded to above if we select them from families that are permuted under a given group action on S. In 'Cube' we work with families of subsets that are permuted under an action of the group of symmetries of our cube.

4. The musical component of 'Cube'.

The musical component of 'Cube' has three movements labelled $\{a\}$, $\{b\}$, and $\{a, b\}$, following one after another. Movement $\{a, b\}$ involves techniques used in movement $\{a\}$ and in movement $\{b\}$.

Let $C = [0, 2]^3$ be a cube with sides of length 2. Inside C we identify the set of vertices $\Omega = \{0, 2\}^3$ and the subset $\Lambda = \{0, 1, 2\}^3$. The cube C has six faces. Each face of C is divided into four squares with sides of length 1, one for each corner of the face. We call these squares the *subfaces* of C. We call an edge of one of these squares an edge of a subface. We define a face (respecively subface, edge) of Λ to be the intersection of a face (respectively subface, edge) of C with Λ.

We have a map m from the set of subfaces of Λ to $\{0, 1, 2, 3, 4\}$ that sends s to the minimum value of $x_1 + x_2 + x_3$ for $(x_1, x_2, x_3) \in s$. See Figure 3.

We write $\epsilon_1 = (1, 0, 0)$, $\epsilon_2 = (0, 1, 0)$, and $\epsilon_3 = (0, 0, 1)$. We fix an orientation of the surface of our cube, consistent with the cyclic ordering $(\epsilon_1, \epsilon_2, \epsilon_3)$.

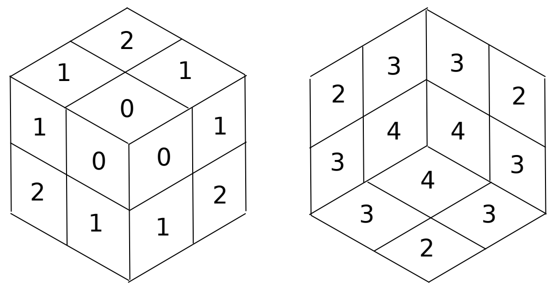

Figure 3: The map m.

The faces of 'Cube: the verbal component' can be naturally indexed by faces of Ω, with elements of Ω corresponding to narrators in the verbal component, the 'other' narrator corresponding to the corner $(0, 0, 0)$. The scenes of 'Cube' are divided into four narratives, one for each subface of Λ.

The musical part of 'Cube' involves a number of numerical and combinatorial choices. For reference, we have made an explicit set of such choices, and recorded the resulting score as 'Cube score' [8]. However, some other choices would presumably work just as well. When making our choices, we applied the obvious restriction that all the notes of our piece should lie in the human hearing range. We also chose the intervals defined by the numbers θ_i to be an octave, a perfect fifth, and a minor sixth, to generate consonances when played on a piano.

In 'Cube score,' movement $\{a\}$ is written in bars 1-54, movement $\{b\}$ is written in bars 58-65, and movement $\{a, b\}$ is written in bars 69-99.

Movement $\{a\}$. We fix integers θ_i between -12 and 12, for $i = 1, 2, 3$. We have a map $\theta : \Lambda \to \mathbb{Z}$ that sends $(\lambda_1, \lambda_2, \lambda_3)$ to $\lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3$. The image of θ gives us the number of semitones by which our motif representatives m_F are shifted, as they appear in our piece. See Figure 4.

We fix a note n on the stave. We then associate a triple of notes to each face of C by the formulas

$$((0, 0, 0) + [0, 2]\epsilon_i + [0, 2]\epsilon_j) \mapsto \{n, n + \theta_i, n + \theta_j\},$$

$$((2, 2, 2) - [0, 2]\epsilon_i - [0, 2]\epsilon_j) \mapsto \{n, n - \theta_i, n - \theta_j\},$$

for $i \neq j$. Here $n + \xi$ denotes the note ξ semitones above n. We fix a three point motif representative m_F for every face F of Ω whose three notes are given by the triple of notes associated to that face.

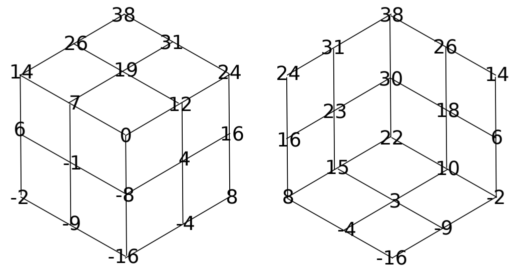

Figure 4: θ on the surface of C, for $\theta_1 = -8$,$\theta_2 = 7$, $\theta_3 = 12$.

In 'Cube score' we take $\theta_1 = -8$, $\theta_2 = 7$, $\theta_3 = 12$, whilst n is given by the C below middle C. Six 3-point motifs are also determined, as described in Section 2.

