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Review: A C*-algebra approach to complex symmetric operators

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A C^*-algebra approach to complex symmetric operators. (English summary)
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This paper not only answers several open questions about complex symmetric operators but also provides a number of new tools and ideas for their study. Let $\mathcal{H}$ denote a separable complex Hilbert space. A conjugate-linear map $C\colon \mathcal{H}\to\mathcal{H}$ is called a *conjugation* if $C^2=I$ and $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathcal{H}$. A bounded linear operator T on $\mathcal{H}$ is a *complex symmetric operator* if $T=CT^*C$.

For a fixed $\mathcal{H}$, let CSO denote the set of all complex symmetric operators on $\mathcal{H}$; note that $T\in$ CSO if and only if there is a conjugation C on $\mathcal{H}$ so that $T=CT^*C$. The *norm closure problem* for complex symmetric operators [S. R. Garcia and W. R. Wogen, J. Funct. Anal. **257** (2009), no. 4, 1251–1260; MR2535469] asked whether or not CSO is norm closed in $\mathcal{B}(\mathcal{H})$; let $\overline{\text{CSO}}$ denote the closure of CSO with respect to the operator norm. In [Proc. Amer. Math. Soc. **140** (2012), no. 5, 1705–1708; MR2869154], S. Zhu, C. G. Li and Y. Q. Ji demonstrated that the Kakutani shift, which is not complex symmetric, belongs to $\overline{\text{CSO}}$. Shortly thereafter, Garcia and Poore [Proc. Amer. Math. Soc. **141** (2013), no. 2, 549; MR2996959] obtained another example of an operator in $\overline{\text{CSO}}\setminus$ CSO.

A unilateral weighted shift T with nonzero weights $\{\alpha_k\}_{k=1}^\infty$ is *approximately Kakutani* if for each $n\geq 1$ and $\epsilon>0$ there exists $N\in\mathbb{N}$ so that $0<|\alpha_n|<\epsilon$ and so that

$$1\leq k\leq n \quad\Longrightarrow\quad \big||\alpha_k|-|\alpha_{N-k}|\big|<\epsilon.$$

This concept was introduced by Garcia and Poore in [J. Funct. Anal. **264** (2013), no. 3, 691–712; MR3003733], where they conjectured that every irreducible weighted shift in $\overline{\text{CSO}}$ is approximately Kakutani. Among many other things, this paper provides a positive resolution to this conjecture. To do this, the authors introduce a variety of C^*-algebraic techniques to the study of complex symmetric operators and their relatives.

A crucial observation is that if $T\in\overline{\text{CSO}}$, then in many important cases T is actually a small *compact* perturbation of an operator in CSO. Given a subset $\mathcal{E}$ of $\mathcal{B}(\mathcal{H})$, let $\overline{\mathcal{E}}^c$ denote the *compact closure* of $\mathcal{E}$. This is the set of all $A\in\mathcal{B}(\mathcal{H})$ for which given any $\epsilon>0$ there is a compact K with $\|K\|<\epsilon$ and $A+K\in\mathcal{E}$. The authors consider the question of whether $\overline{\text{CSO}}$ and $\overline{\text{CSO}}^c$ coincide and they provide positive answers for several special classes of operators.

We say that $A\in\mathcal{B}(\mathcal{H})$ is called a *transpose* of T if $A=CT^*C$ for some conjugation C on $\mathcal{H}$. Representing A and T with respect to an orthonormal basis of $\mathcal{H}$, each of whose elements is fixed by C, confirms that the operator transpose is a simple generalization of the corresponding concept for matrices. We say that T is *UET* if T is unitarily equivalent to T^t (any two transposes are unitarily equivalent, so there is no ambiguity here). This concept was introduced by Garcia and J. E. Tener [J. Operator Theory **68** (2012), no. 1, 179–203; MR2966041]. One result of the authors here is a canonical decomposition of essentially normal operators that are UET. As a natural extension, we say that T is *AUET* if T is approximately unitarily equivalent to T^t. The authors provide the following important connection: if $C^*(T)$ contains no nonzero compact operators, then $T\in\overline{\text{CSO}}$ if and only if T is AUET.

An operator T is *g-normal* if it satisfies $\|p(T^*,T)\|=\|\tilde{p}(T,T^*)\|$ for any polynomial p

in two free variables, in which $\tilde{p}(x,y)$ is obtained from $p(x,y)$ by conjugating each coefficient. If T is a complex symmetric operator, then T is g-normal [S. R. Garcia, B. Lutz and D. Timotin, Proc. Amer. Math. Soc. **142** (2014), no. 5, 1749–1756; MR3168480]. The norm limit of g-normal operators is g-normal, so each operator in $\overline{\mathrm{CSO}}$ is g-normal. Is every g-normal operator on $\mathcal{H}$ in $\overline{\mathrm{CSO}}$? The authors show that this is not the case, although the situation is more interesting than it seems. For instance, if $C^*(T)$ does not contain a nonzero compact operator, then $T \in \overline{\mathrm{CSO}}$ if and only if T is g-normal.

The authors prove that

$$\mathrm{CSO} \subset \overline{\mathrm{CSO}} \cup \mathrm{UET} \subseteq \mathrm{AUET} \subset g\text{-normal operators}$$

and that each inclusion is proper. The main results of the paper concern the relations among these classes and they provide many illuminating examples. For certain classes of operators (e.g., weighted shifts, essentially normal operators) the authors go much further and obtain decompositions for these operators. There is much more in this paper than can be described in a brief summary. It is certainly bound to be one of the more influential articles on the subject of complex symmetric operators and their relatives. *Stephan R. Garcia*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.