3-11-2016

Review: On complex symmetric Toeplitz operators

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Recommended Citation
MR3404546 Ko, E., Lee, J.E., On complex symmetric Toeplitz operators, J. Math. Anal. Appl. 434 (2016), no. 1, 20-34. (Reviewer: Stephan R. Garcia)

MR3404546 (Review) 47B15 30H10
Ko, Eungil (KR-EWHA); **Lee, Ji Eun** (KR-SEJ)
On complex symmetric Toeplitz operators. **(English summary)**
J. Math. Anal. Appl. **434** (2016), *no. 1,* 20–34.

Let $\mathcal{H}$ denote a separable, complex Hilbert space. A *conjugation* on $\mathcal{H}$ is a conjugate-linear, isometric involution $C\colon \mathcal{H}\to\mathcal{H}$. A bounded linear operator $T\colon\mathcal{H}\to\mathcal{H}$ is C-*symmetric* if $T=CT^*C$. If there exists a C with respect to which T is C-symmetric, then T is called a complex symmetric operator (CSO). It is known that T is a CSO if and only if it is unitarily equivalent to a symmetric matrix acting on an ℓ^2 space of the appropriate dimension.

Every normal operator is a complex symmetric operator, as are all *truncated* Toeplitz operators. It is suspected that "most" Toeplitz operators are not CSOs, and index considerations provide many Toeplitz operators that are not CSOs. The unilateral shift T_z is such an example. As another example, if T_ϕ is analytic or coanalytic and a CSO, then ϕ is a constant function (Theorem 2.1 of the paper under review).

If $\phi\in L^\infty$ is real, then T_ϕ is self-adjoint, and hence a CSO. More generally, A. Brown and P. R. Halmos proved that a Toeplitz operator is normal if and only if its symbol is of the form $a\phi+b$, in which $a,b\in\mathbb{C}$ and ϕ is a real-valued function in L^∞ [J. Reine Angew. Math. **213** (1963/1964), 89–102; MR0160136]. For several natural conjugations C on the Hardy space H^2, the authors provide a characterization of the corresponding C-symmetric Toeplitz operators. These characterizations are given in terms of the Fourier coefficients of the symbol. As a consequence, they are able to construct non-normal, complex symmetric Toeplitz operators.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.