To define our movement we define an ordered tree whose root is labelled Λ. We have two vertices of level 1 labelled by the points $(0, 0, 0)$ and $(2, 2, 2)$ of Λ, taken in that order. We have six vertices of level 2 labelled by the faces of our cube. The first three of these are connected by an edge to vertex $(0, 0, 0)$ and correspond to the faces containing $(0, 0, 0)$, ordered $[0, 2] \times [0, 2] \times 0$, $0 \times [0, 2] \times [0, 2]$, $[0, 2] \times 0 \times [0, 2]$. The second three of these are connected by an edge to vertex $(2, 2, 2)$ and correspond to the faces containing $(2, 2, 2)$, ordered $[0, 2] \times 2 \times [0, 2]$, $2 \times [0, 2] \times [0, 2]$, $[0, 2] \times [0, 2] \times 2$. Every face vertex is connected by an edge to four subface vertices of level 3, one for each subface contained in the face. These subface vertices are ordered to respect the orientation of the cube face they are contained in, with the subface containing the corner vertex $(0, 0, 0)$ or $(2, 2, 2)$ coming first in the ordering. Each subface vertex is connected by an edge to four edge vertices of level 4, one for each edge contained in the subface. These edge vertices are ordered to respect the orientation of our cube subface, with the edges containing the corner vertex of the subface coming first and last in the ordering.

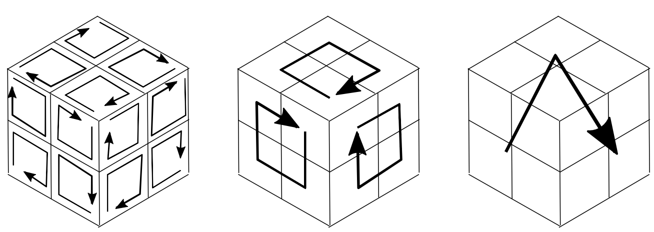

Figure 5: The ordering of passages in the first half of movement $\{a\}$.

Our movement has a motif, sounding in two parts simultaneously, for every edge vertex of our plane tree. The motif is given by m_F, where F is the face containing the subface whose vertex is attached to the edge vertex. The three notes of the motif are sounded as the three notes of m_F, raised by $\theta(v_1)$ semitones, and the three notes of m_F raised by $\theta(v_2)$ semitones, where v_1 and v_2 are the two vertices of Ω in our edge. In 'Cube score,' with $\theta_1 = -8$, $\theta_2 = 7$, $\theta_3 = 12$, the first motif representative appears in the first bar transposed by 0 and -8 semitones. It then appears transposed by -8 and -1 semitones. In the second bar it appears transposed by -1 and 7 semitones, then transposed by 7 and 0 semitones. These motifs correspond to the edges of the subface $\{0,1\} \times \{0,1\} \times 0$, as we move around it following the orientation of C.

To complete the definition of our movement, we need only to define the timing and volume of our motifs. For a feature f of C indexing a vertex of our plane tree, we call the passage corresponding to f the passage corresponding to the portions of the tree branching upwards from the vertex indexed by f. Between the two passages corresponding to $(0,0,0)$ and $(2,2,2)$ we place a two bar pause. Between the three passages corresponding to faces of C containing x we place a bar pause, if x is either $(0,0,0)$ or $(2,2,2)$. Passages corresponding to subfaces of a given face have no pause separating them. The tempo of a passage corresponding to subface s is $60 + 15m(s)$ crotchets per minute. The volume of a passage corresponding to subface s is $wm(s)$, where w is the bijection from $\{0,1,2,3,4\}$ to $\{p, mp, m, mf, f\}$ sending 0, 1, 2, 3 and 4 to p, mp, m, mf, and f, respectively.

Movement $\{b\}$. We fix integers ϕ_{ik} for $i = 0, 2$ and $k = 1, 2, 3$. We fix a note n' on the stave. We fix integers i_0 and i_2.

To define our movement we define an ordered tree. We have eight vertices of level 1 labelled by the elements of Ω, with the ordering on such vertices coinciding with the lexicographic ordering on Ω. We have sixty four vertices of level 2 labelled by the elements of $\Omega \times \Omega$, where a vertex of level 2 indexed by $((\omega_1, \omega_2, \omega_3), (\nu_1, \nu_2, \nu_3)) \in \Omega \times \Omega$ is connected by an edge to a vertex of level 1 indexed by $(\omega_1, \omega_2, \omega_3)$, and the ordering of such vertices by their second Ω coordinate is also lexicographic. The vertices of level 2 of our plane tree correspond to three note chords in our movement.

The three note chord corresponding to $((\omega_1, \omega_2, \omega_3), (\nu_1, \nu_2, \nu_3))$ is given by $n' + \phi_{\nu_1 1} + i_{\omega_1}, n' + \phi_{\nu_2 2} + i_{\omega_1}, n' + \phi_{\nu_3 3} + i_{\omega_1}$. Its duration is a quaver. Its volume is pp if $\omega_2 = 0$ and ff if $\omega_2 = 2$. Its tempo is $60 + 30\omega_3$. Two notes in two consecutive chords are tied whenever they take the same value.

In 'Cube score' we take n' to be middle C, and chose $i_0 = 0, i_2 = 3$, $\phi_{01} = 0, \phi_{21} = 1, \phi_{02} = -12, \phi_{22} = 13, \phi_{03} = -24, \phi_{23} = 25$.

Movement $\{a, b\}$. To define our movement we define an ordered tree whose root is labelled Λ. We have two vertices of level 1 labelled by the points $(0,0,0)$ and $(2,2,2)$ of Λ, taken in that order. We have six vertices of level 2 labelled by the faces of our cube. The first three of these are connected by an edge to vertex $(0,0,0)$ and correspond to the faces containing $(0,0,0)$, ordered $[0,2]\times[0,2]\times 0$, $0\times[0,2]\times[0,2]$, $[0,2]\times 0\times[0,2]$. The second three of these are connected by an edge to vertex $(2,2,2)$ and correspond to the faces containing $(2,2,2)$, ordered $[0,2]\times 2\times[0,2]$, $2\times[0,2]\times[0,2]$, $[0,2]\times[0,2]\times 2$. Every face vertex is connected by an edge to four subface vertices of level 3, one for each subface contained in the face. These subface vertices are ordered to respect the orientation of our cube surface, with the subface containing the corner vertex $(0,0,0)$ or $(2,2,2)$ coming first in the ordering. Each subface vertex is connected by an edge to eight subface vertices of level 4, indexed by the subface and an element of Ω; with the ordering on such vertices coincides with previous ordering of the subfaces, united with the lexicographic ordering on Ω. The vertices of level 4 of our plane tree correspond to three note chords in our movement.

Each subface s of Λ contains a unique corner c, and there is a unique subface of Ω that is contained in the same face of C as s and follows s in the orientation of that face. Let u be the unique common edge of these two subfaces; the edges of these subfaces excluding u, together with their vertices, constitute a hexagonal graph. Beginning with c, and following the orientation of our hexagon inherited from our orientation of the surface of C, we denote the vertices of this graph $v_{01}(s), v_{02}(s), v_{03}(s), v_{21}(s), v_{22}(s), v_{23}(s)$. The set of vertices we call $V(s)$.

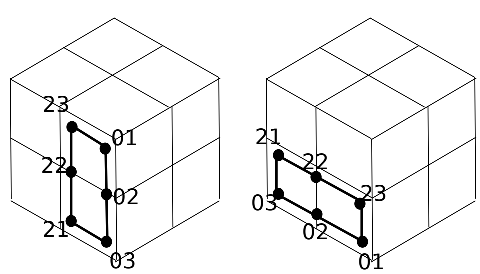

Figure 6: Hexagonal graphs for the first two bars of movement $\{a, b\}$.

We have a map m' from the set of subfaces of Λ to $\{0,1,2,3\}$ that sends s to the minimum value of $x_1+x_2+x_3$ for $(x_1,x_2,x_3) \in V(s)$.

The three note chord indexed by the subface s and element $(\omega_1,\omega_2,\omega_3) \in \Omega$ is given by $n+\theta(v_{\omega_1 1}), n+\theta(v_{\omega_2 2}), n+\theta(v_{\omega_3 3})$. Its duration is a quaver. Two notes in two consecutive chords are tied whenever they take the same value.

To complete the definition of our movement, we need to define the timing and volume of our motifs. Between the two passages corresponding to $(0,0,0)$ and $(2,2,2)$ we place a two bar pause. Between the three passages corresponding to faces of C containing x we place a bar pause, if x is either $(0,0,0)$ or $(2,2,2)$. Passages corresponding to subfaces of a given face have no pause separating them. The tempo of a passage corresponding to subface s is $60+20m'(s)$ crotchets per minute. The volume of a passage corresponding to subface s is $w'm'(s)$, where w' is the bijection from $\{0,1,2,3\}$ to $\{p, mp, mf, f\}$ sending 0, 1, 2, and 3 to p, mp, mf, and f respectively.

5. Further possibilities.

To make it readily accessible, the musical part of 'Cube' was written so it can be notated on a stave, and played through a pair of headphones. There are a couple of natural techniques for associating musical structures Mu to mathematical and verbal structures Ma and V that could combine with the techniques used in 'Cube', and involve the use of synthesised notes emanating from loudspeakers. To conclude our article, we describe these techniques.

Speaker arrangements. A geometrical mathematical structure in Ma may suggest some form of speaker arrangement, from which our musical part Mu then emanates. For example, the mathematical structure of a cube suggests an arrangement of loudspeakers in a cube directed at its centre. An alternative arrangement would be an equilateral triangle of loudspeakers directed at its centre, one loudspeaker for each of the three axes of our cube.

Mathematical functions. (cf. [2], 8.3) We can associate notes to smooth mathematical functions using sine waves. For example, if we have two mathematical functions a and f, and a duration d, then we have a note determined by the function $t \mapsto a(t)\sin(2\pi f(t)t)$. The function a determines how the volume of our note varies over time, whilst the function f determines its frequency, and how it 'bends' over time. There are numerous ways to associate mathematical functions to a narrative. To give one example, one of the transaction narratives of 'Cube' concerns cash. We could use the circular profile of a coin to associate to cash a function that sends t to $\sqrt{1-t^2}$ for $t \in [-1, 1]$ and t to zero otherwise, whose graph contains a semicircle.

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